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CERTAIN SUBCLASSES OF BI-UNIVALENT FUNCTIONS RELATED TO k-FIBONACCI NUMBERS

H. ÖZLEM GÜNEY, G. MURUGUSUNDARAMOORTHY, AND J. SOKÓŁ

Abstract. In this paper, we introduce and investigate new subclasses of bi-univalent functions related to k-Fibonacci numbers. Furthermore, we find estimates of first two coefficients of functions in these classes. Also, we obtain the Fekete-Szegö inequalities for these function classes.

1. Introduction

Let $\mathbb{D} = \{z : |z| < 1\}$ be the unit disc in the complex plane. The class of all analytic functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

in the open unit disc $\mathbb{D}$ with normalization $f(0) = 0,\ f'(0) = 1$ is denoted by $\mathcal{A}$ and the class $\mathcal{S} \subset \mathcal{A}$ is the class which consists of univalent functions in $\mathbb{D}$. We say that f is subordinate to F in $\mathbb{D}$, written as $f \prec F$, if and only if $f(z) = F(\omega(z))$ for some analytic function ω, $|\omega(z)| \leq |z|$, $z \in \mathbb{D}$.

The Koebe one quarter theorem [5] ensures that the image of $\mathbb{D}$ under every univalent function $f \in \mathcal{A}$ contains a disk of radius $1/4$. Thus every univalent function f has an inverse f^{-1} satisfying

$$f^{-1}(f(z)) = z,\ (z \in \mathbb{D}) \ \text{ and } \ f(f^{-1}(w)) = w,\ (|w| < r_0(f),\ r_0(f) \geq \frac{1}{4}).$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{D}$ if f is univalent in $\mathbb{D}$ and f^{-1} has an univalent extension to $\mathbb{D}$. Let Σ denote the class of bi-univalent functions defined in the unit disk $\mathbb{D}$. Someone can see a short history and examples of functions in the class Σ in [14]. Since $f \in \Sigma$ has the Maclaurin series given by (1), a computation shows that its inverse $g = f^{-1}$ has the expansion

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 + \cdots . \tag{2}$$

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The work of Srivastava et al. [14] essentially revived the investigation of various subclasses of the bi-univalent function class in recent years. In a considerably large number of sequels to the aforementioned work of Srivastava et al. [14], several different subclasses of the bi-univalent function class Σ were introduced and studied analogously by many authors (see, for example, [1, 2, 4, 8, 3, 15, 9]), but only non-sharp estimates on the initial coefficients $|a_2|$ and $|a_3|$ in the Taylor-Maclaurin expansion (1) were obtained in these recent papers.

The object of the present work is to introduce a new subclass of the function class Σ and find estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in this new subclass of the function class Σ using the technique of Srivastava et al. [14]

Recently,Yilmaz Özgür and Sokół [10] introduced the class $\mathcal{SL}^k$ of starlike functions connected with $k-$ Fibonacci numbers as the set of functions $f \in \mathcal{A}$ which is described in the following definition.

Definition 1. *Let k be any positive real number. The function $f \in \mathcal{A}$ belongs to the class $\mathcal{SL}^k$ if it satisfies the condition that*

$$\frac{zf'(z)}{f(z)} \prec \widetilde{p}_k(z), \quad z \in \mathbb{D},$$

where

$$\widetilde{p}_k(z) = \frac{1+\tau_k^2 z^2}{1-k\tau_k z - \tau_k^2 z^2}, \; \tau_k = \frac{k-\sqrt{k^2+4}}{2}, \quad z \in \mathbb{D}. \tag{3}$$

Later in [7], Güney et al. defined the class $\mathcal{KSL}^k$ as follows:

Definition 2. *Let k be any positive real number. The function $f \in \mathcal{A}$ belongs to the class $\mathcal{KSL}^k$ if it satisfies the condition that*

$$1 + \frac{zf''(z)}{f'(z)} \prec \widetilde{p}_k(z), \quad z \in \mathbb{D},$$

where the function $\widetilde{p}_k$ is defined in (3).

