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A note on the endomorphism ring of finitely presented modules of the projective dimension ≤ 1

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Abstract

In this paper, we study the behavior of endomorphism rings of a cyclic, finitely presented module of projective dimension ≤ 1. This class of modules extends to arbitrary rings the class of couniformly presented modules over local rings.

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1. Introduction

Throughout this paper, all rings will be associative with identity and modules will be unital right modules. For any ring R, the Jacobson radical of R will be denoted by $J(R)$

Recall that M_R is *couniform* if it has dual Goldie dimension one (if and only if it is non-zero and the sum of any two proper submodules of M_R is a proper submodule of M_R). It is well know that a projective right module P_R is couniform if and only if $\mathrm{E}nd(P_R)$ is a local ring, if and only if there exists an idempotent $e \in R$ with $P_R \cong eR$ and eRe a local ring, if and only if is a finitely generated module with a unique maximal submodule.

In [7], Facchini and Girardi introduced and studied the notion of couniformly presented modules. A module M_R is called *couniformly presented* if it is non-zero and there exists an exact sequence

$$0 \to C_R \xrightarrow{\iota} P_R \to M_R \to 0$$

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with P_R projective and both C_R and P_R couniform modules. In this case, every endomorphism f of M_R lifts to an endomorphism f_0 of its projective cover P_R, and we will denote by f_1 the restriction to C_R of f_0. Hence we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & C_R & \xrightarrow{\iota} & P_R & \to & M_R & \to & 0 \\ & & f_1 \downarrow & & \downarrow f_0 & & \downarrow f & & \\ 0 & \to & C_R & \xrightarrow{\iota} & P_R & \to & M_R & \to & 0. \end{array}$$

In [7, Theorem 2.5], Facchini and Girardi proved that:

- Let $0 \to C_R \to P_R \to M_R \to\ 0$ be a couniform presentation of a couniformly presented module M_R. Set $K := \{ f \in \mathrm{E}nd(M_R) \mid f \text{ is not surjective} \}$ and $I := \{ f \in \mathrm{E}nd(M_R) \mid f_1 \colon C_R \to C_R \text{ is not surjective} \}$. Then K and I are completely prime two-sided ideals of $\mathrm{E}nd(M_R)$, and the union $K \cup I$ is the set of all non-invertible elements of $\mathrm{E}nd(M_R)$. Moreover, one of the following two conditions holds:
(a) Either $\mathrm{E}nd(M_R)$ is a local ring, or
(b) K and I are the two maximal right, maximal left ideals of $\mathrm{E}nd(M_R)$.

If M_R and M'_R are two couniformly presented modules with couniform presentations $0 \to C_R \to P_R \to M_R \to 0$ and $0 \to C'_R \to P'_R \to M'_R \to 0$, we say that M_R and M'_R *have the same lower part,* and we write $[M_R]_\ell = [M'_R]_\ell$, if there are two homomorphisms $f_0 \colon P_R \to P'_R$ and $f'_0 \colon P'_R \to P_R$ such that $f_0(C_R) = C'_R$ and $f'_0(C'_R) = C_R$.

Recall that a ring R is *semilocal* if $R/J(R)$ is semisimple artinian, that is, isomorphic to a finite direct product of rings $M_{n_i}(D_i)$ of $n_i \times n_i$ matrices over division rings D_i. A ring R is *homogeneous semilocal* if $R/J(R)$ is simple artinian, that is, isomorphic to the ring $M_n(D)$ of all $n \times n$ matrices for some positive integer n and some division ring D [2, 4]. Examples of such rings include all local rings and all simple Artinian rings. If R is a homogeneous semilocal ring, then so are the rings eRe and $M_n(R)$, where e is a nonzero idempotent element of R and $M_n(R)$ is the matrix ring over R. Also, homogeneous semilocal rings appear in a natural way when one localizes a right Noetherian ring with respect to a right localizable prime ideal.

