ESKİŞEHİR TECHNICAL UNIVERSITY JOURNAL OF SCIENCE AND TECHNOLOGY
A- APPLIED SCIENCES AND ENGINEERING

2019, 20(1), pp. 54 - 69, DOI: 10.18038/aubtda.466203

THE EFFECT OF A RESIDUAL STRESS ON WAVE PROPAGATION IN A FLUID-FILLED THICK ELASTIC TUBE

Hakan EROL *

Department of Civil Engineering, Faculty of Engineering & Arch., Eskişehir Osmangazi University, Eskişehir, Turkey

ABSTRACT

The propagation of harmonic waves in an elastic tube filled with fluid is presented in this study. The tube material is considered to be incompressible, homogeneous, isotropic, initially axially stretched, inflated, and constructed of thick elastic, like human arteries. The viscous fluid is assumed to be incompressible and Newtonian. The differential equations of both materials are obtained in cylindrical coordinates. The analytical solutions of the equations of motion for the fluid and numerical solutions of the equations of motion for the tube have been found. The residual circumferential strain in the unloaded state of artery causes an opening angle. The dispersion relation is presented as a function of the axial stretch, opening angle, internal pressure, and material parameters. The effects of these parameters are shown and discussed in the graphics.

Keywords: Opening angle, Constitutive equations, Dispersion relation, Blood flow

1. INTRODUCTION

The propagation of pressure waves in a blown-up tube has been investigated since the time of Thomas Young who first showed the speed of pulsatile blood flow in arteries. Womersley [1], Morgan & Kiely [2] and Mirsky [3], have contributed significantly on this problem. Atabey & Lew [4] first considered the initial stretch of an arterial wall. Rachev [5] studied the effect of the transmural pressure of the artery as a membrane. Demiray & Antar [6] investigated the effects on the pressure wave of the thickness of the artery. Cox [7] and Kizilova [8] studied the effects of wall material parameters on the fluid volume flow-rate at thick viscoelastic tube. The blood that combined plasma and blood cells was first considered by Nayfeh [9] who investigated two-phase fluid flow. After that, Nag & Jana [10] studied the oscillating two-phase fluid flow in thin elastic tube without initial stresses. The pulsatile flow of a dusty fluid in an initial stressed thick elastic tube was studied by Ercengiz [11]. Demiray [12,13] investigated nonlinear waves in viscoelastic and elastic tubes. Jagielska et al. [14] determined the dispersion relations for thin-walled flexible fluid-filled tube surrounded by an external viscoelastic tissue. Chaudhry et al. [15] investigated the effects of strain in the oscillating arteries and residual stresses on the stress distribution.

The opening angle indicates the existence of residual stress. The existence of the opening angle was first revealed by Vaishnav & Vossoughi [16]. Huang & Yen [17], in their experimental measurements, observed that the mean opening angle varied between 46° and 82° in the arterial tree. However, there have been no studies that consider tubes subjected to this residual circumferential stress in the course of the wave propagation in the tube.

The purpose of this study is to discover the effect of the opening angle on wave propagation. For this purpose, wave propagation of an inflated, axial stretched, isotropic, and incompressible thick elastic tube filled with viscous fluid was studied. The ratio of arterial wall thickness to arterial radius changes was between 1/6 and ¼; therefore, the tube was considered to be thick. The arteries in the body are inflated with a mean pressure of approximately 13 kPa and subjected to axial stretch of about 1.5.

*Corresponding Author: herol@ogu.edu.tr
Received: 01.10.2018 Accepted: 19.03.2019

Considering this physiological condition, the artery was assumed to have circumferential stretch, an inner pressure of Pi, and axial stretch. These effects create large static stresses. Furthermore, the blood pressure change was around 2.5 kPa. The dynamical effect resulting from this pressure deviation was small compared to the initial static deformation. Therefore, the theory of small deformations superimposed on initial static deformations was used to obtain the governing equations in this study. The nonlinear terms in the equations are neglected because the amplitude of the pressure oscillation is much smaller than the mean pressure.

The analytical solutions of the equations of the fluid can be presented, but a closed-form solution of the equations of the tube cannot be obtained because of the variable coefficients of the governing equations. So, the governing equations of the tube are solved by the finite-difference method. The dispersion relation associated with harmonic wave propagation was derived. How internal pressure, opening angle, and axial stretch affects the dispersion relation were discussed in the graphics.

2. BASIC EQUATIONS

The problem studied here is due to the interactions of the tube with the fluid. Therefore, the motion of the wall, the motion of the fluid, and condition on their interface should be included in the study.

2.1. Equation for Fluid

Blood is known to be an incompressible, non-Newtonian fluid. Heartbeat causes small pressure and velocity increments on large static pressure P_i. It has been assumed that when body forces and initial velocity are absent, blood flow is axially symmetric. Under these assumptions the governing differential equations are written as

$$-\frac{\partial \hat{p}}{\partial r}+\hat{\mu}\left(\frac{\partial^2 \hat{u}}{\partial r^2}+\frac{1}{r}\frac{\partial \hat{u}}{\partial r}-\frac{\hat{u}}{r^2}+\frac{\partial^2 \hat{u}}{\partial z^2}\right)-\hat{\rho}\frac{\partial \hat{u}}{\partial t}=0 \tag{1}$$

$$-\frac{\partial \hat{p}}{\partial z}+\hat{\mu}\left(\frac{\partial^2 \hat{w}}{\partial r^2}+\frac{1}{r}\frac{\partial \hat{w}}{\partial r}+\frac{\partial^2 \hat{w}}{\partial z^2}\right)-\hat{\rho}\frac{\partial \hat{w}}{\partial t}=0\,. \tag{2}$$

In the cylindrical coordinates and the incompressibility equation

$$\frac{\partial \hat{u}}{\partial r}+\frac{\hat{u}}{r}+\frac{\partial \hat{w}}{\partial z}=0\,, \tag{3}$$

where $\hat{\rho}$ denotes the density of the fluid, $\hat{\mu}$ denotes the viscosity, $\hat{p}$ is the pressure and $(\hat{u},\hat{w})$ are the velocity components of fluid. The stresses components which for applying the boundary conditions are given by

$$\hat{t}_{rr}=-\hat{p}+2\hat{\mu}\frac{\partial \hat{u}}{\partial r}\,,\ \hat{t}_{rz}=\hat{\mu}\left(\frac{\partial \hat{u}}{\partial z}+\frac{\partial \hat{w}}{\partial r}\right). \tag{4}$$