For $k = 1$, the classes $\mathcal{SL}$ and $\mathcal{KSL}$ of shell-like functions were defined in [12] (see also [13]).

It was proved in [10] that functions in the class $\mathcal{SL}^k$ are univalent in $\mathbb{D}$. Moreover, the class $\mathcal{SL}^k$ is a subclass of the class of starlike functions $\mathcal{S}^*$, even more, starlike of order $k(k^2+4)^{-1/2}/2$. The name attributed to the class $\mathcal{SL}^k$ is motivated by the shape of the curve

$$\mathcal{C} = \left\{ \widetilde{p}_k(e^{it}) : t \in [0, 2\pi) \setminus \{\pi\} \right\}.$$

Now we define the classes $\mathcal{SLM}_\alpha^k$ and $\mathcal{SLG}_\gamma^k$, as follows:

Definition 3. *Let k be any positive real number. The function $f \in \mathcal{A}$ belongs to the class $\mathcal{SLM}_\alpha^k$, $(0 \le \alpha \le 1)$ if it satisfies the condition that*

$$\alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) + (1-\alpha)\frac{zf'(z)}{f(z)} \prec \widetilde{p}_k(z), \quad z \in \mathbb{D},$$

where the function $\widetilde{p}_k$ *is defined in* (3).

Definition 4. *Let* $0 \leq \gamma \leq 1$, *and* k *be any positive real number. The function* $f \in A$ *belongs to the class* SLG_γ^k *if the following conditions are satisfied:*

$$\left(\frac{zf'(z)}{f(z)}\right)^\gamma \left(1 + \frac{zf''(z)}{f'(z)}\right)^{1-\gamma} \prec \tilde{p}_k(z), \quad z \in \mathbb{D},$$

where the function $\widetilde{p}_k$ *is defined in* (3).

For $k \leq 2$, note that we have

$$\widetilde{p}_k\left(e^{\pm i \arccos\left(k^2/4\right)}\right) = k(k^2+4)^{-1/2},$$

and so the curve $\mathcal{C}$ intersects itself on the real axis at the point $w_1 = k(k^2+4)^{-1/2}$. Thus $\mathcal{C}$ has a loop intersecting the real axis also at the point $w_2 = (k^2+4)/(2k)$. For $k > 2$, the curve $\mathcal{C}$ has no loops and it is like a conchoid, see for details [10]. Moreover, the coefficients of $\widetilde{p}_k$ are connected with k-Fibonacci numbers.

For any positive real number k, the k-Fibonacci number sequence $\{F_{k,n}\}_{n=0}^\infty$ is defined recursively by

$$F_{k,0} = 0,\ F_{k,1} = 1 \text{ and } F_{k,n+1} = kF_{k,n} + F_{k,n-1} \text{ for } n \geq 1.$$

When $k = 1$, we obtain the well-known Fibonacci numbers F_n. It is known that the n^{th} k-Fibonacci number is given by

$$F_{k,n} = \frac{(k-\tau_k)^n - \tau_k^n}{\sqrt{k^2+4}},$$

where $\tau_k = (k - \sqrt{k^2+4})/2$. If $\widetilde{p}_k(z) = 1 + \sum_{n=1}^\infty \widetilde{p}_{k,n} z^n$, then we have

$$\widetilde{p}_{k,n} = (F_{k,n-1} + F_{k,n+1})\tau_k^n,\ n = 1, 2, 3, \ldots.$$

Also, Özgür and Sokół showed in [10] that

$$\begin{aligned}\widetilde{p}_k(z) = \frac{1+\tau_k^2 z^2}{1 - k\tau_k z - \tau_k^2 z^2} &= 1 + \sum_{n=1}^\infty \widetilde{p}_{k,n} z^n \\ &= 1 + (F_{k,0} + F_{k,2})\tau_k z + (F_{k,1} + F_{k,3})\tau_k^2 z^2 + \cdots \\ &= 1 + k\tau_k z + (k^2+2)\tau_k^2 z^2 + (k^3+3k)\tau_k^3 z^3 + \cdots.\end{aligned}$$

where $\tau_k = \frac{k-\sqrt{k^2+4}}{2}$, $\quad z \in \mathbb{D}$, (see [10]).