In [4], Corisello and Facchini showed that:

- a homogeneous semilocal ring has a unique maximal proper two-sided ideal and a unique simple module up to isomorphism. Similarly, as in the case of local rings, a homogeneous semilocal ring has only one indecomposable projective module P_R up to isomorphism, and all projective modules are direct sums of copies of this P_R.
- for a module M over any ring R, the Krull-Schmidt theorem holds for M provided $\mathrm{E}nd_R(M)$ is homogeneous semilocal—that is, the direct sum decomposition of M into indecomposable summands is unique up to isomorphism.

In [2], Barioli-Facchini-Raggi proved that:

- The later result fails to extend to modules M_R with finite direct sum decompositions whose indecomposable summands have homogeneous semilocal endomorphism rings,
- If a module M over a ring R has two decompositions $M = M_1 \oplus \cdots \oplus M_t = N_1 \oplus \cdots \oplus N_s$ where all the summands are indecomposable with homogeneous semilocal endomorphism rings, then these two decompositions are isomorphic.

2. The endomorphism ring

The following results describe the endomorphism ring of a cyclic, finitely presented module of projective dimension ≤ 1 over a local ring. Throughout this paper, we will assume that $M_R \neq 0$.

2.1. Theorem. *Let R be a local ring and let $M_R := R_R/I$ be a cyclic, finitely presented module of projective dimension ≤ 1. Suppose $\mathrm{Ext}^1_R(M_R, R_R) = 0$.*

Assume $0 \neq I \neq R$ and let E be the idealizer of the right ideal I of R, that is, the set of all $r \in R$ with $rI \subseteq I$, so that $\mathrm{End}(M_R) \cong E/I$. Set $L := \{\, r \in R \mid rI \subseteq IJ(R) \,\}$ and $K := E \cap J(R)$. Let $\psi\colon E \to \mathrm{End}_R(I/IJ(R))$ be the ring morphism defined by

$$\psi(e)(x + IJ(R)) = ex + IJ(R),$$

for every $e \in E$ and $x \in I$. Let n be the dimension of the right vector space $I/IJ(R)$ over the division ring $R/J(R)$. Then:

(1) *L and K are prime two-sided ideals of E containing I and K is a completely prime ideal of E;*

(2) *For every $e \in E$, the element $e + I$ of E/I is invertible in E/I if and only if $e + J(R)$ is invertible in $R/J(R)$ and $\psi(e)$ is invertible in $\mathrm{End}_R(I/IJ(R))$.*

(3) *The quotient ring E/L is isomorphic to the ring $M_n(R/J(R))$ of all $n \times n$ matrices over the division ring $R/J(R)$.*

(4) *Exactly one of the following two conditions holds:*
 (a) *Either $K \subseteq L$, in which case E/I is a homogeneous semilocal ring with Jacobson radical L/I, or*
 (b) *L and K are not comparable.*

Proof. (1) and (3). Notice that L is contained in E and is the kernel of ψ, so that L is a two-sided ideal of E. Trivially, I is contained in L. Let us prove that ψ is onto. Let $f\colon I/IJ(R) \to I/IJ(R)$ be a morphism. Since $M_R := R_R/I$ is of projective dimension ≤ 1, the ideal I_R is projective, so that f lifts to a morphism $f'\colon I_R \to I_R$. Apply the functor $\mathrm{H}om(-, R_R)$ to the exact sequence $0 \to I_R \to R_R \to M_R \to 0$, getting a short exact sequence

$$0 \to \mathrm{H}om(M_R, R_R) \to \mathrm{H}om(R_R, R_R) \to \mathrm{H}om(I_R, R_R) \to 0$$

because $\mathrm{Ext}^1_R(M_R, R_R) = 0$. Hence f' can be extended to a morphism $f''\colon R_R \to R_R$, which is necessarily left multiplication by an element $r \in R$. Since f'' restricts to the endomorphism f' of I_R, we get that $r \in E$, and $\psi(e) = f$. This proves that ψ is an onto ring morphism, so that

$$E/L = E/\ker\psi \cong \mathrm{E}nd_R(I/IJ(R)) \cong M_n(R/J(R)).$$

This proves (3).