2.2. Equation for Tube

For the mathematical analysis of the problem, the artery wall is assumed to be incompressible, isotropic and elastic; the tube is subjected to high internal pressure P_i; there is a circumferential and axial stretch; and there is no twist. These conditions are depicted in Figure 1 as Loaded Configuration. This configuration is assumed to be a large static load in the mathematical model. Unloaded configuration describes a condition in which no load applied to the artery is removed from the body. But in this configuration, stress and strain on the artery still exist. If arterial ring is cut in a radial direction, it opens. This state is depicted in Figure 1. as Zero-stress Configuration. Thus, the tube's motion can be described in a cylindrical coordinate system as follows:

$$r = r(R), \qquad \theta = \Gamma\Theta, \qquad z = \lambda Z, \tag{5}$$

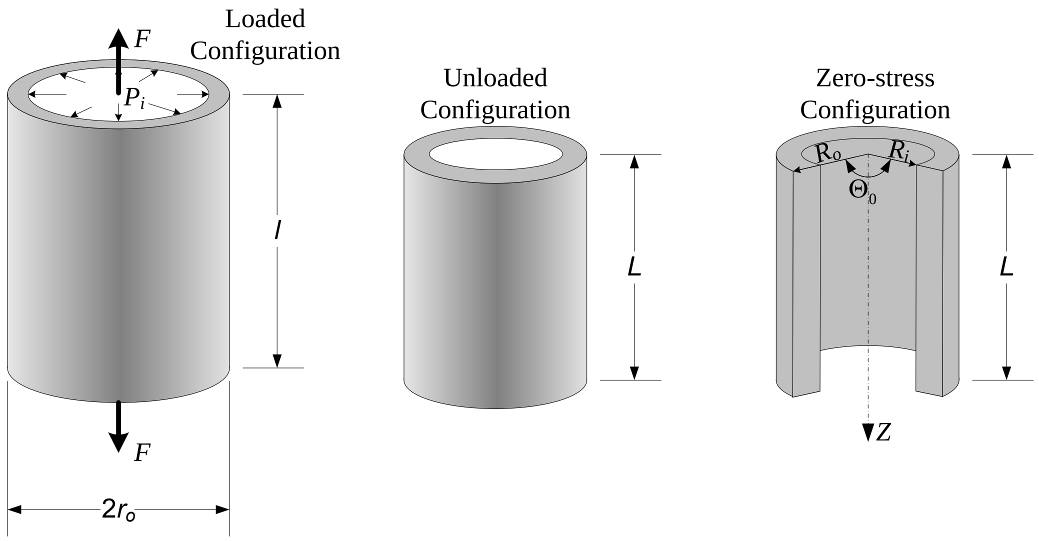

Figure 1. Arterial ring in the various configurations

where (R,Θ,Z), (r,θ,z) are the coordinates in the zero-stress state and loaded state, respectively, Γ is defined by $\Gamma = \pi/(\pi - \Theta_0/2)$, $\lambda_z = \lambda$ is the constant axial stretch, Θ_0 is an opening angle characterizing the residual stresses. The principle stretch ratios in axial, circumferential, radial direction are given as

$$\lambda = \frac{z}{Z}, \qquad \lambda_r = \frac{\partial r}{\partial R}, \qquad \lambda_\theta = \frac{\Gamma r}{R}. \tag{6}$$

Thus, the deformation gradient tensor $F_{kK} \equiv \partial x_k / \partial X_K$ may be expressed as

$$\mathbf{F} = \mathrm{diag}\left[\frac{\partial r}{\partial R}, \frac{\Gamma r}{R}, \lambda_z\right]. \tag{7}$$

From the incompressibility condition $\det \mathbf{F} = 1$ we get

$$r = \sqrt{r_i^2 + \frac{1}{\Gamma\lambda}\left(R^2 - R_i^2\right)}. \tag{8}$$

A set stress-strain relations for the soft biological tissues was proposed by Demiray [18] as

$$t_{kl}^0 = P^0\delta_{kl} + \beta c_{kl}^{-1}\exp\left[\alpha(I-3)\right], \tag{9}$$

where t_{kl}^0 is the stress tensor, P^0 is the hydrostatic pressure, and superscript (^{0}) describes the initial condition, δ_{kl} is the Kronecker delta, α, β are the material constants, and I is the first invariant of the finger deformation tensor c_{kl}^{-1} which is defined as

$$c_{kl}^{-1} = F_{kK}F_{lK} = \text{diag}\left[\frac{x^2}{\lambda^2}, \frac{\Gamma^2}{x^2}, \lambda^2\right], \qquad x \equiv \frac{R}{r}; \tag{10}$$

Introducing (10) into (9), the Cauchy stress tensor may be expressed as

$$\begin{aligned} &t_{rr}^0 = P^0 + \beta\frac{x^2}{\lambda^2}F(x), \ \ t_{\theta\theta}^0 = P^0 + \beta\frac{\Gamma^2}{x^2}F(x), \\ &t_{zz}^0 = P^0 + \beta\lambda^2 F(x), \quad t_{kl}^0 = 0(k \neq l), \qquad . \\ &F(x) = \exp\left[\alpha\left(\frac{x^2}{\lambda^2} + \frac{\Gamma^2}{x^2} + \lambda^2 - 3\right)\right] \end{aligned} \tag{11}$$

These stress components must satisfy the equations of motion in cylindrical the coordinate system, which is given as

$$\frac{\partial t_{rr}^0}{\partial r} + \frac{1}{r}\left(t_{rr}^0 - t_{\theta\theta}^0\right) = 0, \ \ \frac{\partial t_{\theta\theta}^0}{\partial \theta} = 0, \qquad \frac{\partial t_{zz}^0}{\partial z} = 0\,. \tag{12}$$