Let $\mathcal{P}(\beta)$, $0 \leq \beta < 1$, denote the class of analytic functions p in $\mathbb{D}$ with $p(0) = 1$ and $Re\{p(z)\} > \beta$. Especially, we use $\mathcal{P}(0) = \mathcal{P}$ as $\beta = 0$.

Now we give the following lemma which will use in proving.

Lemma 5. *(*[11]*) Let* $p \in \mathcal{P}$ *with* $p(z) = 1 + c_1 z + c_2 z^2 + \cdots$, *then*

$$|c_n| \leq 2 \qquad for \qquad n \geq 1. \tag{4}$$

2. Bi-Univalent function class $\mathcal{SLM}^k_{\alpha,\Sigma}(\widetilde{p}_k(z))$

In this section, we introduce three new subclasses of Σ associated with shell-like functions connected with Fibonacci numbers and obtain the initial Taylor coefficients $|a_2|$ and $|a_3|$ for the function classes by subordination.

Firstly, let $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$, and $p \prec \tilde{p}_k$. Then there exists an analytic function u such that $|u(z)| < 1$ in $\mathbb{U}$ and $p(z) = \tilde{p}_k(u(z))$. Therefore, the function

$$h(z) = \frac{1 + u(z)}{1 - u(z)} = 1 + c_1 z + c_2 z^2 + \dots \tag{5}$$

is in the class $\mathcal{P}(0)$. It follows that

$$u(z) = \frac{c_1 z}{2} + \left(c_2 - \frac{c_1^2}{2}\right)\frac{z^2}{2} + \left(c_3 - c_1 c_2 + \frac{c_1^3}{4}\right)\frac{z^3}{2} + \cdots \tag{6}$$

and

$$\begin{aligned}
\tilde{p}_k(u(z)) = 1 &+ \tilde{p}_{k,1}\left\{\frac{c_1 z}{2} + \left(c_2 - \frac{c_1^2}{2}\right)\frac{z^2}{2} + \left(c_3 - c_1 c_2 + \frac{c_1^3}{4}\right)\frac{z^3}{2} + \cdots\right\} \\
&+ \tilde{p}_{k,2}\left\{\frac{c_1 z}{2} + \left(c_2 - \frac{c_1^2}{2}\right)\frac{z^2}{2} + \left(c_3 - c_1 c_2 + \frac{c_1^3}{4}\right)\frac{z^3}{2} + \cdots\right\}^2 \\
&+ \tilde{p}_{k,3}\left\{\frac{c_1 z}{2} + \left(c_2 - \frac{c_1^2}{2}\right)\frac{z^2}{2} + \left(c_3 - c_1 c_2 + \frac{c_1^3}{4}\right)\frac{z^3}{2} + \cdots\right\}^3 + \cdots \\
&= 1 + \frac{\tilde{p}_{k,1} c_1 z}{2} + \left\{\frac{1}{2}\left(c_2 - \frac{c_1^2}{2}\right)\tilde{p}_{k,1} + \frac{c_1^2}{4}\tilde{p}_{k,2}\right\} z^2 \\
&+ \left\{\frac{1}{2}\left(c_3 - c_1 c_2 + \frac{c_1^3}{4}\right)\tilde{p}_{k,1} + \frac{1}{2}c_1\left(c_2 - \frac{c_1^2}{2}\right)\tilde{p}_{k,2} + \frac{c_1^3}{8}\tilde{p}_{k,3}\right\} z^3 + \cdots .
\end{aligned} \tag{7}$$