As $\mathrm{E}nd_R(I/IJ(R)) \cong M_n(R/J(R))$ is a simple ring, it follows that L is a prime ideal and a maximal two-sided ideal. Similarly, K is the kernel of the composite morphism $\varphi\colon E \to R/J(R)$ of the embedding $E \to R$ and the canonical projection $R \to R/J(R)$. Since $R/J(R)$ is a division ring, we get that K is a completely prime, two-sided ideal of E containing I. This concludes the proof of (1).

(2). (:⇒) Since $\varphi(I) = 0$ and $\psi(I) = 0$, the morphisms φ and ψ induce morphisms $\widetilde{\varphi}\colon E/I \to R/J(R)$ and $\widetilde{\psi}\colon E/I \to \mathrm{E}nd(I/IJ(R))$, respectively. Hence $e + I$ invertible implies $\varphi(e) = e + J(R)$ invertible in $R/J(R)$ and $\psi(e)$ is invertible in $\mathrm{E}nd_R(I/IJ(R))$.
(⇐:) Assume that $e \in E$ and that $\varphi(e)$ and $\psi(e)$ are invertible in $R/J(R)$ and $\mathrm{E}nd_R(I/IJ(R))$, respectively. Then we have a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I & \longrightarrow & R_R & \xrightarrow{\pi} & R_R/I & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle e} & & \downarrow{\scriptstyle e} & & \downarrow{\scriptstyle e} & & \\ 0 & \longrightarrow & I & \longrightarrow & R_R & \xrightarrow{\pi} & R_R/I & \longrightarrow & 0. \end{array}$$

Now $\varphi(e) = e + J(R)$ invertible implies that $e \in R \setminus J(R)$, and so e is invertible in R. Hence the middle vertical arrow is an isomorphism. Since $\psi(e)$ is invertible, it is an automorphism of $I/IJ(R)$, and so $e(I/IJ(R)) = I/IJ(R)$, that is, $eI + IJ(R) = I$. By Nakayama's Lemma, $eI = I$. Hence the left vertical arrow is an epimorphism. By the Snake Lemma, the right vertical arrow is a monomorphism, hence an isomorphism. That is, $e + I$ is invertible in E/I.
(4) We have the three cases (a) $L \subset K$, (b) $K \subseteq L$, and (c) $L \not\subseteq K$ and $K \not\subseteq L$.

Assume $L \subset K$. In this case, $L \subset K \subset E$ implies that $0 \subset K/L \subset E/L$, so that $E/L \cong M_n(R/J(R))$ has a proper non-zero two-sided ideal. This is impossible, because $M_n(R/J)$ is a simple ring. Hence this case cannot occur.

Assume $K \subseteq L$. From (2), it follows that an element $e+I$ of E/I is invertible in E/I if and only if $e + J(R)$ is invertible in $R/J(R)$ and $e + L$ is invertible in E/L. Hence, in order to prove (4) in this case $K \subseteq L$, it suffices to prove that $J(E/I) = L/I$.
($\subseteq$) If $e + I \in J(E/I)$, then $1 - xey + I$ is invertible in E/I for every $x, y \in E$. Thus $1 - xey + L$ is invertible in E/L for all $x, y \in E$, so that $e + L \in J(E/L)$.But $E/L \cong M_n(R/J(R))$ has Jacobson radical 0 so that $e \in L$.
($\supseteq$) Take $l + I \in L/I$ with $l \in L$. Then $1 - xly + L = 1 + L$ in E/L for every $x, y \in E$. Hence $1 - xly + L$ is invertible in E/L. In particular, $1 - xly \notin L$. Thus $1 - xly \notin K$, so that $1 - xly \notin J(R)$. As $R/J(R)$ is a division ring, it follows that $1 - xly + J(R)$ is invertible in $R/J(R)$. Thus $1 - xly + I$ is invertible in E/I, and $l \in J(E/I)$. □

It is known that a finitely presented module over a semilocal ring always has a semilocal endomorphism ring. We have the following natural question.