Introducing (11) into (12), we get,

$$\begin{aligned} &t_{rr}^0 = \frac{\beta}{\lambda^2}\int_{x_o}^{x}\left(\zeta + \frac{\lambda}{\zeta}\right)d\zeta, \qquad t_{\theta\theta}^0 = t_{rr}^0 + \frac{\beta}{\lambda^2}\left(\frac{\Gamma^2\lambda^2}{x^2} - x^2\right)F(x), \\ &t_{zz}^0 = t_{rr}^0 + \frac{\beta}{\lambda^2}\left(\lambda^4 - x^2\right)F(x), \quad P^0 = t_{rr}^0 - \frac{\beta}{\lambda^2}x^2F(x), \\ &P_i = \frac{\beta}{\lambda^2}\int_{x_i}^{x_o}\left(\zeta + \frac{\lambda}{\zeta}\right)F(\zeta)\,d\zeta, \ F(x) = \exp\left[\alpha\left(I - 3\right)\right] \end{aligned} \tag{13}$$

where subscripts (i) and (o) denote the value of a quantity written on the inner and outer surfaces of the tube, respectively. These stress components correspond to a large static stress field due to mean internal pressure, axial stretch, and opening angle. More information about the theory of superimposing small deformations on large initial static deformations can be found in Baek et al. [19]. To investigate small pressure oscillation at this initial state, the incremental Piola-Kichhoff stress tensor was proposed by Demiray [20] as

$$T_{kl} = \bar{t}_{kl} + t_{kl}^0 u_{l,m}\,, \tag{14}$$

where u_l is incremental displacement components and $\bar{t}_{kl}$ is defined as

$$\bar{t}_{kl} = \bar{p}\delta_{kl} - 2\bar{P}^0 e_{kl} + 2\alpha\beta\exp\left[\alpha\left(I - 3\right)\right]c_{mn}^{-1}e_{mn}c_{kl}^{-1}\,. \tag{15}$$

where e_{mn} is defined as $e_{mn} = \left(u_{m,n} + u_{n,m}\right)/2$. Hereafter, we may write the equations of incremental motion. For this purpose, incremental motion for the tube is considered to be an axially symmetric and there is no twist on the tube. Therefore, this motion is described as

$$u_1 = u(r,z,t), \ \ u_2 = 0, \ u_3 = w(r,z,t). \tag{16}$$

Introducing (16), (15) and (13) into (14) and recalling the incompressibility condition $u_{k,k} = 0$ non-zero, the components of the Piola-Kichhoff stress tensor are written as

$$T_{rr}=t_{rr}^{0}\frac{\partial u}{\partial r}+\bar{p}+\frac{\partial u}{\partial r}\bar{\alpha}_1+\frac{u}{r}\bar{\alpha}_2, \qquad T_{\theta\theta}=t_{\theta\theta}^{0}\frac{u}{r}+\bar{p}+\frac{\partial u}{\partial r}\bar{\alpha}_3+\frac{u}{r}\bar{\alpha}_4$$
$$T_{zz}=t_{zz}^{0}\frac{\partial w}{\partial z}+\bar{p}+\frac{\partial u}{\partial r}\bar{\alpha}_5+\frac{\partial w}{\partial z}\bar{\alpha}_6, \qquad T_{rz}=t_{rr}^{0}\frac{\partial w}{\partial r}-\bar{P}^{0}\left(\frac{\partial w}{\partial r}+\frac{\partial u}{\partial z}\right) \tag{17}$$
$$T_{zr}=t_{zz}^{0}\frac{\partial w}{\partial z}-\bar{P}^{0}\left(\frac{\partial w}{\partial r}+\frac{\partial u}{\partial z}\right)$$

where $\bar{\alpha}_j,\left(j=1,\ldots,6\right)$ are described as

$$\bar{\alpha}_1=\left(\frac{x^4}{\lambda^4}-x^2\right)2\alpha\beta F\left(x\right)-2\bar{P}^0, \qquad \bar{\alpha}_2=\left(\frac{\Gamma^2}{\lambda^2}-x^2\right)2\alpha\beta F\left(x\right),$$
$$\bar{\alpha}_3=\left(\frac{\Gamma^2}{\lambda^2}-\frac{\lambda^2\Gamma^2}{x^2}\right)2\alpha\beta F\left(x\right), \qquad \bar{\alpha}_4=\left(\frac{x^4}{\lambda^4}-\frac{\lambda^2\Gamma^2}{x^2}\right)2\alpha\beta F\left(x\right)-2\bar{P}^0, \tag{18}$$
$$\bar{\alpha}_5=\left(x^2-\frac{\lambda^2\Gamma^2}{x^2}\right)2\alpha\beta F\left(x\right), \qquad \bar{\alpha}_6=\left(\lambda^4-\frac{\lambda^2\Gamma^2}{x^2}\right)2\alpha\beta F\left(x\right)-2\bar{P}^0,$$

These stress field satisfy the governing differential equations which may be written as (Eringen & Suhubi [8])

$$T_{kl,k}+\rho\bar{f}_l-\frac{\partial^2 u_l}{\partial t^2}=0 \tag{19}$$

where ρ is the mass density of the tube material and $\bar{f}_l$ is the body forces vector. If body forces are vanished and (17) written into (19), we get,

$$\frac{\partial p}{\partial r}+\frac{\partial^2 u}{\partial r^2}\bar{\beta}_1\left(r\right)+\frac{\bar{\beta}_2}{r}\frac{\partial u}{\partial r}-\bar{\beta}_3\frac{u}{r^2}+\bar{\beta}_4\frac{\partial^2 u}{\partial z^2}-\rho\frac{\partial^2 u}{\partial t^2}=0,$$
$$\frac{\partial p}{\partial z}+\bar{\beta}_5\frac{\partial^2 w}{\partial r^2}+\frac{\bar{\beta}_6}{r}\frac{\partial w}{\partial r}+\bar{\beta}_7\frac{\partial^2 w}{\partial z^2}+\frac{\bar{\beta}_8}{r}\frac{\partial u}{\partial z}-\rho\frac{\partial^2 w}{\partial t^2}=0, \tag{20}$$
$$\frac{\partial u}{\partial r}+\frac{u}{r}+\frac{\partial w}{\partial z}=0,$$

where $\bar{\beta}_j,\left(j=1,\ldots,8\right)$ are described as

$$\bar{\beta}_1=t_{rr}^0+\bar{\alpha}_1+\bar{P}^0, \qquad \bar{\beta}_2=t_{\theta\theta}^0+\bar{\alpha}_2+\frac{d}{dr}\left(r\bar{\alpha}_1\right)-\bar{\alpha}_3+\bar{P}^0$$
$$\bar{\beta}_3=-r\frac{d\alpha_2}{dr}+t_{\theta\theta}^0+\bar{\alpha}_4+\bar{P}^0, \quad \bar{\beta}_4=t_{zz}^0-\bar{P}^0$$
$$\bar{\beta}_5=t_{rr}^0-\bar{P}^0, \quad \bar{\beta}_6=t_{\theta\theta}^0-\frac{d}{dr}\left(r\bar{P}^0\right), \quad \bar{\beta}_7=t_{zz}^0+\bar{\alpha}_6-\bar{\alpha}_5+\bar{P}^0 \tag{21}$$
$$\bar{\beta}_8=-\bar{\alpha}_5-r\frac{d\bar{P}^0}{dr}$$