And similarly, there exists an analytic function v such that $|v(w)| < 1$ in $\mathbb{D}$ and $p(w) = \tilde{p}_k(v(w))$. Therefore, the function

$$k(w) = \frac{1 + v(w)}{1 - v(w)} = 1 + d_1 w + d_2 w^2 + \dots \tag{8}$$

is in the class $\mathcal{P}(0)$. It follows that

$$v(w) = \frac{d_1 w}{2} + \left(d_2 - \frac{d_1^2}{2}\right)\frac{w^2}{2} + \left(d_3 - d_1 d_2 + \frac{d_1^3}{4}\right)\frac{w^3}{2} + \cdots \tag{9}$$

and

$$\begin{aligned}
\tilde{p}_k(v(w)) = 1 &+ \frac{\tilde{p}_{k,1} d_1 w}{2} + \left\{\frac{1}{2}\left(d_2 - \frac{d_1^2}{2}\right)\tilde{p}_{k,1} + \frac{d_1^2}{4}\tilde{p}_{k,2}\right\} w^2 \\
&+ \left\{\frac{1}{2}\left(d_3 - d_1 d_2 + \frac{d_1^3}{4}\right)\tilde{p}_{k,1} + \frac{1}{2}d_1\left(d_2 - \frac{d_1^2}{2}\right)\tilde{p}_{k,2} + \frac{d_1^3}{8}\tilde{p}_{k,3}\right\} w^3 + \cdots .
\end{aligned} \tag{10}$$

Definition 6. *For $0 \leq \alpha \leq 1$, a function $f \in \Sigma$ of the form (1) is said to be in the class $\mathcal{SLM}^k_{\alpha,\Sigma}(\widetilde{p}_k(z))$ if the following subordination hold:*

$$\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)+(1-\alpha)\left(\frac{zf'(z)}{f(z)}\right)\prec \widetilde{p}_k(z)=\frac{1+\tau_k^2z^2}{1-k\tau_kz-\tau_k^2z^2}, \tag{11}$$

and

$$\alpha\left(1+\frac{wg''(w)}{g'(w)}\right)+(1-\alpha)\left(\frac{wg'(w)}{g(w)}\right)\prec \widetilde{p}_k(w)=\frac{1+\tau_k^2w^2}{1-k\tau_kw-\tau_k^2w^2}, \tag{12}$$

where $\tau_k=\frac{k-\sqrt{k^2+4}}{2}$ where $z,w\in\mathbb{D}$ and g is given by (2).

Specializing the parameter $\alpha=0$ and $\alpha=1$ we have the following:

Definition 7. *A function $f \in \Sigma$ of the form (1) is said to be in the class $\mathcal{SL}^k_{\Sigma}(\tilde{p}_k(z))$ if the following subordination hold:*

$$\frac{zf'(z)}{f(z)}\prec \widetilde{p}_k(z)=\frac{1+\tau_k^2z^2}{1-k\tau_kz-\tau_k^2z^2}, \tag{13}$$

and

$$\frac{wg'(w)}{g(w)}\prec \widetilde{p}_k(w)=\frac{1+\tau_k^2w^2}{1-k\tau_kw-\tau_k^2w^2}, \tag{14}$$

where $\tau_k=\frac{k-\sqrt{k^2+4}}{2}$, $z,w\in\mathbb{D}$ and g is given by (2).

Definition 8. *A function $f \in \Sigma$ of the form (1) is said to be in the class $\mathcal{KSL}^k_{\Sigma}(\tilde{p}_k(z))$ if the following subordination hold:*

$$1+\frac{zf''(z)}{f'(z)}\prec \widetilde{p}_k(z)=\frac{1+\tau_k^2z^2}{1-k\tau_kz-\tau_k^2z^2}, \tag{15}$$

and

$$1+\frac{wg''(w)}{g'(w)}\prec \widetilde{p}_k(w)=\frac{1+\tau_k^2w^2}{1-k\tau_kw-\tau_k^2w^2}, \tag{16}$$

where $\tau_k=\frac{k-\sqrt{k^2+4}}{2}$, $z,w\in\mathbb{D}$ and g is given by (2).

In the following theorem we determine the initial Taylor coefficients $|a_2|$ and $|a_3|$ for the function class $\mathcal{SLM}^k_{\alpha,\Sigma}(\widetilde{p}_k(z))$. Later we state the bounds to other classes as a special cases.

