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BEŞİNCİ MERTEBEDEN FARK DENKLEM SİSTEMİNİN ÇÖZÜLEBİLİRLİĞİ ÜZERİNE

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ÖZET

Bu çalışmada, $x_n = \dfrac{x_{n-4}y_{n-5}}{y_{n-1}\left(a_n + b_n x_{n-2}y_{n-3}x_{n-4}y_{n-5}\right)}, y_n = \dfrac{y_{n-4}x_{n-5}}{x_{n-1}\left(\alpha_n + \beta_n y_{n-2}x_{n-3}y_{n-4}x_{n-5}\right)}, n \in N_0,$ fark denklem sisteminin kapalı formda çözülebildiği gösterildi. Burada $(a_n)_{n\in N_0}, (b_n)_{n\in N_0}, (\alpha_n)_{n\in N_0}, (\beta_n)_{n\in N_0}$ birer dizi ve $j \in \{1,2,3,4,5\}$ için x_{-j}, y_{-j}, başlangıç şartları reel sayılardır. Ayrıca $(a_n)_{n\in N_0}, (b_n)_{n\in N_0}, (\alpha_n)_{n\in N_0}, (\beta_n)_{n\in N_0}$ dizilerinin sabit olduğu durumda bahsi geçen sistemin çözümlerinin periyodikliği ve asimptotik davranışı da incelendi.

Anahtar Kelimeler: Fark denklem sistemi, Asimptotik davranış

ON A SOLVABLE SYSTEM OF DIFFERENCE EQUATIONS OF FIFTH-ORDER

ABSTRACT

In this paper, we show that the difference equation system $x_n = \dfrac{x_{n-4}y_{n-5}}{y_{n-1}\left(a_n + b_n x_{n-2}y_{n-3}x_{n-4}y_{n-5}\right)}, y_n = \dfrac{y_{n-4}x_{n-5}}{x_{n-1}\left(\alpha_n + \beta_n y_{n-2}x_{n-3}y_{n-4}x_{n-5}\right)}, n \in N_0,$ where the sequences $(a_n)_{n\in N_0}, (b_n)_{n\in N_0}, (\alpha_n)_{n\in N_0}, (\beta_n)_{n\in N_0}$ and the initial values x_{-j}, y_{-j}, $j \in \{1,2,3,4,5\}$, are real numbers, can be solved in the closed form. Also, for the case when all the sequences $(a_n)_{n\in N_0}, (b_n)_{n\in N_0}, (\alpha_n)_{n\in N_0}, (\beta_n)_{n\in N_0}$ are constant, the asymptotic behavior and periodicity of solutions of aforementioned system are investigated.

Keywords: System of difference equation, Asymptotic behavior

1. GİRİŞ

Son zamanlarda lineer olmayan fark denklemeleri ve bu denklemlerin sistemleri ile ilgili literatürde çok fazla sayıda çalışma vardır [1-36]. Lineer olmayan fark denklemleri ya da fark denklem sistemlerinin dikkat çeken bir yönü de kapalı ya da açık formda çözülebilen denklem ya da sistemler bulabilmektir. Bu tipteki fark denklemlerinin ya da onların sistemlerinin çözümleri için elde edilen formüller, denklemlerin ya da onların sistemlerinin çözümlerinin bir çok özelliğini belirlemede kullanılabileceği açıktır. Bu nedenle lineer olmayan bir fark denklemi veya sisteminin çözümlerini elde etmek ilgi çekici olmasının yanında oldukça önemlidir.

$(a_n)_{n\in N_0}, (b_n)_{n\in N_0}$ diziler ve x_{-j}, $j \in \{1,2,3,4\}$, başlangıç şartları reel sayılar olmak üzere;

$$x_n = \frac{x_{n-3}x_{n-4}}{x_{n-1}\left(a_n + b_n x_{n-2}x_{n-3}x_{n-4}\right)}, \quad n \in N_0, \tag{1}$$

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fark denkleminin çözümleri elde edilmiştir. Burada $(a_n)_{n\in N_0}$, $(b_n)_{n\in N_0}$ dizilerinin sabit olduğu durumda çözümlerin davranışı da incelenmiştir [24].

$$x_n = \frac{x_{n-4}x_{n-5}}{x_{n-1}\left(\pm 1 \pm x_{n-2}x_{n-3}x_{n-4}x_{n-5}\right)},\ n \in N_0, \tag{2}$$

beşinci mertebeden bazı rasyonel fark denklemlerinin çözümleri tümevarım yöntemi kullanarak elde edilmiştir. Burada başlangıç şartları $x_{-j},\ j \in \{1,2,3,4,5\}$, reel sayılardır [15].

Bu çalışmada ise, (2) denkleminin hem genellemesi hemde sisteme genişletilmesi olan aşağıdaki fark denklem sisteminin,

$$\begin{aligned} x_n &= \frac{x_{n-4}y_{n-5}}{y_{n-1}\left(a_n + b_n x_{n-2}y_{n-3}x_{n-4}y_{n-5}\right)} \\ y_n &= \frac{y_{n-4}x_{n-5}}{x_{n-1}\left(\alpha_n + \beta_n y_{n-2}x_{n-3}y_{n-4}x_{n-5}\right)},\ n \in N_0, \end{aligned} \tag{3}$$

çözümleri, dönüşümler yardımıyla teorik bir şekilde açıklanarak elde edilmiştir. Burada $(a_n)_{n\in N_0}, (b_n)_{n\in N_0}, (\alpha_n)_{n\in N_0}, (\beta_n)_{n\in N_0}$ birer dizi ve $x_{-j}, y_{-j},\ j \in \{1,2,3,4,5\}$, başlangıç şartları sıfırdan farklı reel sayılardır. Ayrıca, $(a_n)_{n\in N_0}$, $(b_n)_{n\in N_0}$, $(\alpha_n)_{n\in N_0}$, $(\beta_n)_{n\in N_0}$ dizilerinin sabit olduğu durumda yukarıda verilen sistemin çözümlerinin periyodikliği ve asimptotik davranışı da incelenmiştir.

2. (3) SİSTEMİNİN KAPALI FORMDA ÇÖZÜMLERİ

Tanım 2.1. [5] $(x_n, y_n)_{n\geq -5}$ (3) sisteminin bir çözümü olsun. Eğer $\forall n \geq n_0$ için $x_{n+p} = x_n,\ y_{n+p} = y_n$ koşulunu sağlayan bir $n_0 \geq -5$ tam sayısı varsa $(x_n, y_n)_{n\geq -5}$ çözümüne er-geç *p*-periyotludur denir. Eğer $n_0 = -5$ ise, $(x_n, y_n)_{n\geq -5}$ çözümüne *p*-periyotludur denir.

Lemma 2.2. [5] $(a_n)_{n\in N_0}$, $(b_n)_{n\in N_0}$ reel sayı dizileri, $y_{2m+i},\ i \in \{0,1\}$, $y_{2m+i} = a_{2m+i}y_{2(m-1)+i} + b_{2m+i}, m \in N_0$, denkleminin çözümü olsun. O zaman $i \in \{0,1\},\ m \geq -1$ için verilen denklemin genel çözümü

$$y_{2m+i} = y_{i-2}\prod_{j=0}^{m} a_{2j+i} + \sum_{l=0}^{m} b_{2l+i}\prod_{j=l+1}^{m} a_{2j+i}$$

dür. Dahası eğer $(a_n)_{n\in N_0}$, $(b_n)_{n\in N_0}$ dizileri sabit ise $i \in \{0,1\},\ m \geq -1$ için

$$y_{2m+i} = \begin{cases} a^{m+1}y_{i-2} + b\dfrac{1-a^{m+1}}{1-a}, & a \neq 1 \\ y_{i-2} + b(m+1), & a = 1 \end{cases} \quad \text{dir.}$$

$(x_n, y_n)_{n\geq -5}$, (3) sisteminin bir çözümü olsun. Eğer $x_{-j},\ y_{-j},\ j = 1,2,3,4,5,$ başlangıç şartlarından en az biri sıfır ise, o zaman $\exists n_0 \in N_0$ vardır öyle ki $x_{n_0} = 0$ ya da $y_{n_0} = 0$ olup x_{n_0+1} ya da y_{n_0+1} terimlerinden biri tanımsızdır. Örneğin $x_{-5} = 0$ ise o zaman $y_0 = 0$ olup, (3) sisteminden x_1'in tanımsız olduğu kolayca görülebilir.

Tersine $n_1 \in N_0$ ve $0 \le n \le n_1 - 1$ için $x_n \ne 0$ ve $x_{n_1} = 0$ olduğunu kabul edelim. Bu durumda (3) sisteminin ilk denkleminden $y_{n_1-5} = 0$ olduğu, (3) sisteminin ikinci denklemi ve kabulümüzden de $y_{n_1-9} = 0$ olduğu görülür. Bu işlemler tekrar edilirse $\exists j_1 \in \{1,2,3,4,5\}$ için $y_{-j_1} = 0$ olduğu kolayca görülür. Benzer şekilde $n_2 \in N_0$ ve $0 \le n \le n_2 - 1$ için $y_n \ne 0$ ve $y_{n_2} = 0$ olduğu kabul edilirse $\exists j_2 \in \{1,2,3,4,5\}$ için $x_{-j_2} = 0$ olacağı, (3) sistemi ve kabulden kolayca görülür. Dolayısıyla, (3) sisteminin her iyi tanımlı çözümü için,

$$x_n y_n \ne 0,\ n \ge -5, \tag{4}$$

olması için gerek ve yeter şartın $x_{-j} y_{-j} \ne 0,\ j \in \{1,2,3,4,5\}$, olmasıdır. Bundan sonra (3) sisteminin $(x_n, y_n)_{n \ge -5}$ çözümünün (4) eşitsizliğini sağlayan çözümleri olduğunu kabul edeceğiz. $\forall n \ge -5$ için, $x_n y_n \ne 0$ olduğu dikkate alınırsa

$$u_n = \frac{1}{x_n y_{n-1} x_{n-2} y_{n-3}}, v_n = \frac{1}{y_n x_{n-1} y_{n-2} x_{n-3}},\ n \ge -2, \tag{5}$$

dönüşümünden (3) sistemi

$$u_n = a_n u_{n-2} + b_n,\ v_n = \alpha_n v_{n-2} + \beta_n,\ n \in N_0, \tag{6}$$

ikinci mertebeden değişken katsayılı lineer denklemlere indirgenir.
Lemma 2.2' den (6)' daki denklemlerin genel çözümleri

$$u_{2m+i} = u_{i-2} \prod_{j=0}^{m} a_{2j+i} + \sum_{l=0}^{m} b_{2l+i} \prod_{j=l+1}^{m} a_{2j+i} \tag{7}$$

$$v_{2m+i} = v_{i-2} \prod_{j=0}^{m} \alpha_{2j+i} + \sum_{l=0}^{m} \beta_{2l+i} \prod_{j=l+1}^{m} \alpha_{2j+i} \tag{8}$$