The above system of differential equations must satisfy boundary conditions. The stresses must be zero on the outside of the tube. Stresses and displacement components on the fluid-tube interface inside the tube because of the continuity of the points.

$$
\begin{aligned}
&T_{rr}\big|_{r_i} = \hat{t}_{rr}(r_i,t) - t_{rr}^0(r_i)\frac{\partial u}{\partial r}(r_i,t), \qquad T_{rz}\big|_{r_i} = \hat{t}_{rz}(r_i,t),\\
&T_{rr}\big|_{r_o} = 0, \qquad T_{rz}\big|_{r_o} = 0, \qquad .\\
&\frac{\partial u}{\partial t}(r_i,t) = \hat{u}(r_i,t), \qquad \frac{\partial w}{\partial t}(r_i,t) = \hat{w}(r_i,t).
\end{aligned} \tag{22}
$$

3. SOLUTION OF THE FIELD EQUATIONS

In this section, with regard to the pulsatile motion of the heart, we seek harmonic type solution to the field equations given in (1)-(3) for fluid and (20) for the tube. One can see from equations (1)-(3) that a closed-form solution can be presented for the field equation of fluid. However, a numerical solution for the equations of tube can be given by the finite-difference method.

3.1. Solution of the Field Equations for Fluid

As noted previously, when harmonic type solution is sought, we set

$$
(\hat{p},\hat{u},\hat{w}) = \left\{\hat{P}(r),\, \hat{U}(r),\, \hat{W}(r)\right\} \exp\left[i(\omega t - kz)\right] \tag{23}
$$

where ω denotes the angular frequency, k is axial wave number, and $\hat{P}(r),\hat{U}(r),\hat{W}(r)$ are complex amplitudes to be determined from solution of differential equations and boundary conditions. Introducing (23) into equations (1)-(3), the solution of the differential equations for $\hat{P}(r),\hat{U}(r),\hat{W}(r)$ may be given as

$$
\begin{aligned}
&\hat{P}(r) = AI_0(kr), \qquad \hat{U}(r) = \frac{AkI_1(kr)}{\mu\left(k^2+s^2\right)} + \frac{iBk}{s}J_1(rs)\\
&\hat{W}(r) = -\frac{iAkI_0(kr)}{\mu\left(k^2+s^2\right)} + BJ_0(rs), \qquad s^2 = -\frac{i\rho\omega}{\mu} - k^2
\end{aligned} \quad . \tag{24}
$$

Where A and B are integration constants to be calculated by boundary conditions, $J_n(rs)$ and $I_n(kr)$ are the first kind of Bessel and modified Bessel functions of order n, respectively. To get stress components in equation (4), (24) is substituted in (4),

$$
\begin{aligned}
&\hat{t}_{rr} = \left\{ \begin{array}{l} A\left[-I_0(kr) + \dfrac{2k}{r\left(k^2+s^2\right)}\left(krI_0(kr) - I_1(kr)\right)\right] + \\ \dfrac{2Bik\mu}{rs}\left[rsJ_0(rs) - J_1(rs)\right] \end{array} \right\} \exp\left[i(\omega t - kz)\right],\\
&\hat{t}_{rz} = \left\{-\frac{2Aik^2}{k^2+s^2}I_1(kr) + \left(\frac{k^2-s^2}{s}\right)B\mu J_1(rs)\right\}\exp\left[i(\omega t - kz)\right],
\end{aligned} \quad . \tag{25}
$$

These stress components will be used in the solution for governing equations for the tube as boundary conditions.

3.2. Solution of the Field Equations for the Tube

As in the solution of the field equations for fluid, a harmonic type solution is sought. For that we set again,

$$(p,u,w)=\left\{\bar{P}(r),\bar{U}(r),\bar{W}(r)\right\}\exp\left[i(\omega t-kz)\right]. \tag{26}$$

If (26) is introduced in (20), ordinary differential equations for the tube are obtained as

$$\begin{aligned}
&\frac{dP}{d\xi}+\beta_1\frac{d^2U}{d\xi^2}+\frac{\beta_2}{\xi}\frac{dU}{d\xi}+\left(\Omega^2-\beta_4\eta^2-\frac{\beta_3}{\xi^2}\right)U=0\\
&-i\eta P+\beta_5\frac{d^2W}{d\xi^2}+\frac{\beta_6}{\xi}\frac{dW}{d\xi}+\left(\Omega^2-\beta_7\eta^2\right)W-\frac{i\eta\beta_8}{\xi}U=0 \quad .\\
&\frac{dU}{d\xi}+\frac{U}{\xi}-i\eta W=0
\end{aligned} \tag{27}$$

In order to normalize the above equations, the following non-dimensional parameters are introduced:

$$\begin{aligned}
&r=\bar{r}\xi, \qquad \bar{r}=\frac{r_i+r_o}{2}, \qquad \eta=\bar{r}k,\ \bar{\beta}_j=\beta\beta_j(\xi),(j=1,...,8)\\
&\bar{P}=\beta P, \qquad \bar{U}=\bar{r}U, \qquad \bar{W}=\bar{r}W, \qquad \Omega^2=\frac{\bar{r}^2\rho\omega^2}{\beta}
\end{aligned} \quad . \tag{28}$$

Using similar procedures, the boundary conditions in (22) can be normalized into the following forms:

$$\begin{aligned}
&\left[P+\frac{\alpha_2}{\xi}U+(\beta_1+\beta_5)\frac{dU}{d\xi}\right]_{\xi=\xi_i}+\left(\frac{f(\gamma^2-\eta^2)}{\eta(\gamma^2+\eta^2)}+\frac{2\eta}{\xi(\gamma^2+\eta^2)}\right)\bar{A}+\left(\frac{2i\eta}{\gamma\xi}(1-g\xi)\right)\bar{B}=0,\\
&\left[i\eta P^0U+\beta_5\frac{dW}{d\xi}\right]_{\xi=\xi_i}+\frac{2i\eta^2}{\gamma^2+\eta^2}\bar{A}+\frac{\gamma^2-\eta^2}{\gamma}\bar{B}=0,\\
&\left[P+\frac{\alpha_2U}{\xi}+(\alpha_1+t_{rr}^0)\frac{dU}{d\xi}\right]_{\xi=\xi_o}=0,\\
&\left[iP^0\eta U+\beta_5\frac{dW}{d\xi}\right]_{\xi=\xi_o}=0,\\
&\left[i\Omega^2U\right]_{\xi=\xi_i}-\frac{\bar{\alpha}^2q\eta}{\gamma^2+\eta^2}\bar{A}-\frac{iq\bar{\alpha}^2\eta}{\gamma}\bar{B}=0,\\
&i\Omega^2W+\frac{ifq\bar{\alpha}^2}{\gamma^2+\eta^2}\bar{A}-\frac{gq\bar{\alpha}^2}{\gamma}\bar{B}=0,
\end{aligned} \tag{29}$$