2.2. Question. Characterize $J(E/I)$. This was done in [1] for cyclically presented modules.

As far as Question 2.2 is concerned, notice that, in the proof of Theorem 2.1(2), we have seen that the mapping

$$\widetilde{\varphi} \times \widetilde{\psi}\colon E/J \to R/J(R) \times \mathrm{E}nd(I/IJ(R))$$

is a local morphism, so that its kernel $K/I \cap L/I$ is contained in $J(E/I)$. In particular, when $K \subseteq L$, we have that $L/I = J(E/I)$ as we have seen in Theorem 2.1(4)(a). We are not able to describe $J(E/I)$ when K and K are not comparable.

2.3. Remark. Let R be a local right self-injective ring. Let M_R be a cyclic and finitely presented module of projective dimension ≤ 1. Since R_R is injective, we have that $\mathrm{Ext}^1_R(M_R, R_R) = 0$. Thus, Theorem 2.1 can be applied.

Let A and B be two modules. We say that:
- A and B *have the same monogeny class*, and write $[A]_m = [B]_m$, if there exist a monomorphism $A \to B$ and a monomorphism $B \to A$ [5];

- A and B *have the same epigeny class*, and write $[A]_e = [B]_e$, if there exist an epimorphism $A \to B$ and an epimorphism $B \to A$;

It is clear that a module A has the same monogeny (epigeny) class as the zero module if and only if $A = 0$.

- Two cyclically presented modules R/aR and R/bR over a local ring R are said to *have the same lower part*, denoted $[R/aR]_l = [R/bR]_l$, if there exist $r, s \in R$ such that $raR = bR$ and $sbR = aR$ [1].

- If M_R and M'_R are two couniformly presented modules with couniform presentations

$$0 \to C_R \to P_R \to M_R \to 0$$

and

$$0 \to C'_R \to P'_R \to M'_R \to 0,$$

we say that M_R and M'_R *have the same lower part,* and we write $[M_R]_\ell = [M'_R]_\ell$, if there are two homomorphisms $f_0 \colon P_R \to P'_R$ and $f'_0 \colon P'_R \to P_R$ such that $f_0(C_R) = C'_R$ and $f'_0(C'_R) = C_R$ [7].

2.4. Theorem. *Let R be a semiperfect ring and let R_R/L be a cyclic uniform right R-module with $L \neq 0$. Let E be the idealizer of the right ideal L of R, that is, the set of all $r \in R$ with $rL \subseteq L$, so that*

$$\mathrm{E}nd(R_R/L) \cong E/L.$$

Similarly, let E' be the idealizer of the right ideal $L + J(R)$ of R, so that

$$\mathrm{E}nd(R_R/(L + J(R))) \cong E'/(L + J(R)).$$

Set $I := \{\, e \in E \mid$ left multiplication by $e + I$ is a non-injective endomorphism of $R_R/L \,\}$ and $K := E \cap (L + J(R))$. Then:

(1) *I and K are two two-sided ideals of E containing L, and I is completely prime in E.*
(2) *For every $e \in E$, the element $e + L$ of E/L is invertible in E/L if and only if $e + L + J(R)$ is invertible in $E'/L + J(R)$ and $e \notin I$.*
(3) *Moreover:*
   (a) *If $I \subseteq K$, then every epimorphism $R_R/L \to R_R/L$ is an automorphism of R_R/L,*
   (b) *$K \nsubseteq I$ if and only if $[R_R/L]_m = [L + J(R)/L]_m$.*