şeklindedir. (5) dönüşümünden, $i_1 \in \{0,1,2,3\}$ $n \ge -2$

$$\begin{aligned} x_{4n+i_1} &= \frac{1}{u_{4n+i_1} y_{4n+i_1-1} x_{4n+i_1-2} y_{4n+i_1-3}} = \frac{v_{4n+i_1-1}}{u_{4n+i_1}} x_{4(n-1)+i_1} \\ y_{4n+i_1} &= \frac{1}{v_{4n+i_1} x_{4n+i_1-1} y_{4n+i_1-2} x_{4n+i_1-3}} = \frac{u_{4n+i_1-1}}{v_{4n+i_1}} y_{4(n-1)+i_1} \end{aligned} \tag{9}$$

eşitlikleri yazılabilir.
(9)' daki denklemlerde (7) ve (8) çözümleri yerlerine yazılırsa, $m \ge -1,\ j \in \{0,1\},\ i \in \{-1,0\}$ için

$$\begin{aligned} x_{4m+2j+i} &= \frac{v_{4m+2j+i-1}}{u_{4m+2j+i}} x_{4(m-1)+2j+i} \\ &= x_{2j+i-4} \prod_{s=0}^{m} \frac{v_{4s+2j+i-1}}{u_{4s+2j+i}} \end{aligned} \tag{10}$$

$$\begin{aligned} y_{4m+2j+i} &= \frac{u_{4m+2j+i-1}}{v_{4m+2j+i}} y_{4(m-1)+2j+i} \\ &= y_{2j+i-4} \prod_{s=0}^{m} \frac{u_{4s+2j+i-1}}{v_{4s+2j+i}} \end{aligned} \tag{11}$$

elde edilir. (7), (8), (10) ve (11) denklemlerinden $(x_n, y_n)_{n\geq -5}$ fark denklem sisteminin çözümleri

$$x_{4m+2j-1} = x_{2j-5} \prod_{s=0}^{m} \frac{v_{-2} \prod_{i=0}^{2s+j-1} \alpha_{2i} + \sum_{l=0}^{2s+j-1} \beta_{2l} \prod_{i=l+1}^{2s+j-1} \alpha_{2i}}{u_{-1} \prod_{i=0}^{2s+j-1} a_{2i+1} + \sum_{l=0}^{2s+j-1} b_{2l+1} \prod_{i=l+1}^{2s+j-1} a_{2i+1}}$$

$$= x_{2j-5} \prod_{s=0}^{m} \frac{x_{-1}\left(\prod_{i=0}^{2s+j-1} \alpha_{2i} + y_{-2}x_{-3}y_{-4}x_{-5} \sum_{l=0}^{2s+j-1} \beta_{2l} \prod_{i=l+1}^{2s+j-1} \alpha_{2i} \right)}{x_{-5}\left(\prod_{i=0}^{2s+j-1} a_{2i+1} + x_{-1}y_{-2}x_{-3}y_{-4} \sum_{l=0}^{2s+j-1} b_{2l+1} \prod_{i=l+1}^{2s+j-1} a_{2i+1} \right)}$$

$$x_{4m+2j} = x_{2j-4} \prod_{s=0}^{m} \frac{v_{-1} \prod_{i=0}^{2s+j-1} \alpha_{2i+1} + \sum_{l=0}^{2s+j-1} \beta_{2l+1} \prod_{i=l+1}^{2s+j-1} \alpha_{2i+1}}{u_{-2} \prod_{i=0}^{2s+j} a_{2i} + \sum_{l=0}^{2s+j} b_{2l} \prod_{i=l+1}^{2s+j} a_{2i}}$$

$$= x_{2j-4} \prod_{s=0}^{m} \frac{y_{-5}\left(\prod_{i=0}^{2s+j-1} \alpha_{2i+1} + y_{-1}x_{-2}y_{-3}x_{-4} \sum_{l=0}^{2s+j-1} \beta_{2l+1} \prod_{i=l+1}^{2s+j-1} \alpha_{2i+1} \right)}{y_{-1}\left(\prod_{i=0}^{2s+j} a_{2i} + x_{-2}y_{-3}x_{-4}y_{-5} \sum_{l=0}^{2s+j} b_{2l} \prod_{i=l+1}^{2s+j} a_{2i} \right)}$$

$$y_{4m+2j-1} = y_{2j-5} \prod_{s=0}^{m} \frac{u_{-2} \prod_{i=0}^{2s+j-1} a_{2i} + \sum_{l=0}^{2s+j-1} b_{2l} \prod_{i=l+1}^{2s+j-1} a_{2i}}{v_{-1} \prod_{i=0}^{2s+j-1} \alpha_{2i+1} + \sum_{l=0}^{2s+j-1} \beta_{2l+1} \prod_{i=l+1}^{2s+j-1} \alpha_{2i+1}}$$

$$= y_{2j-5} \prod_{s=0}^{m} \frac{y_{-1}\left(\prod_{i=0}^{2s+j-1} a_{2i} + x_{-2}y_{-3}x_{-4}y_{-5} \sum_{l=0}^{2s+j-1} b_{2l} \prod_{i=l+1}^{2s+j-1} a_{2i} \right)}{y_{-5}\left(\prod_{i=0}^{2s+j-1} \alpha_{2i+1} + y_{-1}x_{-2}y_{-3}x_{-4} \sum_{l=0}^{2s+j-1} \beta_{2l+1} \prod_{i=l+1}^{2s+j-1} \alpha_{2i+1} \right)}$$

$$y_{4m+2j} = y_{2j-4} \prod_{s=0}^{m} \frac{u_{-1} \prod_{i=0}^{2s+j-1} a_{2i+1} + \sum_{l=0}^{2s+j-1} b_{2l+1} \prod_{i=l+1}^{2s+j-1} a_{2i+1}}{v_{-2} \prod_{i=0}^{2s+j} \alpha_{2i} + \sum_{l=0}^{2s+j} \beta_{2l} \prod_{i=l+1}^{2s+j} \alpha_{2i}}$$

$$= y_{2j-4} \prod_{s=0}^{m} \frac{x_{-5}\left(\prod_{i=0}^{2s+j-1} a_{2i+1} + x_{-1}y_{-2}x_{-3}y_{-4} \sum_{l=0}^{2s+j-1} b_{2l+1} \prod_{i=l+1}^{2s+j-1} a_{2i+1} \right)}{x_{-1}\left(\prod_{i=0}^{2s+j} \alpha_{2i} + y_{-2}x_{-3}y_{-4}x_{-5} \sum_{l=0}^{2s+j} \beta_{2l} \prod_{i=l+1}^{2s+j} \alpha_{2i} \right)}$$

şeklindedir. Dolayısıyla aşağıdaki teorem ispatlanmış olur.

Teorem 2.3. $(x_n, y_n)_{n\geq -5}$ (3) sisteminin iyi tanımlı çözümü olsun. O zaman (3) sisteminin çözümleri $m\geq -1,\ j\in\{0,1\}$ için

$$x_{4m+2j-1}=x_{2j-5}\prod_{s=0}^{m}\frac{x_{-1}\left(\prod_{i=0}^{2s+j-1}\alpha_{2i}+y_{-2}x_{-3}y_{-4}x_{-5}\sum_{l=0}^{2s+j-1}\beta_{2l}\prod_{i=l+1}^{2s+j-1}\alpha_{2i}\right)}{x_{-5}\left(\prod_{i=0}^{2s+j-1}a_{2i+1}+x_{-1}y_{-2}x_{-3}y_{-4}\sum_{l=0}^{2s+j-1}b_{2l+1}\prod_{i=l+1}^{2s+j-1}a_{2i+1}\right)}$$

$$x_{4m+2j}=x_{2j-4}\prod_{s=0}^{m}\frac{y_{-5}\left(\prod_{i=0}^{2s+j-1}\alpha_{2i+1}+y_{-1}x_{-2}y_{-3}x_{-4}\sum_{l=0}^{2s+j-1}\beta_{2l+1}\prod_{i=l+1}^{2s+j-1}\alpha_{2i+1}\right)}{y_{-1}\left(\prod_{i=0}^{2s+j}a_{2i}+x_{-2}y_{-3}x_{-4}y_{-5}\sum_{l=0}^{2s+j}b_{2l}\prod_{i=l+1}^{2s+j}a_{2i}\right)}$$

$$y_{4m+2j-1}=y_{2j-5}\prod_{s=0}^{m}\frac{y_{-1}\left(\prod_{i=0}^{2s+j-1}a_{2i}+x_{-2}y_{-3}x_{-4}y_{-5}\sum_{l=0}^{2s+j-1}b_{2l}\prod_{i=l+1}^{2s+j-1}a_{2i}\right)}{y_{-5}\left(\prod_{i=0}^{2s+j-1}\alpha_{2i+1}+y_{-1}x_{-2}y_{-3}x_{-4}\sum_{l=0}^{2s+j-1}\beta_{2l+1}\prod_{i=l+1}^{2s+j-1}\alpha_{2i+1}\right)}$$

$$y_{4m+2j}=y_{2j-4}\prod_{s=0}^{m}\frac{x_{-5}\left(\prod_{i=0}^{2s+j-1}a_{2i+1}+x_{-1}y_{-2}x_{-3}y_{-4}\sum_{l=0}^{2s+j-1}b_{2l+1}\prod_{i=l+1}^{2s+j-1}a_{2i+1}\right)}{x_{-1}\left(\prod_{i=0}^{2s+j}\alpha_{2i}+y_{-2}x_{-3}y_{-4}x_{-5}\sum_{l=0}^{2s+j}\beta_{2l}\prod_{i=l+1}^{2s+j}\alpha_{2i}\right)}$$

şeklindedir.