where

$$\gamma = \bar{r}s,\ q = \frac{\rho}{\rho_0}, \qquad \bar{P}^0 = \beta P^0, \qquad \bar{\alpha}^2 = \frac{\omega \bar{r}^2}{\nu}, \qquad \xi_i = \frac{r_i}{\bar{r}},$$

$$\xi_o = \frac{r_o}{\bar{r}}, \qquad \bar{\alpha}_1 = \beta\alpha_1, \quad \bar{\alpha}_2 = \beta\alpha_2, \qquad f = \frac{\eta I_0(\eta\xi)}{I_1(\eta\xi)},\ g = \frac{\gamma J_0(\gamma\xi)}{J_1(\gamma\xi)},. \tag{30}$$

$$\bar{A} = \frac{I_1(\eta\xi)}{\beta}A, \qquad \bar{B} = \frac{J_1(\gamma\xi)}{\bar{r}\beta}B,$$

The variable $\bar{\alpha}$, known as the Womersley number, denotes the ratio of unsteady internal forces to viscous forces in the flow. Because the coefficients of the equations are complex, it is almost impossible to achieve a closed form solution. Therefore, the finite-difference method is used to solve equations (27) and (29). To obtain dimensionless quantities, we divide the thickness of the tube, $h = r_o - r_i$, into *n* equal intervals, defining thus:

$$\xi_j = \xi_0 + j\frac{h}{n\bar{r}},\ j = 0,1,...,n, \quad \xi_0 = \xi_{\text{int.}} = 1 - \frac{h}{2\bar{r}}, \qquad \xi_n = \xi_o. \tag{31}$$

By introducing these terms for various derivatives, we get the finite difference equations given in the appendix for the equations (27) and (29). This system is known as the eigensystem. Therefore, the determinant of the coefficient matrix must be zero. Thus, the condition of the determinate of the coefficient matrix gives the dispersion relations for the Young mode and the Lamb mode.

4. NUMERICAL ANALYSIS AND DISCUSSION

First, some special cases will be investigated for the verification of the present study. To this end, special cases in the literature will be examined, by making some simplifications in the equations obtained for the general case.

4.1. Long Wave Approximation and Thin Tube with Inviscid Fluid

The wavelength of the oscillations is much larger than the radius of the artery (Atabek & Lew) [22]. Therefore, to simplify the frequency equation, we will assume that the wave number *k* is very small and $\mu \ll 1$ and f denoted by (30) approaches 2. Using the definitions $\gamma = s\bar{r}$ and $\bar{\alpha}^2 = \hat{\rho}i\omega\bar{r}^2/\hat{\mu}$ in (24), we get

$$\gamma^2 \approx -\bar{\alpha}^2. \tag{32}$$

To compare the obtained results with previous studies, the wall of tube is assumed to be a tube filled with a non-viscous fluid. As a result, the dispersion relation may be written in term of the complex phase velocity $c = \Omega/\eta$ by setting $n = 1$, $\gamma \to \infty$ and $g \to \infty$ become as follows:

$$B1c^4 + B2c^2 + B3 = 0 \tag{33}$$

where *B1*, *B2,* and *B3* coefficients are determined from the equation (27) under the above assumptions. In the absence of an initial deformation $(\lambda = 1, \Gamma = 1)$, these coefficients can be obtained:

$$\begin{aligned} B1 &= -2hqi \\ B2 &= 8hqi + 4ih^2q^2\ . \\ B3 &= -12ih^2q^2 \end{aligned} \tag{34}$$

The approximate roots of the equation (33) can be calculated under the assumption that *h* is quiet for the thin tube as

$$c_1^2 = 4 + O(h), \qquad c_2^2 = \frac{3qh}{2} + O(h^2). \tag{35}$$

The complex roots are represented as

$$c = X + iY, \tag{36}$$

proposed by Atabek & Lew [22], the velocity of wave propagation v and transmission per wavelength χ are defined by

$$v = \frac{X^2 + Y^2}{X}, \quad \chi = \exp\left[-2\pi \frac{Y}{X}\right]. \tag{37}$$

Using the real physical quantities and setting $\beta = E/3$, where E is the Young modulus of the tube, the speeds of the propagation may be expressed as

$$v_1^2 = \frac{4E}{3\rho}, \qquad \chi_1 = 1, \qquad v_2^2 = \frac{Eh}{2\hat{\rho}r}, \qquad \chi_2 = 1 \tag{38}$$

where v_1 denotes Lamb mode wave speed and v_2 denotes Moens-Korteweg wave speed.

4.2. Long-Wave Approach in a Thick Tube Filled With a Viscous Fluid

To examine the dispersion relation numerically, the inner and outer radii $R_i = 3.1mm, R_o = 3.8mm$ presented by Simon et al. [23] and material constants $\alpha = 1.948, \beta = 9.9kPa$ calculated by Demiray [24] will be used. The thickness of the arterial wall is divided into four parts (n=4). There have not been major changes in the result for much larger values of n. The ratio of the density of the arterial wall to density of the blood q is approximately equal to 1 ($q \approx 1$).

The dispersion relation is derived for the given assumptions. The speeds of wave propagation and transmission coefficients are calculated for various axial stretch, internal pressure, and opening angles. The results are discussed in figures 2-9.

In Figures 2 and 3, changes in wave propagation velocity and transmission coefficient for a different axial stretch (λ) and internal pressure (P_i) related to the Womersley number are shown. Figure 2 shows that the primary wave speed v_1 increases with the axial stretch and decreases with the P_i for all Womersley number values. Figure 3 shows that the transmission coefficient χ_1 of the primary wave speed decreases for $\bar{\alpha}$ <3 with the increase of $\bar{\alpha}$ and increases of the λ decreases of P_i. But the effect of internal pressure and axial stretch was not observed sufficiently on transmission coefficient χ_1. These results are in good agreement with Demiray & Antar [6], Demiray & Ercengiz [25], and Demiray & Akgün [26].