Proof. (1) We know that $\mathrm{E}nd(R_R/L) \cong E/L$. Every endomorphism $e + L$ of R_R/L extends to an endomorphism e_1 of the injective envelope $E(R_R/L)$. Define a ring morphism

$$\varphi \colon E \to \mathrm{E}nd(E(R_R/L))/J(\mathrm{E}nd(E(R_R/L)))$$

by $\varphi(e) = e_1 + J(\mathrm{E}nd(E(R_R/L)))$ for every $e \in E$. Since R_R/L is uniform, the injective envelope $E(R_R/L)$ is indecomposable, the endomorphism ring $\mathrm{E}nd(E(R_R/L))$ is a local ring, and the Jacobson radical $J(\mathrm{E}nd(E(R_R/L)))$ consists of all non-injective endomorphisms of $E(R_R/L)$. It follows that I, which is equal to the kernel of the ring morphism φ, whose range is the division ring

$$\mathrm{E}nd(E(R_R/L))/J(\mathrm{E}nd(E(R_R/L))),$$

must be a completely prime two-sided ideal of E. The remaining part of statement (1) is easily checked.
(2) We have already seen that there is a ring morphism

$$\varphi \colon E \to \mathrm{E}nd(E(R_R/L))/J(\mathrm{E}nd(E(R_R/L)))$$

whose kernel is I. Hence if $e \in E$ and $e + L$ is invertible in E/L, then $\varphi(e)$ must be invertible in the division ring $\mathrm{E}nd(E(R_R/L))/J(\mathrm{E}nd(E(R_R/L)))$. Thus $\varphi(e) \neq 0$, that is, $e \notin \ker\varphi = I$. Similarly, we can consider the ring morphism

$$\psi\colon E \to \mathrm{E}nd(R_R/L + J(R))$$

defined by $\psi(e)(r + L + J(R)) = er + L + J(R)$ for every $e \in E$ and every $r \in R$. Its kernel is K, which contains L. Hence $e + L$ invertible in E/L implies $\psi(e)$ invertible in $\mathrm{E}nd(R_R/L + J(R))$. But

$$\mathrm{E}nd(R_R/(L + J(R))) \cong E'/(L + J(R)),$$

so that $e + L + J(R)$ must be invertible in $E'/L + J(R)$.

Conversely, assume $e \in E$, $e+L+J(R)$ invertible in $E'/L+J(R)$ and $e \notin I$. We want to show that $e + L$ is invertible in E/L. Since $E/L \cong \mathrm{E}nd(R_R/L)$, this is equivalent to showing that left multiplication $\mu_e\colon R_R/L \to R_R/L$ by e is an automorphism of R_R/L. Now $e \notin I$ is equivalent to μ_e is injective by definition of I. In order to show that μ_e is onto as well, it suffices to prove that μ_e induces an onto endomorphism

$$(R_R/L)/(R_R/L)J(R) \to (R_R/L)/(R_R/L)J(R)$$

by Nakayama's Lemma. But $(R_R/L)J(R) = L + J(R)/L$, so that

$$(R_R/L)/(R_R/L)J(R) \cong R_R/L + J(R).$$

Hence $e + L + J(R)$ invertible in $E'/L + J(R) \cong \mathrm{E}nd(R_R/(L + J(R)))$ means that the endomorphism $\psi(e)$ of $R_R/L + J(R)$ induced by μ_e is onto, as desired.
(3) (a) Assume $I \subseteq K$. Let $e+L\colon R_R/L \to R_R/L$ be an epimorphism with $e \in E$. Then the induced morphism $\psi(e)\colon R_R/L + J(R) \to R_R/L + J(R)$ is also an epimorphism, so that it is an automorphism because $R_R/L+J(R)$ is a semisimple module of finite Goldie dimension. In the isomorphism

$$\mathrm{E}nd(R_R/(L + J(R))) \cong E'/(L + J(R)),$$

we obtain that $e+L+J(R)$ is invertible in the ring $E'/(L+J(R))$. Thus $e \notin K$. Hence $e \notin I$. It follows from (2) that $e+L$ is invertible, that is, it is an automorphism of R_R/L.
(b) Assume $K \not\subseteq I$. Then there is an element $f \in K$, $f \notin I$. Thus $f \in E$ induces an endomorphism f of R_R/L. Now $f \notin I$ means that f is injective, and $f \in K$ means that the image of f is contained in $L+J(R)/L$. Hence $[R_R/L]_m = [L+J(R)/L]_m$. Conversely, if $[R_R/L]_m = [L + J(R)/L]_m$, then there is a monomorphism $f\colon R_R/L \to L + J(R)/L$. If we compose it with the inclusion $L + J(R)/L \to R_R/L$ we get an endomorphism of R_R/L which is in K but not in I. Hence $K \not\subseteq I$. □

We finish this study with the following result.