3. KATSAYILARIN SABİT OLDUĞU DURUM

Bu bölümde (3) denklem sisteminde, $\forall n\in N_0\ \ a_n=a,\ b_n=b,\ \alpha_n=\alpha,\ \beta_n=\beta$ sabit katsayılı lineer olmayan

$$x_n=\frac{x_{n-4}y_{n-5}}{y_{n-1}\left(a+bx_{n-2}y_{n-3}x_{n-4}y_{n-5}\right)}$$
$$y_n=\frac{y_{n-4}x_{n-5}}{x_{n-1}\left(\alpha+\beta y_{n-2}x_{n-3}y_{n-4}x_{n-5}\right)},\quad n\in N_0, \tag{12}$$

rasyonel fark denklem sisteminin çözümü ve çözümlerinin asimptotik davranışları incelenmiştir. Burada a, b, α, β reel sabitlerdir. Bu alt bölümde çözümlerin iyi tanımlı yani $\forall n\in N_0,\ y_{n-1}\left(a+bx_{n-2}y_{n-3}x_{n-4}y_{n-5}\right)\neq 0,\ \ x_{n-1}\left(\alpha+\beta y_{n-2}x_{n-3}y_{n-4}x_{n-5}\right)\neq 0$ olduğu kabul edilmiştir.

Teorem 3.1. $(x_n, y_n)_{n\geq -5}$ (12) denklem sisteminin iyi tanımlı çözümü, $a\neq 1,\ \alpha\neq 1,\ i=1,2,3,4,5$ için $x_{-i}\neq 0, y_{-i}\neq 0$ olsun. Bu durumda (12) denklem sisteminin çözümleri

$$x_{4m+2j-1}=x_{2j-5}\prod_{s=0}^{m}\frac{x_{-1}(1-a)\left(\beta y_{-2}x_{-3}y_{-4}x_{-5}+\left(1-\alpha-\beta y_{-2}x_{-3}y_{-4}x_{-5}\right)\alpha^{2s+j}\right)}{x_{-5}(1-\alpha)\left(bx_{-1}y_{-2}x_{-3}y_{-4}+\left(1-a-bx_{-1}y_{-2}x_{-3}y_{-4}\right)a^{2s+j}\right)}$$

$$x_{4m+2j}=x_{2j-4}\prod_{s=0}^{m}\frac{y_{-5}(1-a)\left(\beta y_{-1}x_{-2}y_{-3}x_{-4}+\left(1-\alpha-\beta y_{-1}x_{-2}y_{-3}x_{-4}\right)\alpha^{2s+j}\right)}{y_{-1}(1-\alpha)\left(bx_{-2}y_{-3}x_{-4}y_{-5}+\left(1-a-bx_{-2}y_{-3}x_{-4}y_{-5}\right)a^{2s+j+1}\right)}$$

$$y_{4m+2j-1} = y_{2j-5}\prod_{s=0}^{m}\frac{y_{-1}(1-\alpha)\left(bx_{-2}y_{-3}x_{-4}y_{-5}+(1-a-bx_{-2}y_{-3}x_{-4}y_{-5})a^{2s+j}\right)}{y_{-5}(1-a)\left(\beta y_{-1}x_{-2}y_{-3}x_{-4}+(1-\alpha-\beta y_{-1}x_{-2}y_{-3}x_{-4})\alpha^{2s+j}\right)}$$

$$y_{4m+2j} = y_{2j-4}\prod_{s=0}^{m}\frac{x_{-5}(1-\alpha)\left(bx_{-1}y_{-2}x_{-3}y_{-4}+(1-a-bx_{-1}y_{-2}x_{-3}y_{-4})a^{2s+j}\right)}{x_{-1}(1-a)\left(\beta y_{-2}x_{-3}y_{-4}x_{-5}+(1-\alpha-\beta y_{-2}x_{-3}y_{-4}x_{-5})\alpha^{2s+j+1}\right)}$$

şeklindedir. Burada $j\in\{0,1\}$'dir.

İspat: (7) ve (8) denklemlerinde $\forall n\in N_0$ için $a_n=a, b_n=b, \quad \alpha_n=\alpha, \beta_n=\beta$ alınırsa $(u_n)_{n\ge -2}$, $(v_n)_{n\ge -2}$ denklemlerinin çözümü, $m\in N_0$, $i\in\{0,1\}$ için,

$$u_{2m+i}=u_{i-2}a^{m+1}+b\frac{1-a^{m+1}}{1-a}, \qquad v_{2m+i}=v_{i-2}\alpha^{m+1}+\beta\frac{1-\alpha^{m+1}}{1-\alpha},$$

şeklinde olacaktır. Öte yandan (10), (11) denklemleri ve $u_{-1}=(x_{-1}y_{-2}x_{-3}y_{-4})^{-1}$, $u_{-2}=(x_{-2}y_{-3}x_{-4}y_{-5})^{-1}$, $v_{-1}=(y_{-1}x_{-2}y_{-3}x_{-4})^{-1}$, $v_{-2}=(y_{-2}x_{-3}y_{-4}x_{-5})^{-1}$ eşitlikleri dikkate alınırsa, $j\in\{0,1\}$ için (12) denklem sisteminin çözümleri

$$x_{4m+2j-1} = x_{2j-5}\prod_{s=0}^{m}\frac{x_{-1}(1-a)\left(\beta y_{-2}x_{-3}y_{-4}x_{-5}+(1-\alpha-\beta y_{-2}x_{-3}y_{-4}x_{-5})\alpha^{2s+j}\right)}{x_{-5}(1-\alpha)\left(bx_{-1}y_{-2}x_{-3}y_{-4}+(1-a-bx_{-1}y_{-2}x_{-3}y_{-4})a^{2s+j}\right)}$$

$$x_{4m+2j} = x_{2j-4}\prod_{s=0}^{m}\frac{y_{-5}(1-a)\left(\beta y_{-1}x_{-2}y_{-3}x_{-4}+(1-\alpha-\beta y_{-1}x_{-2}y_{-3}x_{-4})\alpha^{2s+j}\right)}{y_{-1}(1-\alpha)\left(bx_{-2}y_{-3}x_{-4}y_{-5}+(1-a-bx_{-2}y_{-3}x_{-4}y_{-5})a^{2s+j+1}\right)}$$

$$y_{4m+2j-1} = y_{2j-5}\prod_{s=0}^{m}\frac{y_{-1}(1-\alpha)\left(bx_{-2}y_{-3}x_{-4}y_{-5}+(1-a-bx_{-2}y_{-3}x_{-4}y_{-5})a^{2s+j}\right)}{y_{-5}(1-a)\left(\beta y_{-1}x_{-2}y_{-3}x_{-4}+(1-\alpha-\beta y_{-1}x_{-2}y_{-3}x_{-4})\alpha^{2s+j}\right)}$$

$$y_{4m+2j} = y_{2j-4}\prod_{s=0}^{m}\frac{x_{-5}(1-\alpha)\left(bx_{-1}y_{-2}x_{-3}y_{-4}+(1-a-bx_{-1}y_{-2}x_{-3}y_{-4})a^{2s+j}\right)}{x_{-1}(1-a)\left(\beta y_{-2}x_{-3}y_{-4}x_{-5}+(1-\alpha-\beta y_{-2}x_{-3}y_{-4}x_{-5})\alpha^{2s+j+1}\right)}$$

elde edilir.

Teorem 3.2. $(x_n,y_n)_{n\ge -5}$ (12) denklem sisteminin çözümü, $a=1,\ \alpha=1$ ve $i=1,2,3,4,5$ için $x_{-i}\ne 0, y_{-i}\ne 0$ olsun. Bu durumda (12) denklem sisteminin çözümleri

$$x_{4m+2j-1}=x_{2j-5}\prod_{s=0}^{m}\frac{x_{-1}}{x_{-5}}\frac{1+(2s+j)\beta y_{-2}x_{-3}y_{-4}x_{-5}}{1+(2s+j)bx_{-1}y_{-2}x_{-3}y_{-4}}$$

$$x_{4m+2j}=x_{2j-4}\prod_{s=0}^{m}\frac{y_{-5}}{y_{-1}}\frac{1+(2s+j)\beta y_{-1}x_{-2}y_{-3}x_{-4}}{1+(2s+j+1)bx_{-2}y_{-3}x_{-4}y_{-5}}$$

$$y_{4m+2j-1}=y_{2j-5}\prod_{s=0}^{m}\frac{y_{-1}}{y_{-5}}\frac{1+(2s+j)bx_{-2}y_{-3}x_{-4}y_{-5}}{1+(2s+j)\beta y_{-1}x_{-2}y_{-3}x_{-4}}$$

$$y_{4m+2j}=y_{2j-4}\prod_{s=0}^{m}\frac{x_{-5}}{x_{-1}}\frac{1+(2s+j)bx_{-1}y_{-2}x_{-3}y_{-4}}{1+(2s+j+1)\beta y_{-2}x_{-3}y_{-4}x_{-5}}$$

şeklindedir. Burada $j \in \{0,1\}$'dir.