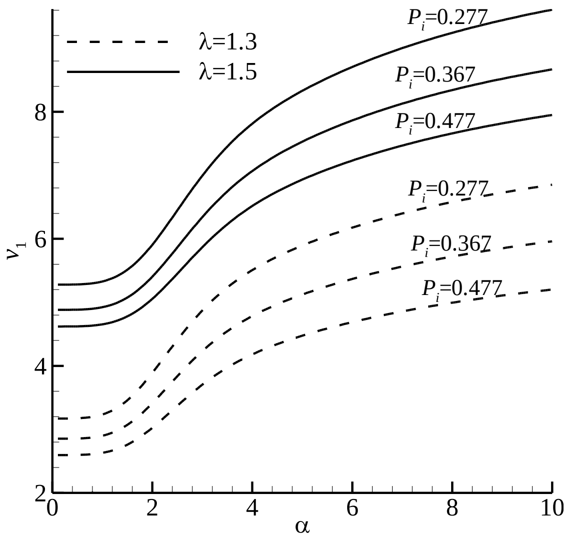

Figure 2. Change in v_1 related to $\bar{\alpha}$ for different P_i and λ values.

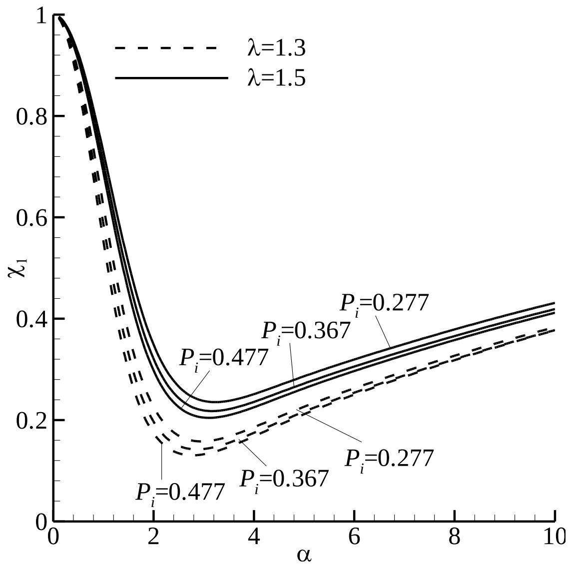

Figure 3. Change in χ_1 related to $\bar{\alpha}$ for different P_i and λ values.

In Figure 4, secondary wave speed v_2 increases for all values of $\bar{\alpha}$ with increases of λ and increases of P_i. The effect of the internal pressure was more noticeable when the axial stretch was greater. Figure 5 shows that the transmission coefficient χ_2 of the secondary wave speed increases with the decreases of λ and the increases of P_i. These results are in good agreement with Demiray & Antar [6], Demiray & Ercengiz [25], and Demiray (Akgün [26].

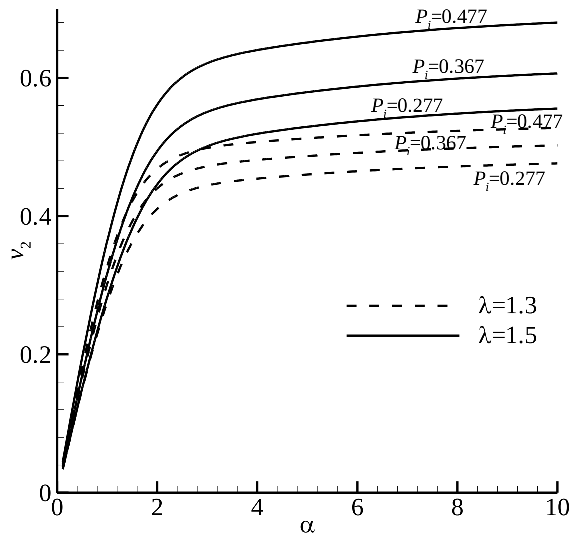

Figure 4. Change in v_2 related to $\bar{\alpha}$ for different P_i and λ values.

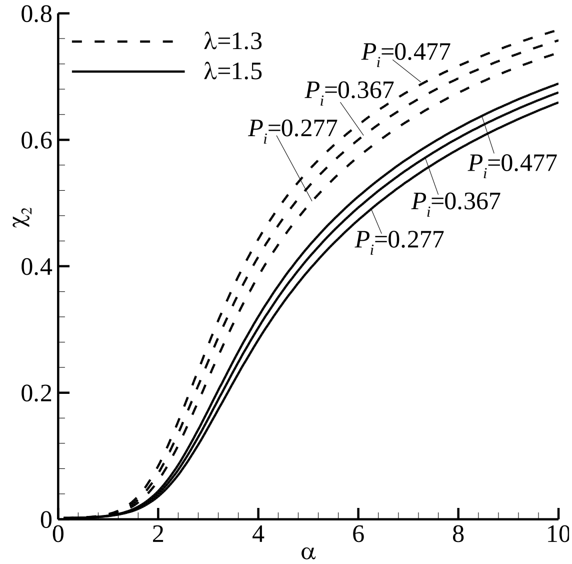

Figure 5. Change in χ_2 related to $\bar{\alpha}$ for different P_i and λ values.

The effect of the opening angle on wave characteristics was depicted in Figures 6-9. In Figure 6, an increase in the opening angle caused increased primary wave speed, but the opening angle was more effective at the lower axial stretch. The transmission coefficient was not sufficiently affected from opening angle in Figure 7.

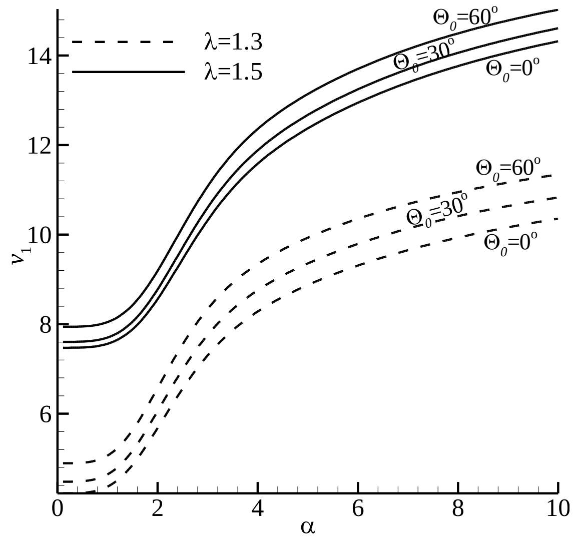

Figure 6. Change in v_1 related to $\bar{\alpha}$ for different Θ_0 and λ values.