2.5. Theorem. *Let R be a semiperfect ring, let $R/L, R/L'$ be two cyclic uniform modules with $L \neq 0$ and $L' \neq 0$ proper right ideals of R. Assume that either*

(1) *every monomorphism $R_R/L \to R_R/L$ is an automorphism of R_R/L, or*
(2) *every epimorphism $R_R/L \to R_R/L$ is an automorphism of R_R/L, or*
(3) $[R_R/L]_m = [L + J(R)/L]_m$.

Then the followings are equivalent.

(*a*) $R_R/L \cong R_R/L'$
(*b*) $[R_R/L]_m = [R_R/L']_m$ *and* $[R_R/L]_e = [R_R/L']_e$.

Proof. Assume $[R_R/L]_m = [R_R/L']_m$ and $[R_R/L]_e = [R_R/L']_e$. Then there are monomorphisms $\alpha\colon R_R/L \to R_R/L'$ and $\beta\colon R_R/L' \to R_R/L$ and epimorphisms $\alpha\colon R_R/L \to R_R/L'$ and $\beta\colon R_R/L' \to R_R/L$. Then $\beta\alpha$ is a monomorphism $R_R/L \to R_R/L$ and $\beta'\alpha'$ is an epimorphism $R_R/L \to R_R/L$. If hypothesis (a) holds, then $\beta\alpha$ is an automorphism of R_R/L that factors through R_R/L', so that R_R/L is isomorphic to a direct summand of R_R/L'. But $R_R/L \neq 0$ and R_R/L' is uniform, so that $R_R/L \cong R_R/L'$. This proves our theorem under hypothesis (a). Dually one proves that the theorem holds when hypothesis (b) holds.

Assume now that hypothesis (c) holds, i.e., $[R_R/L]_m = [L + J(R)/L]_m$. Equivalently, there exists a monomorphism $\gamma\colon R_R/L \to R_R/L$ whose image is contained in $L+J(R)/L$. Now if either α or α' are isomorphisms, then the existence of α or α' shows that $R_R/L \cong R_R/L'$. This allows us to conclude. Thus we can assume that α is not an epimorphism and α' is not a monomorphism. Then $\alpha' + \alpha\gamma\colon R_R/L \to R_R/L'$ is an isomorphism, because:

(1) It is injective, because it is the sum of the injective morphism $\alpha\gamma\colon R_R/L \to R_R/L'$ and the non-injective morphism $\alpha'\colon R_R/L \to R_R/L'$, and R_R/L is uniform.

(2) The ideal $J(R)$ is superfluous in R_R by Nakayama's Lemma. Considering the canonical projection $R_R \to R_R/L$, it follows that $L + J(R)/L$ is superfluous in R_R/L. Applying the morphism $\alpha\colon R/L \to R/L'$, we get that the image of $\alpha\gamma$ is contained in $\alpha(L + J(R)/L)$, hence is a superfluous submodule of R/L'. Thus the sum of $\alpha\gamma$ and the surjective morphism $\alpha'\colon R/L \to R/L'$ is a surjective morphism $\alpha'+\alpha\gamma\colon R_R/L \to R_R/L'$.

Thus $\alpha + \alpha'\gamma$ is an isomorphism of R_R/L onto R_R/L'. □

2.6. Remark. By Theorem 2.4, the only case in which we cannot apply Theorem 2.5 is when K is properly contained in I. Namely, if $K \not\subseteq I$, then $[R_R/L]_m = [L + J(R)/L]_m$ and we can apply Theorem 2.5(a); if $K \subseteq I$, then either K is properly contained in I, which is the case still unknown, or $K = I$, but in the latter case every epimorphism $R_R/L \to R_R/L$ is an automorphism of R_R/L by Theorem 2.4(1).

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