İspat: $a=1$ ve $\alpha=1$ için Lemma 2.2'den $(u_n)_{n\geq -2}$ ve $(v_n)_{n\geq -2}$ denklemlerinin çözümleri

$$u_{2m+i} = u_{i-2} + (m+1)b,$$
$$v_{2m+i} = v_{i-2} + (m+1)\beta,$$

şeklinde olacaktır. Öte yandan (10) ve (11) denklemlerinde yukarıdaki eşitlikler yerlerine yazılır ve $u_{-1} = (x_{-1}y_{-2}x_{-3}y_{-4})^{-1}$, $u_{-2} = (x_{-2}y_{-3}x_{-4}y_{-5})^{-1}$, $v_{-1} = (y_{-1}x_{-2}y_{-3}x_{-4})^{-1}$, $v_{-2} = (y_{-2}x_{-3}y_{-4}x_{-5})^{-1}$ ifadeleri dikkate alınırsa, $j \in \{0,1\}$ için (12) denklem sisteminin çözümleri

$$x_{4m+2j-1} = x_{2j-5}\prod_{s=0}^{m}\frac{x_{-1}\left(1+(2s+j)\beta y_{-2}x_{-3}y_{-4}x_{-5}\right)}{x_{-5}\left(1+(2s+j)bx_{-1}y_{-2}x_{-3}y_{-4}\right)}$$

$$x_{4m+2j} = x_{2j-4}\prod_{s=0}^{m}\frac{y_{-5}\left(1+(2s+j)\beta y_{-1}x_{-2}y_{-3}x_{-4}\right)}{y_{-1}\left(1+(2s+j+1)bx_{-2}y_{-3}x_{-4}y_{-5}\right)}$$

$$y_{4m+2j-1} = y_{2j-5}\prod_{s=0}^{m}\frac{y_{-1}\left(1+(2s+j)bx_{-2}y_{-3}x_{-4}y_{-5}\right)}{y_{-5}\left(1+(2s+j)\beta y_{-1}x_{-2}y_{-3}x_{-4}\right)}$$

$$y_{4m+2j} = y_{2j-4}\prod_{s=0}^{m}\frac{x_{-5}\left(1+(2s+j)bx_{-1}y_{-2}x_{-3}y_{-4}\right)}{x_{-1}\left(1+(2s+j+1)\beta y_{-2}x_{-3}y_{-4}x_{-5}\right)}$$

şeklindedir.

Teorem 3.3. $(x_n, y_n)_{n\geq -5}$ (12) denklem sisteminin çözümü, $a=-1$, $\alpha=-1$, $b\neq 0$, $\beta\neq 0$ ve $i=1,2,3,4,5$ için $x_{-i}\neq 0$, $y_{-i}\neq 0$ olsun. Bu durumda (12) denklem sisteminin çözümleri

$$x_{4m+2j-1} = x_{2j-5}\prod_{s=0}^{m}\frac{x_{-1}\left(\beta y_{-2}x_{-3}y_{-4}x_{-5} + (2-\beta y_{-2}x_{-3}y_{-4}x_{-5})(-1)^{2s+j}\right)}{x_{-5}\left(bx_{-1}y_{-2}x_{-3}y_{-4} + (2-bx_{-1}y_{-2}x_{-3}y_{-4})(-1)^{2s+j}\right)}$$

$$x_{4m+2j} = x_{2j-4}\prod_{s=0}^{m}\frac{y_{-5}\left(\beta y_{-1}x_{-2}y_{-3}x_{-4} + (2-\beta y_{-1}x_{-2}y_{-3}x_{-4})(-1)^{2s+j}\right)}{y_{-1}\left(bx_{-2}y_{-3}x_{-4}y_{-5} + (2-bx_{-2}y_{-3}x_{-4}y_{-5})(-1)^{2s+j+1}\right)}$$

$$y_{4m+2j-1} = y_{2j-5}\prod_{s=0}^{m}\frac{y_{-1}\left(bx_{-2}y_{-3}x_{-4}y_{-5} + (2-bx_{-2}y_{-3}x_{-4}y_{-5})(-1)^{2s+j}\right)}{y_{-5}\left(\beta y_{-1}x_{-2}y_{-3}x_{-4} + (2-\beta y_{-1}x_{-2}y_{-3}x_{-4})(-1)^{2s+j}\right)}$$

$$y_{4m+2j} = y_{2j-4}\prod_{s=0}^{m}\frac{x_{-5}\left(bx_{-1}y_{-2}x_{-3}y_{-4} + (2-bx_{-1}y_{-2}x_{-3}y_{-4})(-1)^{2s+j}\right)}{x_{-1}\left(\beta y_{-2}x_{-3}y_{-4}x_{-5} + (2-\beta y_{-2}x_{-3}y_{-4}x_{-5})(-1)^{2s+j+1}\right)}$$

şeklindedir. Burada $j \in \{0,1\}$'dir.

İspat: Teorem 3.1'den ispat açıktır.

Teorem 3.4. $(x_n, y_n)_{n\geq -5}$ (12) denklem sisteminin çözümü, $a\neq 0,\ \alpha\neq 0,\ b=0,\ \beta=0$ ve $i=1,2,3,4,5$ için $x_{-i}\neq 0, y_{-i}\neq 0$ olsun. Bu durumda (12) denklem sisteminin çözümleri

$$x_{4m+2j-1}=x_{2j-5}\prod_{s=0}^{m}\frac{x_{-1}}{x_{-5}}\left(\frac{\alpha}{a}\right)^{2s+j}$$

$$x_{4m+2j}=x_{2j-4}\frac{1}{a}\prod_{s=0}^{m}\frac{y_{-5}}{y_{-1}}\left(\frac{\alpha}{a}\right)^{2s+j}$$

$$y_{4m+2j-1}=y_{2j-5}\prod_{s=0}^{m}\frac{y_{-1}}{y_{-5}}\left(\frac{a}{\alpha}\right)^{2s+j}$$

$$y_{4m+2j}=y_{2j-4}\frac{1}{\alpha}\prod_{s=0}^{m}\frac{x_{-5}}{x_{-1}}\left(\frac{a}{\alpha}\right)^{2s+j}$$

şeklindedir.

İspat: Varsayalım ki $a\neq 0,\ \alpha\neq 0,\ b=0,\ \beta=0$ olsun. Bu durumda (12) denklem sistemi $x_n=\frac{x_{n-4}y_{n-5}}{ay_{n-1}},\ y_n=\frac{y_{n-4}x_{n-5}}{\alpha x_{n-1}}$ şekline indirgenir. Öte yandan Teorem 3.1'deki $(x_n, y_n)_{n\geq -5}$ çözümlerinde $b=0,\ \beta=0$ alınırsa

$$x_{4m+2j-1}=x_{2j-5}\prod_{s=0}^{m}\frac{x_{-1}}{x_{-5}}\left(\frac{\alpha}{a}\right)^{2s+j}$$

$$x_{4m+2j}=x_{2j-4}\frac{1}{a}\prod_{s=0}^{m}\frac{y_{-5}}{y_{-1}}\left(\frac{\alpha}{a}\right)^{2s+j}$$

$$y_{4m+2j-1}=y_{2j-5}\prod_{s=0}^{m}\frac{y_{-1}}{y_{-5}}\left(\frac{a}{\alpha}\right)^{2s+j}$$

$$y_{4m+2j}=y_{2j-4}\frac{1}{\alpha}\prod_{s=0}^{m}\frac{x_{-5}}{x_{-1}}\left(\frac{a}{\alpha}\right)^{2s+j}$$

çözümleri elde edilir. Burada $j\in\{0,1\}$'dir.

Teorem 3.5. Varsayalım ki $a\neq -1,\ \alpha\neq -1,\ b\neq 0,\ \beta\neq 0,\ (x_n, y_n)_{n\geq -5}$ (12) denklem sisteminin iyi tanımlı bir çözümü olsun. Bu durumda aşağıdaki ifadeler doğrudur.

a) Eğer $|a|>1,\ x_{-1}y_{-2}x_{-3}y_{-4}\neq\frac{1-a}{b}\neq x_{-2}y_{-3}x_{-4}y_{-5},\quad y_{-1}x_{-2}y_{-3}x_{-4}=\frac{1-\alpha}{\beta}=y_{-2}x_{-3}y_{-4}x_{-5}$ ise, o zaman $m\to\infty$ iken $x_m\to 0, |y_m|\to\infty$'dur.

b) Eğer $|a|<1,\ x_{-1}y_{-2}x_{-3}y_{-4}\neq\frac{1-a}{b}\neq x_{-2}y_{-3}x_{-4}y_{-5},\quad y_{-1}x_{-2}y_{-3}x_{-4}=\frac{1-\alpha}{\beta}=y_{-2}x_{-3}y_{-4}x_{-5}$ ise, o zaman $m\to\infty$ iken $|x_m|\to\infty, y_m\to 0$'dır.

c) Eğer $|\alpha|>1,\ x_{-1}y_{-2}x_{-3}y_{-4}=\dfrac{1-a}{b}=x_{-2}y_{-3}x_{-4}y_{-5},\quad y_{-1}x_{-2}y_{-3}x_{-4}\neq\dfrac{1-\alpha}{\beta}\neq y_{-2}x_{-3}y_{-4}x_{-5}$ ise, o zaman $m\to\infty$ iken $|x_m|\to\infty, y_m\to 0$'dır.

d) Eğer $|\alpha|<1,\ x_{-1}y_{-2}x_{-3}y_{-4}=\dfrac{1-a}{b}=x_{-2}y_{-3}x_{-4}y_{-5},\quad y_{-1}x_{-2}y_{-3}x_{-4}\neq\dfrac{1-\alpha}{\beta}\neq y_{-2}x_{-3}y_{-4}x_{-5}$ ise, o zaman $m\to\infty$ iken $x_m\to 0, |y_m|\to\infty$'dur.