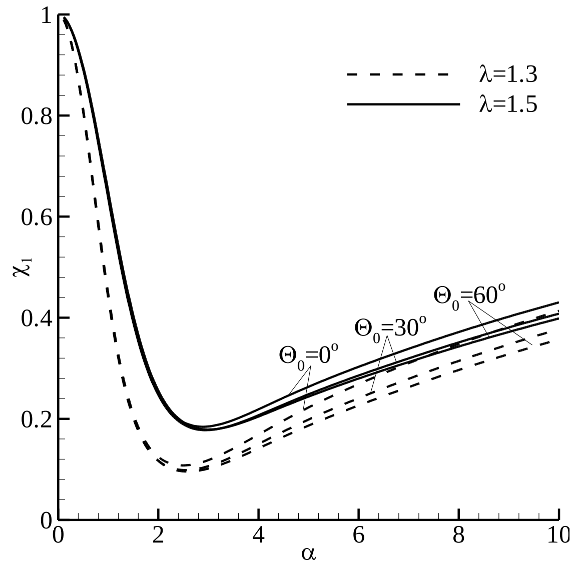

Figure 7. Change in χ_1 related to $\bar{\alpha}$ for different Θ_0 and λ values.

The opening angle had a major impact on the secondary wave speed when axial stretch was smaller, as in Figure 8. On the other hand, it did not affect the transmission coefficient χ_2, as in Figure 9.

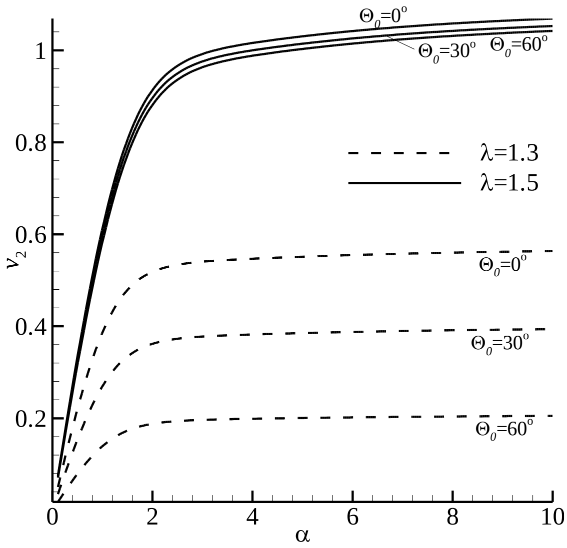

Figure 8. Change in v_2 related to $\bar{\alpha}$ for different Θ_0 and λ values.

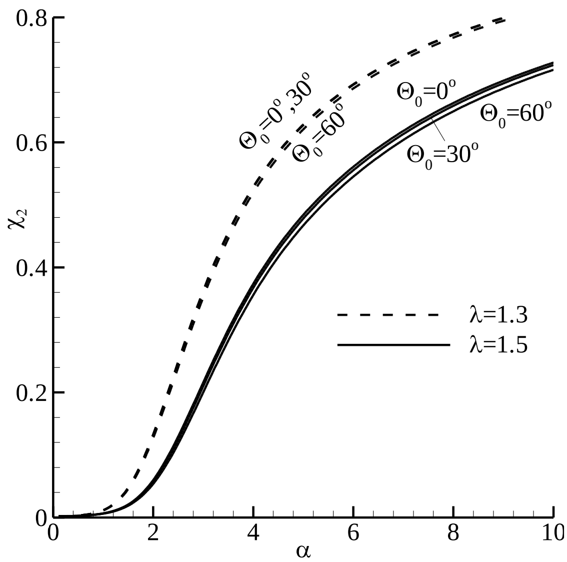

Figure 9. Change in χ_2 related to $\bar{\alpha}$ for different Θ_0 and λ values.

5. CONCLUSIONS AND RESULTS

In the present study, harmonic wave propagation in an initially stressed, thick elastic tube filled with viscous fluid was investigated. The effects of the axial inflation, internal pressure, and opening angle on the wave propagation characteristics was examined. The computational results show that the effects of internal pressure and axial stretch were consistent with the previous studies. However, the effect of the opening angle on the wave propagation characteristics has not previously been reported in the literature.As mentioned by Demiray & Antar [6], Demiray & Ercengiz [25], and Demiray & Akgün [26], axial stretch affected all wave propagation characteristics strongly, and the same results were obtained in this study.

The obtained results for the opening angle shown in Figures 6-9, emphasize the importance of the opening angle, particularly on the secondary wave speed. The effect of the opening angle that characterized the residual strain in the unloaded state must be taken into consideration in the new study. Additionally, recent experimental studies have revealed the importance of the opening angle (Bustamante & Holzapfel) [26].

The results obtained from the simplification of the governing equations presented in this study are similar to the results in the literature. The result confirms the validity of this study.

Appendix. Terms in the equations

$$(\bullet)\equiv(\bullet)_j, \qquad \frac{d(\bullet)}{d\xi}\equiv\frac{(\bullet)_{j+1}-(\bullet)_j}{h}, \quad \frac{d^2(\bullet)}{d\xi^2}\equiv\frac{(\bullet)_{j+2}-2(\bullet)_{j+1}+(\bullet)_j}{h^2} \tag{A.1}$$

If we use appropriate finite difference expressions given above in equation (27), we get finite difference equations

$$-hP_j+hP_{j+1}+h^2\left(\frac{\beta_{1j}}{h^2}-\beta_{4j}\eta^2-\frac{\beta_{3j}}{\xi_j^2}+\Omega^2\right)U_j+h^2\left(\frac{\beta_{2j}}{h\xi_j}-\frac{2\beta_{1j}}{h^2}\right)U_{j+1}+\beta_{1j}U_{j+2}=0 \tag{A.2}$$

$$ih^3\eta P_j-\frac{ih^3\beta_{8j}\eta}{\xi_j}U_j+h^3\left(\frac{\beta_{5j}}{h^2}-\beta_{7j}\eta^2-\frac{\beta_{6j}}{h\xi_j}+\Omega^2\right)W_j+h^3\left(\frac{\beta_{6j}}{h\xi_j}-\frac{2\beta_{5j}}{h^2}\right)W_{j+1}+ \tag{A3}$$
$$h\beta_{5j}W_{j+2}=0$$

$$\left(\frac{h}{\xi_0}-1\right)U_j+U_{j+1}-ih\eta W_j=0 \tag{A.4}$$

$$\left(1-\frac{1}{\xi_j^2}-\frac{h}{\xi_0}\right)U_j+\left(\frac{h}{\xi_0}-2\right)U_{j+1}+U_{j+2}+ih\eta W_j-ih\eta W_{j+1}=0\,. \tag{A.5}$$

Using the similar procedures, the boundary conditions in (29) written as

$$\mathrm{P}_0+\left(\frac{\alpha_{20}}{\xi_0}-\frac{\beta_{10}+\beta_{50}}{h}\right)U_0+\frac{\beta_{10}+\beta_{50}}{h}U_1+\frac{f\gamma^2\xi_0+\eta^2\left(2-f\xi_0\right)}{\eta\left(\gamma^2+\eta^2\right)\xi_0}A-\frac{2i\eta\left(g\xi_0-1\right)}{\gamma\xi_0}B=0 \tag{A.6}$$

$$ih^2P_0^0\eta U_0-h\beta_{50}W_0+h\beta_{50}W_1+\frac{2ih^2\eta^2}{\gamma^2+\eta^2}A+h^2\left(\frac{\gamma^2-\eta^2}{\gamma}\right)B=0 \tag{A.7}$$

$$P_n+\frac{\alpha_{2n}}{\xi_n}+\frac{\left(\beta_{1n}-P_n^0\right)}{h}\left(-U_n+U_{n+1}\right)=0 \tag{A.8}$$

$$ih^2\eta P_n^0U_n+h\beta_{5n}\left(-W_n+W_{n+1}\right)=0 \tag{A.9}$$

$$i\Omega^2U_0-\left(\frac{q\eta\bar{\alpha}^2}{\gamma^2+\eta^2}\right)A-\left(\frac{iq\eta\bar{\alpha}^2}{\gamma}\right)B=0 \tag{A.10}$$

$$i\Omega^2W_0+\left(\frac{iqf\bar{\alpha}^2}{\gamma^2+\eta^2}\right)A-\left(\frac{qg\bar{\alpha}^2}{\gamma}\right)B=0 \tag{A.11}$$

where subscripts ($_0$) and ($_n$) are described to the inner surface and the outer surface of a thick elastic tube, respectively. Determinant of the coefficients U_j, W_j, P_k, A and B $\left(j=1,\ldots,n+2\right)$, $\left(k=1,\ldots n\right)$ matrix must vanish. This gives the dispersion relationship of the problem.

REFERENCES

[1] Womersley JR. 1957. An elastic tube theory of pulse transmission and oscillatory flow in mammalian arteries, WADC. Tech. Rep. TR., 56-614.

[2] Morgan GW, Kiely, JP. 1954. Wave propagation in a viscous liquid contained in a flexible tube, J. Acoust. Soc. Am. 26, 323–328.

[3] Mirsky I. 1967. Wave propagation in a viscous fluid contained in an orthotropic elastic tube, Biophys. J. 7, 165–186.

[4] Atabek HB, Lew HS. 1966. Wave propagation through a viscous incompressible fluid contained in an initially stressed elastic tube, Biophys. J. 6, 481–503.

[5] Rachev AI. 1980. Effect of transmural pressure and muscular activity on pulse waves in arteries, J. Biomech. Eng. ASME 102, 119–123.

[6] Demiray H, Antar N. 1996. Effects of initial stresses and wall thickness on wave characteristics in elastic tubes. ZAMM 76, 521–530.

[7] Cox RH. 1968. Wave propagation through a Newtonian fluid contained within a thick-walled viscoelastic tube, Biophys. J. 8, 691–709.

[8] Kizilova NN. 2006. Pressure wave propagation in liquid-filled tubes of viscoelastic material, Fluid Dynam. 41, 434–446.

[9] Nayfeh AH. 1966. Oscillating two-phase flow through a rigid pipe, AIAA. J. 4 1868–1870.

[10] Nag SK, Jana RN. 1981. Oscillating two-phase flow in an elastic tube, Acta Mech. 41 121–128.

[11] Ercengiz A. 2005. Oscillating two-phase flow in a prestressed thick elastic tube, Acta Mech. 179, 169–185.

[12] Demiray H. 2005. Weakly nonlinear waves in a viscous fluid contained in a viscoelastic tube, European Journal of Mechanics A/ Solids. 24, 337-347.

[13] Demiray H. 2006. Weakly nonlinear waves in a fluid with variable viscosity contained in a prestressed thin elastic tube. Chaos Solitons & Fractails, 36 196-202.

[14] Jagielska K, Trzupek D, Lepers M, Pelc A and Zieliński P. 2007. Effect of surrounding tissue on propagation of axisymmetric waves in arteries, Physical Review E 76.

[15] Chaudhry HR, Bukiet B, Davis A, Ritterj AB and Findley T. J. 1997. Residual stresses in oscillating thoracic arteries reduce circumferential stress gradiants, J. Biomech., 30, 57-62.

[16] Vaishnav RN, and Vossoughi J. Estimation of residual strains in aortic segments. In: Biomedical Engineering. II. Recent Developments, edited by C. W. Hall. New York: Pergamon, 1983, vol. 2, p.330–333.

[17] Huang W, Yen RT. 1998. Zero-stress states of human pulmonary arteries and veins, J Appl Physiol 85 867-873.

[18] Demiray H. 1972. A note on the elasticity of soft biological tissue. J. Biomech. 5, 309–311.

[19] Baek S, Gleason RL, Rajagopal KR, Humphrey JD, 2007. Theory of small on large: Potential utility in computations of fluid-solid interactions in arteries, Computer Methods in Applied Mechanics and Engineering, 196, 3070-3078.

[20] Demiray H, Ercengiz A. 1995. Harmonic waves in a prestressed viscoelastic tube filled with a viscous fluid Int. J. Engng Sci. 33, 105-107.

[21] Eringen AC, Suhubi ES. Elastodynamics, vol. I. New York: Academic Press 1974.

[22] Atabek HB, Lew HS, 1966. Wave propagation through a viscous incompressible fluid contained in an initially stressed elastic tube, Biophys. J. 6, 481–503.

[23] Simon BR, Kobayashi AS, Strandness DE, Wiederhielm CA. 1972. Re-evaluation of arterial constitutive laws. Circulation Res. 30, 491–500.

[24] Demiray H. 1976. Some basic problems in biophysics. Bull. Math. Biol. 38, 701–711.

[25] Demiray H, Ercengiz A. 1991. Wave propagation in a prestressed elastic tube filled with a viscous fluid, 29, 575-585.

[26] Bustamante R, Holzapfel GA. 2010. Methods to compute 3D residual stress distributions in hyperelastic tubes with application to arterial walls, International Journal of Engineering Science, 48, 1066-1082.

[27] Demiray H, Akgün G. 1997. Wave propagation in a viscous fluid contained in a prestressed viscoelastic thin tube, Int. J. Engng Sci. 35 1065-1079.