e) Eğer $\left|\dfrac{a}{\alpha}\right|>1,\ x_{-1}y_{-2}x_{-3}y_{-4}\neq\dfrac{1-a}{b}\neq x_{-2}y_{-3}x_{-4}y_{-5},\quad y_{-1}x_{-2}y_{-3}x_{-4}\neq\dfrac{1-\alpha}{\beta}\neq y_{-2}x_{-3}y_{-4}x_{-5}$ ise, o zaman $m\to\infty$ iken $x_m\to 0, |y_m|\to\infty$'dur.

f) Eğer $\left|\dfrac{a}{\alpha}\right|<1,\ x_{-1}y_{-2}x_{-3}y_{-4}\neq\dfrac{1-a}{b}\neq x_{-2}y_{-3}x_{-4}y_{-5},\quad y_{-1}x_{-2}y_{-3}x_{-4}\neq\dfrac{1-\alpha}{\beta}\neq y_{-2}x_{-3}y_{-4}x_{-5}$ ise, o zaman $m\to\infty$ iken $|x_m|\to\infty, y_m\to 0$'dır.

g) Eğer $a=\alpha,\ |a|<1,\ b=\beta,\ x_{-1}=x_{-5},\quad y_{-1}=y_{-5}$ ise o zaman $\forall j\in\{0,1\}$ için $\left(x_{4m+2j-1}\right)_{n\in N_0},\left(y_{4m+2j-1}\right)_{n\in N_0}$ dizileri sabit ve $\left(x_{4m+2j}\right)_{n\in N_0},\left(y_{4m+2j}\right)_{n\in N_0}$ dizileri yakınsaktır.

h) Eğer $x_{-1}y_{-2}x_{-3}y_{-4}=\dfrac{1-a}{b}=x_{-2}y_{-3}x_{-4}y_{-5},\quad y_{-1}x_{-2}y_{-3}x_{-4}=\dfrac{1-\alpha}{\beta}=y_{-2}x_{-3}y_{-4}x_{-5}$ ya da $a=0,\ \alpha=0$ ise $x_{4m+j}=x_{j-4}, y_{4m+j}=y_{j-4}$' dür. Burada $m\in N_0$ ve $j\in\{-1,0,1,2\}$'dir.

i) Eğer $a=\alpha=1,\ b=\beta,\ b\neq 0, x_{-1}=x_{-5},\quad y_{-1}=y_{-5}$ ise o zaman $m\to\infty$ iken $x_{4m+2j-1}\to x_{2j-5}, x_{4m+2j}\to 0,\ \ y_{4m+2j-1}\to y_{2j-5}, y_{4m+2j}\to 0$' dır. Burada $m\in N_0$ ve $j\in\{0,1\}$'dir.

İspat: $m\geq -1\ a\neq -1,\ \alpha\neq -1, b\neq 0,\ \beta\neq 0$ ve $\forall j\in\{0,1\}$ için $p_m^{2j-1},\ p_m^{2j},\ q_m^{2j-1},\ q_m^{2j}$ ifadeleri aşağıdaki gibi tanımlansın.

$$p_m^{2j-1}=\frac{x_{-1}(1-a)\left(\beta y_{-2}x_{-3}y_{-4}x_{-5}+\left(1-\alpha-\beta y_{-2}x_{-3}y_{-4}x_{-5}\right)\alpha^{2m+j}\right)}{x_{-5}(1-\alpha)\left(bx_{-1}y_{-2}x_{-3}y_{-4}+\left(1-a-bx_{-1}y_{-2}x_{-3}y_{-4}\right)a^{2m+j}\right)} \tag{13}$$

$$p_m^{2j}=\frac{y_{-5}(1-a)\left(\beta y_{-1}x_{-2}y_{-3}x_{-4}+\left(1-\alpha-\beta y_{-1}x_{-2}y_{-3}x_{-4}\right)\alpha^{2m+j}\right)}{y_{-1}(1-\alpha)\left(bx_{-2}y_{-3}x_{-4}y_{-5}+\left(1-a-bx_{-2}y_{-3}x_{-4}y_{-5}\right)a^{2m+j+1}\right)} \tag{14}$$

$$q_m^{2j-1}=\frac{y_{-1}(1-\alpha)\left(bx_{-2}y_{-3}x_{-4}y_{-5}+\left(1-a-bx_{-2}y_{-3}x_{-4}y_{-5}\right)a^{2m+j}\right)}{y_{-5}(1-a)\left(\beta y_{-1}x_{-2}y_{-3}x_{-4}+\left(1-\alpha-\beta y_{-1}x_{-2}y_{-3}x_{-4}\right)\alpha^{2m+j}\right)} \tag{15}$$

$$q_m^{2j}=\frac{x_{-5}(1-\alpha)\left(bx_{-1}y_{-2}x_{-3}y_{-4}+\left(1-a-bx_{-1}y_{-2}x_{-3}y_{-4}\right)a^{2m+j}\right)}{x_{-1}(1-a)\left(\beta y_{-2}x_{-3}y_{-4}x_{-5}+\left(1-\alpha-\beta y_{-2}x_{-3}y_{-4}x_{-5}\right)\alpha^{2s+j+1}\right)} \tag{16}$$

aşağıdaki gibi tanımlansın.

a) Varsayalım ki $a>1,\ \ x_{-1}y_{-2}x_{-3}y_{-4}\neq\dfrac{1-a}{b}\neq x_{-2}y_{-3}x_{-4}y_{-5},\ y_{-1}x_{-2}y_{-3}x_{-4}=\dfrac{1-\alpha}{\beta}=y_{-2}x_{-3}y_{-4}x_{-5}$ olsun.

(13), (14), (15) ve (16) denklemlerinden $\forall j\in\{0,1\}$ için

$$\begin{aligned}\lim_{m\to\infty}p_m^{2j-1}&=\lim_{m\to\infty}\frac{x_{-1}(1-a)}{x_{-5}\left(bx_{-1}y_{-2}x_{-3}y_{-4}+\left(1-a-bx_{-1}y_{-2}x_{-3}y_{-4}\right)a^{2m+j}\right)}\\&=0\end{aligned}$$

$$\lim_{m\to\infty} p_m^{2j} = \lim_{m\to\infty} \frac{y_{-5}(1-a)}{y_{-1}\left(bx_{-2}y_{-3}x_{-4}y_{-5} + (1-a-bx_{-2}y_{-3}x_{-4}y_{-5})a^{2m+j+1}\right)}$$
$$=0$$
$$\lim_{m\to\infty} q_m^{2j-1} = \lim_{m\to\infty} \frac{y_{-1}\left(bx_{-2}y_{-3}x_{-4}y_{-5} + (1-a-bx_{-2}y_{-3}x_{-4}y_{-5})a^{2m+j}\right)}{y_{-5}(1-a)}$$
$$=\infty$$
$$\lim_{m\to\infty} q_m^{2j} = \lim_{m\to\infty} \frac{x_{-5}\left(bx_{-1}y_{-2}x_{-3}y_{-4} + (1-a-bx_{-1}y_{-2}x_{-3}y_{-4})a^{2m+j}\right)}{x_{-1}(1-a)}$$
$$=\infty$$

elde edilir.

b) (a)'nın ispatına benzer şekilde yapılır.

c) Varsayalım ki $|\alpha|>1,\ x_{-1}y_{-2}x_{-3}y_{-4} = \frac{1-a}{b} = x_{-2}y_{-3}x_{-4}y_{-5},\quad y_{-1}x_{-2}y_{-3}x_{-4} \neq \frac{1-\alpha}{\beta} \neq y_{-2}x_{-3}y_{-4}x_{-5}$ olsun. (13), (14), (15) ve (16) denklemlerinden $\forall j \in \{0,1\}$ için

$$\lim_{m\to\infty} p_m^{2j-1} = \lim_{m\to\infty} \frac{x_{-1}\left(\beta y_{-2}x_{-3}y_{-4}x_{-5} + (1-\alpha-\beta y_{-2}x_{-3}y_{-4}x_{-5})\alpha^{2m+j}\right)}{x_{-5}(1-\alpha)}$$
$$=\infty$$
$$\lim_{m\to\infty} p_m^{2j} = \lim_{m\to\infty} \frac{y_{-5}\left(\beta y_{-1}x_{-2}y_{-3}x_{-4} + (1-\alpha-\beta y_{-1}x_{-2}y_{-3}x_{-4})\alpha^{2m+j+1}\right)}{y_{-1}(1-\alpha)}$$
$$=\infty$$
$$\lim_{m\to\infty} q_m^{2j-1} = \lim_{m\to\infty} \frac{y_{-1}(1-\alpha)}{y_{-5}\left(\beta y_{-1}x_{-2}y_{-3}x_{-4} + (1-\alpha-\beta y_{-1}x_{-2}y_{-3}x_{-4})\alpha^{2m+j}\right)}$$
$$=0$$
$$\lim_{m\to\infty} q_m^{2j} = \lim_{m\to\infty} \frac{x_{-5}(1-\alpha)}{x_{-1}\left(\beta y_{-2}x_{-3}y_{-4}x_{-5} + (1-\alpha-\beta y_{-2}x_{-3}y_{-4}x_{-5})\alpha^{2m+j+1}\right)}$$
$$=0$$

elde edilir.

d) (c)'nin ispatına benzer şekilde yapılır.

e) Varsayalım ki $\left|\frac{a}{\alpha}\right|>1,\ x_{-1}y_{-2}x_{-3}y_{-4} \neq \frac{1-a}{b} \neq x_{-2}y_{-3}x_{-4}y_{-5},\quad y_{-1}x_{-2}y_{-3}x_{-4} \neq \frac{1-\alpha}{\beta} \neq y_{-2}x_{-3}y_{-4}x_{-5}$ olsun. (13), (14), (15) ve (16) denklemlerinden $\forall j \in \{0,1\}$ için

$$\lim_{m\to\infty} p_m^{2j-1} = \lim_{m\to\infty} \frac{x_{-1}(1-a)\left(\dfrac{\beta y_{-2}x_{-3}y_{-4}x_{-5}}{\alpha^{2m+j}} + 1-\alpha-\beta y_{-2}x_{-3}y_{-4}x_{-5}\right)}{x_{-5}(1-\alpha)\left(\dfrac{bx_{-1}y_{-2}x_{-3}y_{-4}}{a^{2m+j}} + 1-a-bx_{-1}y_{-2}x_{-3}y_{-4}\right)}\left(\frac{\alpha}{a}\right)^{2m+j} \quad (17)$$

$$=0$$

$$\lim_{m\to\infty} p_m^{2j} = \lim_{m\to\infty} \frac{y_{-5}(1-a)\left(\dfrac{\beta y_{-1}x_{-2}y_{-3}x_{-4}}{\alpha^{2m+j}} + 1-\alpha-\beta y_{-1}x_{-2}y_{-3}x_{-4}\right)}{y_{-1}(1-\alpha)\left(\dfrac{bx_{-2}y_{-3}x_{-4}y_{-5}}{a^{2m+j}} + \left(1-a-bx_{-2}y_{-3}x_{-4}y_{-5}\right)a\right)}\left(\frac{\alpha}{a}\right)^{2m+j} \quad (18)$$

$$=0$$

$$\lim_{m\to\infty} q_m^{2j-1} = \lim_{m\to\infty} \frac{y_{-1}(1-\alpha)\left(\dfrac{bx_{-2}y_{-3}x_{-4}y_{-5}}{a^{2m+j}} + 1-a-bx_{-2}y_{-3}x_{-4}y_{-5}\right)}{y_{-5}(1-a)\left(\dfrac{\beta y_{-1}x_{-2}y_{-3}x_{-4}}{\alpha^{2m+j}} + 1-\alpha-\beta y_{-1}x_{-2}y_{-3}x_{-4}\right)}\left(\frac{a}{\alpha}\right)^{2m+j} \quad (19)$$

$$=\infty$$

$$\lim_{m\to\infty} q_m^{2j} = \lim_{m\to\infty} \frac{x_{-5}(1-\alpha)\left(\dfrac{bx_{-1}y_{-2}x_{-3}y_{-4}}{a^{2m+j}} + 1-a-bx_{-1}y_{-2}x_{-3}y_{-4}\right)}{x_{-1}(1-a)\left(\dfrac{\beta y_{-2}x_{-3}y_{-4}x_{-5}}{\alpha^{2m+j}} + \left(1-\alpha-\beta y_{-2}x_{-3}y_{-4}x_{-5}\right)\alpha\right)}\left(\frac{a}{\alpha}\right)^{2m+j} \quad (20)$$

$$=\infty$$

elde edilir. Öte yandan (17), (18), (19), (20) denklemleri, Teorem 3.1 ve $\left|\dfrac{a}{\alpha}\right| > 1,$ kabulünden ispat kolayca görülür.

f) (e)'nin ispatına benzer şekilde yapılır.

g) Varsayalım ki $a=\alpha,\ |a|<1,\ b=\beta,\quad x_{-1}=x_{-5},\ y_{-1}=y_{-5}$ olsun. $x\in(-\varepsilon,\varepsilon)$ olmak üzere $(1+x)^{-1}=1-x+O\left(x^2\right)$ asimptotik bağıntısı, $\forall j\in\{0,1\}$ ve yeterince büyük m değerleri için

$$p_m^{2j-1} = \frac{bx_{-1}y_{-2}x_{-3}y_{-4} + \left(1-a-bx_{-1}y_{-2}x_{-3}y_{-4}\right)a^{2m+j}}{bx_{-1}y_{-2}x_{-3}y_{-4} + \left(1-a-bx_{-1}y_{-2}x_{-3}y_{-4}\right)a^{2m+j}} = 1 \quad (21)$$

$$\begin{aligned} p_m^{2j} &= \frac{by_{-1}x_{-2}y_{-3}x_{-4} + \left(1-a-by_{-1}x_{-2}y_{-3}x_{-4}\right)a^{2m+j}}{by_{-1}x_{-2}y_{-3}x_{-4} + \left(1-a-by_{-1}x_{-2}y_{-3}x_{-4}\right)a^{2m+j+1}} \\ &= 1+\frac{1-a-by_{-1}x_{-2}y_{-3}x_{-4}}{by_{-1}x_{-2}y_{-3}x_{-4}}(1-a)a^{2m+j}+O\left(a^{2m}\right) \end{aligned} \quad (22)$$

$$q_m^{2j-1} = \frac{by_{-1}x_{-2}y_{-3}x_{-4} + \left(1-a-by_{-1}x_{-2}y_{-3}x_{-4}\right)a^{2m+j}}{by_{-1}x_{-2}y_{-3}x_{-4} + \left(1-a-by_{-1}x_{-2}y_{-3}x_{-4}\right)a^{2m+j}} = 1 \quad (23)$$

$$\begin{aligned} q_m^{2j} &= \frac{bx_{-1}y_{-2}x_{-3}y_{-4} + \left(1-a-bx_{-1}y_{-2}x_{-3}y_{-4}\right)a^{2m+j}}{bx_{-1}y_{-2}x_{-3}y_{-4} + \left(1-a-bx_{-1}y_{-2}x_{-3}y_{-4}\right)a^{2m+j+1}} \\ &= 1+\frac{1-a-bx_{-1}y_{-2}x_{-3}y_{-4}}{bx_{-1}y_{-2}x_{-3}y_{-4}}(1-a)a^{2m+j}+O\left(a^{2m}\right) \end{aligned} \quad (24)$$

yazılabilir. Öte yandan (21), (22), (23), (24) denklemleri, $|a|<1$ kabulü,

$\left(\prod_{j=0}^{m} p_m^{2j-1}\right)_{m\in N_0}, \left(\prod_{j=0}^{m} q_m^{2j-1}\right)_{m\in N_0}$ dizilerinin sabitliği ve $\left(\prod_{j=0}^{m} p_m^{2j}\right)_{m\in N_0}, \left(\prod_{j=0}^{m} q_m^{2j}\right)_{m\in N_0}$ dizilerinin yakınsaklığından $\forall j \in \{0,1\}$ $\left(x_{4m+2j-1}\right)_{n\in N_0}, \left(y_{4m+2j-1}\right)_{n\in N_0}$ dizilerinin sabit ve $\left(x_{4m+2j}\right)_{n\in N_0}, \left(y_{4m+2j}\right)_{n\in N_0}$ dizilerinin de yakınsak olduğu kolayca görülür.

h) Teorem 3.1'den ispat açıktır.

i) $m \ge -1, a = \alpha = 1, x_{-1} = x_{-5}, y_{-1} = y_{-5}$, $b = \beta$ ve $j \in \{0,1\}$ olsun. O zaman

$$\begin{aligned} x_{4m+2j-1} &= x_{2j-5}\prod_{s=0}^{m}\frac{1+(2s+j)bx_{-1}y_{-2}x_{-3}y_{-4}}{1+(2s+j)bx_{-1}y_{-2}x_{-3}y_{-4}} \\ &= x_{2j-5} \end{aligned} \tag{25}$$

$$\begin{aligned} x_{4m+2j} &= x_{2j-4}\prod_{s=0}^{m}\frac{1+(2s+j)by_{-1}x_{-2}y_{-3}x_{-4}}{1+(2s+j+1)by_{-1}x_{-2}y_{-3}x_{-4}} \\ &= x_{2j-4}C_1(m_0)\prod_{s=m_0+1}^{m}\left(1-\frac{1}{2s}+O(s^2)\right) \\ &= x_{2j-4}C_1(m_0)e^{\sum_{s=m_0+1}^{m}\ln\left(1-\frac{1}{2s}+O(s^2)\right)} \\ &= x_{2j-4}C_1(m_0)e^{\left(-\frac{1}{2}\sum_{s=m_0+1}^{m}\left(\frac{1}{s}+O(s^2)\right)\right)} \end{aligned} \tag{26}$$

$$\begin{aligned} y_{4m+2j-1} &= y_{2j-5}\prod_{s=0}^{m}\frac{1+(2s+j)by_{-1}x_{-2}y_{-3}x_{-4}}{1+(2s+j)by_{-1}x_{-2}y_{-3}x_{-4}} \\ &= y_{2j-5} \end{aligned} \tag{27}$$

$$\begin{aligned} y_{4m+2j} &= y_{2j-4}\prod_{s=0}^{m}\frac{1+(2s+j)bx_{-1}y_{-2}x_{-3}y_{-4}}{1+(2s+j+1)bx_{-1}y_{-2}x_{-3}y_{-4}} \\ &= y_{2j-4}C_1(m_0)\prod_{s=m_0+1}^{m}\left(1-\frac{1}{2s}+O(s^2)\right) \\ &= y_{2j-4}C_1(m_0)e^{\sum_{s=m_0+1}^{m}\ln\left(1-\frac{1}{2s}+O(s^2)\right)} \\ &= y_{2j-4}C_1(m_0)e^{\left(-\frac{1}{2}\sum_{s=m_0+1}^{m}\left(\frac{1}{s}+O(s^2)\right)\right)}. \end{aligned} \tag{28}$$

ayrıca $m \to \infty$ iken $\sum_{s=m_0+1}^{m}\frac{1}{s}$ serisi ıraksak olduğundan $m \to \infty$ iken $\sum_{s=m_0+1}^{m} O\left(\frac{1}{s^2}\right)$ serisi ise yakınsak olacaktır. Öte yandan (25), (26), (27) ve (28) ifadelerinde $m \to \infty$ iken limit alındığında $x_{4m+2j-1} \to x_{2j-5}, x_{4m+2j} \to 0,\ y_{4m+2j-1} \to y_{2j-5},\ y_{4m+2j} \to 0$ olduğu kolaylıkla görülür.

Teorem 3.6. $(x_n, y_n)_{n\ge -5}$ (12) denklem sisteminin çözümü, $a=-1,\ \alpha=-1,\ b=\beta, b\neq 0$, $i_1 \in \{1,2,3,4,5\}$ için $x_{-i_1} \neq 0, y_{-i_1} \neq 0$, $N = bx_{-1}y_{-2}x_{-3}y_{-4} - 1$, $M = by_{-1}x_{-2}y_{-3}x_{-4} - 1$ olsun. Bu durumda aşağıdakiler doğrudur.

a) Eğer $x_{-1}y_{-2}x_{-3}y_{-4} = x_{-2}y_{-3}x_{-4}y_{-5} = \frac{2}{b}$, $y_{-1}x_{-2}y_{-3}x_{-4} = y_{-2}x_{-3}y_{-4}x_{-5} = \frac{2}{b}$ ise $\left(x_n, y_n\right)_{n\geq -5}$ dizisi 4 periyotlu çözümlere sahiptir.

b) Eğer $x_{-1} = x_{-5}, y_{-1} = y_{-5}, \left|N\right| < 1$ ve $\left|M\right| < 1$ ise $m \to \infty$ iken $x_{4m+2j-1} \to x_{2j-5}$, $\left|x_{4m+4t}\right| \to \infty$, $x_{4m+4t+2} \to 0$, $y_{4m+2j-1} \to y_{2j-5}$, $\left|y_{4m+4t}\right| \to \infty$, $y_{4m+4t+2} \to 0$'dır.

c) Eğer $x_{-1} = x_{-5}, y_{-1} = y_{-5}, \left|N\right| > 1$ ve $\left|M\right| > 1$ ise $m \to \infty$ iken $x_{4m+2j-1} \to x_{2j-5}$, $x_{4m+4t} \to 0$, $\left|x_{4m+4t+2}\right| \to \infty$, $y_{4m+2j-1} \to y_{2j-5}$, $y_{4m+4t} \to 0$, $\left|y_{4m+4t+2}\right| \to \infty$'dur.

d) Eğer $x_{-1} = x_{-5}, y_{-1} = y_{-5}, N = 1$ ve $M = 1$ ise $\left(x_n, y_n\right)_{n\geq -5}$ dizisi 4 periyotlu çözümlere sahiptir.

e) Eğer $x_{-1} = x_{-5}, y_{-1} = y_{-5}, N = -1$ ve $M = -1$ ise $\left(x_n, y_n\right)_{n\geq -5}$ dizisi 8 periyotlu çözümlere sahiptir.

İspat: Varsayalım ki $a = -1$, $\alpha = -1$, $b = \beta$, $b \neq 0$ ve $i_1 = 1,2,3,4,5$ için $x_{-i_1} \neq 0, y_{-i_1} \neq 0$ ve $N = bx_{-1}y_{-2}x_{-3}y_{-4} - 1$, $M = by_{-1}x_{-2}y_{-3}x_{-4} - 1$ olsun. Bu durumda

(a)'nın ispatı: Teorem 3.3'te $b = \beta$ alınırsa sonuç açıktır.

(b)-(c)'nin ispatı: Varsayalım ki $x_{-1} = x_{-5}, y_{-1} = y_{-5}$ olsun. Teorem 3.3'ten

$$x_{4m+2j-1} = x_{2j-5}\prod_{s=0}^{m}\frac{bx_{-1}y_{-2}x_{-3}y_{-4} + \left(2 - bx_{-1}y_{-2}x_{-3}y_{-4}\right)\left(-1\right)^{2s+j}}{bx_{-1}y_{-2}x_{-3}y_{-4} + \left(2 - bx_{-1}y_{-2}x_{-3}y_{-4}\right)\left(-1\right)^{2s+j}} = x_{2j-5} \tag{29}$$

$$x_{4m+2j} = x_{2j-4}\prod_{s=0}^{m}\frac{by_{-1}x_{-2}y_{-3}x_{-4} + \left(2 - by_{-1}x_{-2}y_{-3}x_{-4}\right)\left(-1\right)^{2s+j}}{by_{-1}x_{-2}y_{-3}x_{-4} + \left(2 - by_{-1}x_{-2}y_{-3}x_{-4}\right)\left(-1\right)^{2s+j+1}} = \frac{x_{2j-4}}{\left(\dfrac{by_{-1}x_{-2}y_{-3}x_{-4} + \left(2 - by_{-1}x_{-2}y_{-3}x_{-4}\right)\left(-1\right)^{j+1}}{by_{-1}x_{-2}y_{-3}x_{-4} + \left(2 - by_{-1}x_{-2}y_{-3}x_{-4}\right)\left(-1\right)^{j}}\right)^{m+1}} \tag{30}$$

$$y_{4m+2j-1} = y_{2j-5}\prod_{s=0}^{m}\frac{y_{-1}}{y_{-5}}\frac{by_{-1}x_{-2}y_{-3}x_{-4} + \left(2 - by_{-1}x_{-2}y_{-3}x_{-4}\right)\left(-1\right)^{2s+j}}{by_{-1}x_{-2}y_{-3}x_{-4} + \left(2 - by_{-1}x_{-2}y_{-3}x_{-4}\right)\left(-1\right)^{2s+j}} = y_{2j-5} \tag{31}$$

$$y_{4m+2j} = y_{2j-4}\prod_{s=0}^{m}\frac{bx_{-1}y_{-2}x_{-3}y_{-4} + \left(2 - bx_{-1}y_{-2}x_{-3}y_{-4}\right)\left(-1\right)^{2s+j}}{bx_{-1}y_{-2}x_{-3}y_{-4} + \left(2 - bx_{-1}y_{-2}x_{-3}y_{-4}\right)\left(-1\right)^{2s+j+1}} = \frac{y_{2j-4}}{\left(\dfrac{bx_{-1}y_{-2}x_{-3}y_{-4} + \left(2 - bx_{-1}y_{-2}x_{-3}y_{-4}\right)\left(-1\right)^{j+1}}{bx_{-1}y_{-2}x_{-3}y_{-4} + \left(2 - bx_{-1}y_{-2}x_{-3}y_{-4}\right)\left(-1\right)^{j}}\right)^{m+1}} \tag{32}$$

yazılabilir. Öte yandan $j \in \{0,1\}$ olduğundan $t = 0$ için $j = 2t, j = 2t+1$ dönüşümleri (30) ve (32) denklemlerinde yerine yazılırsa; sırasıyla,

$$\begin{aligned} x_{4m+4t} &= \frac{x_{4t-4}}{\left(\dfrac{by_{-1}x_{-2}y_{-3}x_{-4}+\left(2-by_{-1}x_{-2}y_{-3}x_{-4}\right)\left(-1\right)^{2t+1}}{by_{-1}x_{-2}y_{-3}x_{-4}+\left(2-by_{-1}x_{-2}y_{-3}x_{-4}\right)\left(-1\right)^{2t}}\right)^{m+1}} \\ &= \frac{x_{4t-4}}{\left(by_{-1}x_{-2}y_{-3}x_{-4}-1\right)^{m+1}} \end{aligned} \tag{33}$$

$$\begin{aligned} x_{4m+4t+2} &= \frac{x_{4t-2}}{\left(\dfrac{by_{-1}x_{-2}y_{-3}x_{-4}+\left(2-by_{-1}x_{-2}y_{-3}x_{-4}\right)\left(-1\right)^{2t+2}}{by_{-1}x_{-2}y_{-3}x_{-4}+\left(2-by_{-1}x_{-2}y_{-3}x_{-4}\right)\left(-1\right)^{2t+1}}\right)^{m+1}} \\ &= \frac{x_{4t-4}}{\left(\dfrac{1}{by_{-1}x_{-2}y_{-3}x_{-4}-1}\right)^{m+1}} \end{aligned} \tag{34}$$

$$\begin{aligned} y_{4m+4t} &= \frac{y_{4t-4}}{\left(\dfrac{bx_{-1}y_{-2}x_{-3}y_{-4}+\left(2-bx_{-1}y_{-2}x_{-3}y_{-4}\right)\left(-1\right)^{2t+1}}{bx_{-1}y_{-2}x_{-3}y_{-4}+\left(2-x_{-1}y_{-2}x_{-3}y_{-4}\right)\left(-1\right)^{2t}}\right)^{m+1}} \\ &= \frac{y_{4t-4}}{\left(bx_{-1}y_{-2}x_{-3}y_{-4}-1\right)^{m+1}} \end{aligned} \tag{35}$$

$$\begin{aligned} y_{4m+4t+2} &= \frac{y_{4t-2}}{\left(\dfrac{bx_{-1}y_{-2}x_{-3}y_{-4}+\left(2-bx_{-1}y_{-2}x_{-3}y_{-4}\right)\left(-1\right)^{2t+2}}{bx_{-1}y_{-2}x_{-3}y_{-4}+\left(2-bx_{-1}y_{-2}x_{-3}y_{-4}\right)\left(-1\right)^{2t+1}}\right)^{m+1}} \\ &= \frac{y_{4t-4}}{\left(\dfrac{1}{bx_{-1}y_{-2}x_{-3}y_{-4}-1}\right)^{m+1}} \end{aligned} \tag{36}$$

elde edilir. (29), (31), (33)-(36) denklemlerinden (b)-(c) öncüllerinin geçerliliği kolaylıkla görülür.

(d)-(e)'nin ispatı: (29), (31), (33)-(36) denklemlerinden (d)-(e) öncüllerinin geçerliliği kolaylıkla görülür.

Teorem 3.7. $\left(x_n, y_n\right)_{n\geq -5}$ (12) denklem sisteminin iyi tanımlanmış çözümü, $a\alpha \neq 0,\ b=0=\beta$ ve $i=1,2,3,4,5$ için $x_{-i}\neq 0, y_{-i}\neq 0$ olsun. Bu durumda aşağıdakiler doğrudur.

a) Eğer $\left|\dfrac{a}{\alpha}\right|>1$ ise $m\to\infty$ iken $x_m\to 0,\ \left|y_m\right|\to\infty$'dur.

b) Eğer $\left|\dfrac{a}{\alpha}\right|<1$ ise $m\to\infty$ iken $\left|x_m\right|\to\infty,\ y_m\to 0$'dır.

c) Eğer $\dfrac{a}{\alpha}=1,\left(a=\alpha=1\right), x_{-1}=x_{-5},\ y_{-1}=y_{-5}$ ise $\left(x_n, y_n\right)_{n\geq -5}$ dizisi 4-periyotludur.

d) Eğer $\frac{a}{\alpha}=-1,(a=1\wedge\alpha=-1\vee a=-1\wedge\alpha=1)$, $x_{-1}=x_{-5}$, $y_{-1}=y_{-5}$ ise $(x_n,y_n)_{n\geq-5}$ dizisi 8 periyotludur.

İspat: Varsayalım ki $a\alpha\neq 0$, $b=0=\beta$ olsun. Teorem 3.4'ten $(x_n,y_n)_{n\geq-5}$ ' in çözümü

$$x_{4m+2j-1}=x_{2j-5}\prod_{s=0}^{m}\frac{x_{-1}}{x_{-5}}\left(\frac{\alpha}{a}\right)^{2s+j}$$

$$x_{4m+2j}=x_{2j-4}\frac{1}{a}\prod_{s=0}^{m}\frac{y_{-5}}{y_{-1}}\left(\frac{\alpha}{a}\right)^{2s+j}$$

$$y_{4m+2j-1}=y_{2j-5}\prod_{s=0}^{m}\frac{y_{-1}}{y_{-5}}\left(\frac{a}{\alpha}\right)^{2s+j}$$

$$y_{4m+2j}=y_{2j-4}\frac{1}{\alpha}\prod_{s=0}^{m}\frac{x_{-5}}{x_{-1}}\left(\frac{a}{\alpha}\right)^{2s+j}$$

şeklindedir. Burada $j\in\{0,1\}$ ' dir. Teoremin ispatı $(x_n,y_n)_{n\geq-5}$ ' in çözümünden açıktır.

4. SONUÇ

Bu çalışmada $j\in\{1,2,3,4,5\}$ için x_{-j},y_{-j}, başlangıç şartları reel sayılar ve $(a_n)_{n\in N_0}$, $(b_n)_{n\in N_0}$, $(\alpha_n)_{n\in N_0}$, $(\beta_n)_{n\in N_0}$ birer dizi olmak üzere;

$$x_n=\frac{x_{n-4}y_{n-5}}{y_{n-1}\left(a_n+b_nx_{n-2}y_{n-3}x_{n-4}y_{n-5}\right)},$$

$$y_n=\frac{y_{n-4}x_{n-5}}{x_{n-1}\left(\alpha_n+\beta_ny_{n-2}x_{n-3}y_{n-4}x_{n-5}\right)},\quad n\in N_0,$$

rasyonel fark denklem sisteminin iyi tanımlanmış çözümleri elde edilmiştir. Daha sonra $a_n=a,\ b_n=b,\ \alpha_n=\alpha,\ \beta_n=\beta$ olmak üzere sabit katsayılı lineer olmayan

$$x_n=\frac{x_{n-4}y_{n-5}}{y_{n-1}\left(a+bx_{n-2}y_{n-3}x_{n-4}y_{n-5}\right)}$$

$$y_n=\frac{y_{n-4}x_{n-5}}{x_{n-1}\left(\alpha+\beta y_{n-2}x_{n-3}y_{n-4}x_{n-5}\right)},\quad n\in N_0,$$

fark denklem sisteminin çözümleri $a, b, \alpha,$ ve β ' nın durumlarına göre elde edilmiştir. Elde edilen çözümlerin asimptotik davranışı ve periyodikliği incelenmiştir.

KAYNAKLAR

[1] Alzahrani E O, El-Dessoky M M, Elsayed E M and Kuang Y. Solutions and properties of some degernerate systems of difference equations. J. Computational Analysis and Applications, 2015; 3: 321-333.

[2] Çınar C and Elsayed EM. On the positive solutions of the difference equation $x_{n+1}=\frac{x_{n-1}}{1+x_nx_{n-1}}$. Applied Mathematics and Computation, 2004; 150: 21-24.

[3] Dekkar I. Touafek N and Yazlik Y. Global stability of a third-order non- linear system of difference equations with period two coefficients. Revista de la Real Academia de Ciencias Exactas, Fisicas y Naturales. Serie A. Matematicas, 2017; 111: 325-347.

[4] Din Q. On a system of fourth-order rational difference equations. Acta Univ. Apulensis, Math. Inform, 2014; 39: 137-150.

[5] Elaydi S. An Introduction to Difference Equations. Springer, New York, 1996.

[6] El-Dessoky M M, Elsayed E M and Alghamdi M Solutions and periodicity for some systems of fourth order rational difference equations. J. Computational Analysis and Applications, 2015; 18: 179-194.

[7] El-Metwally H and Elsayed E M. Qualitative study of solutions of some difference equatiaons. Abstract and Applied Analysis, Article ID 248291, 2012, 16 pages.

[8] El-Metwally H and Elsayed E M. Qualitative behavior of some rational difference equatiaons. J. Computational Analysis and Applications, 2016.; 20: 226-236.

[9] Elsayed EM. Behavior and expression of the solutions of some rational difference equations. J. Computational Analysis and Applications, 2013; 15: 73-81.

[10] Elsayed EM. Solution of a rational recursive sequences of order three. Funct. Approx. Comment. Math., 2013; 48: 7-17.

[11] Elsayed EM, El-Dessoky MM and Alzahrani EO. The form of solution and dynamics of a rational recursive sequence. J. Computational Analysis and Applications, 2014; 17: 172-186.

[12] Elsayed EM and İbrahim TF. Solutions and periodicity of a rational recursive sequences of order five. Bulletin of the Malaysian Mathematical Sciences Society, 2014; 38: 95-112.

[13] Elsayed EM. Mahmoud SR and Ali AT. Expression and dynamics of the soltions of some rational recursive sequences. Iranian Journal of Science & Technology, 2014; 38: 295-303.

[14] Elsayed EM and İbrahim TF. Periodicity and solutions for some systems of nonlinear rational difference equations. Hacettepe Journal of Mathematics and Statistics, 2015; 44: 1361-1390.

[15] Elsayed EM. Expression and behavior of the solutions of some rational recursive sequences. Mathematical Methods in the Applied Sciences, 2016; 39: 5682-5694.

[16] Gelisken A and Kara M. Some General Systems of Rational Difference Equations. Journal of Difference Equations, 2015; 396757, 1-7.

[17] Grove EA and Ladas G. Periodicities in Nonlinear Difference Equations. Advances in Discrete Mathematics and Applications, 2005; 4: Chapman & Hall/CRC, London.

[18] Halim Y, Touafek N and Yazlik Y. Dynamic behavior of a second-order nonlinear rational difference equation. Turk. J. Math, 2015; 39: 1004-1018.

[19] İbrahim TF. On the third order rational difference equation $x_{n+1} = \dfrac{x_n x_{n-2}}{x_{n-1}(a + b x_n x_{n-2})}$. Int. J. Contemp. Math. Sciences, 2009; 4: 1321-1334.

[20] İbrahim TF and Touafek N. On a third order rational difference equation with variable coefficients. Dynamics of Discrete and Impulsive Systems Series B Applications Algorithms, 2013; 20: 251-264.

[21] Steviç S. Diblik J. Iricanin B and Smarda Z. On the difference equation $x_{n+1}=\frac{x_n x_{n-k}}{x_{n-k+1}\left(a+bx_n x_{n-k}\right)}$. Hindawi Publishing Corporation Abstract and Applied Analysis, 2012, 9 pages.

[22] Steviç S. On some solvable systems of difference equations. Applied Mathematics and Computation, 2012; 218: 5010-5018.

[23] Steviç S. On a solvable rational system of difference equations. Applied Mathematics and Computation, 2012; 219: 2896-2908.

[24] Steviç S. Diblik J. Iricanin B and Smarda Z. Solvability of nonlinear difference equations of fourth order. Electronic Journal of Differential Equations, 2014; 264: 1-14.

[25] Steviç S. Iricanin B and Smarda Z. On a close to symmetric system of difference equations of second order. Advences in Difference Equations, 2015; 264: 1-34.

[26] Steviç S. Diblik J. Iricanin B and Smarda Z. On a fifth-order difference equation. J. Computational Analysis and Applications, 2016; 20: 1214-1227.

[27] Stevic S. Alghamdi MA. Alotaibi A and Elsayed EM. On a class of solvable higher order difference equations. Filomat, 2017; 31: 461-477.

[28] Tollu DT. Yazlik Y and Taskara N. On the solutions of two special types of riccati difference equation via fibonacci numbers. Advances in Difference Equations, 2013; 174: 1-7.

[29] Tollu DT. Yazlik Y and Taskara N. On fourteen solvable systems of difference equations. Applied Mathematics and Computation, 2014; 233: 310-319.

[30] Tollu DT. Yazlik Y and Taskara N. Behavior of positive solutions of a difference equation. Journal of Applied Mathematics and Informatics, 2017; 35: 217-230.

[31] Touafek N and Elsayed E M. Periodicity of some systems of nonlinear difference equations. Bull. Math. Soc. Sci. Math. Roumanie Tome, 2012; 55: 217-224.

[32] Yalcinkaya I and Tollu D T. Global behavior of a second order system of difference equations. Advanced Studies in Contemporary Mathematics, 2016; 26: 653-667.

[33] Yazlik Y. On the solutions and behavior of rational difference equations. J. Comput. Anal. Appl, 2014; 17: 584-594.

[34] Yazlik Y. Elsayed E M and Taskara N. On the behaviour of the solutions the solutions of difference equation system. J. Comput. Anal. Appl, 2014; 16: 932-941.

[35] Yazlik Y. Tollu DT and Taskara N. On the behaviour of solutions for some systems of difference equations. J. Comput. Anal. Appl, 2015; 18: 166-178.

[36] Yazlik Y. Tollu DT and Taskara N. On the solutions of a three-dimensional system of difference equations. Kuwait Journal of Science, 2015; 43: 95-111.