Theorem 9. *Let f given by (1) be in the class $\mathcal{SLM}^k_{\alpha,\Sigma}(\widetilde{p}_k(z))$. Then*

$$|a_2|\leq \frac{k\sqrt{k}|\tau_k|}{\sqrt{(1+\alpha)^2k-(1+\alpha)(2(1+\alpha)+\alpha k^2)\tau_k}} \tag{17}$$

and

$$|a_3|\leq \frac{k|\tau_k|\left\{(1+\alpha)^2k-\left[(k^2+2)\alpha^2+(5k^2+4)\alpha+2(k^2+1)\right]\tau_k\right\}}{2(1+2\alpha)(1+\alpha)\left[(1+\alpha)k-(2(1+\alpha)+\alpha k^2)\tau_k\right]}. \tag{18}$$

Proof. Let $f \in \mathcal{SLM}^k_{\alpha,\Sigma}(\widetilde{p}_k(z))$ and $g = f^{-1}$. Considering (11) and (12), we have

$$\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)+(1-\alpha)\left(\frac{zf'(z)}{f(z)}\right)=\tilde{p_k}(u(z)) \tag{19}$$

and

$$\alpha\left(1+\frac{wg''(w)}{g'(w)}\right)+(1-\alpha)\left(\frac{wg'(w)}{g(w)}\right)=\tilde{p_k}(v(w)), \tag{20}$$

where $\tau_k = \frac{k-\sqrt{k^2+4}}{2}$, $z, w \in \mathbb{D}$ and g is given by (2). We have also

$$\begin{aligned} &\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)+(1-\alpha)\left(\frac{zf'(z)}{f(z)}\right)\\ =\;& 1+(1+\alpha)a_2z+(2(1+2\alpha)a_3-(1+3\alpha)a_2^2)z^2+\dots\\ =\;& 1+\frac{\tilde{p}_{k,1}c_1z}{2}+\left[\frac{1}{2}\left(c_2-\frac{c_1^2}{2}\right)\tilde{p}_{k,1}+\frac{c_1^2}{4}\tilde{p}_{k,2}\right]z^2\\ &+\left[\frac{1}{2}\left(c_3-c_1c_2+\frac{c_1^3}{4}\right)\tilde{p}_{k,1}+\frac{1}{2}c_1\left(c_2-\frac{c_1^2}{2}\right)\tilde{p}_{k,2}+\frac{c_1^3}{8}\tilde{p}_{k,3}\right]z^3+\cdots \end{aligned} \tag{21}$$

and

$$\begin{aligned} &\alpha\left(1+\frac{wg''(w)}{g'(w)}\right)+(1-\alpha)\left(\frac{wg'(w)}{g(w)}\right)\\ =\;& 1-(1+\alpha)a_2w+((3+5\alpha)a_2^2-2(1+2\alpha)a_3)w^2+\dots\\ =\;& 1+\frac{\tilde{p}_{k,1}d_1w}{2}+\left[\frac{1}{2}\left(d_2-\frac{d_1^2}{2}\right)\tilde{p}_{k,1}+\frac{d_1^2}{4}\tilde{p}_{k,2}\right]w^2\\ +\;&\left[\frac{1}{2}\left(d_3-d_1d_2+\frac{d_1^3}{4}\right)\tilde{p}_{k,1}+\frac{1}{2}d_1\left(d_2-\frac{d_1^2}{2}\right)\tilde{p}_{k,2}+\frac{d_1^3}{8}\tilde{p}_{k,3}\right]w^3+\cdots. \end{aligned} \tag{22}$$

It follows from (21) and (22) that

$$(1+\alpha)a_2=\frac{c_1k\tau_k}{2}, \tag{23}$$

$$2(1+2\alpha)a_3-(1+3\alpha)a_2^2=\frac{1}{2}\left(c_2-\frac{c_1^2}{2}\right)k\tau_k+\frac{c_1^2}{4}(k^2+2)\tau_k^2, \tag{24}$$

and

$$-(1+\alpha)a_2=\frac{d_1k\tau_k}{2}, \tag{25}$$

$$(3+5\alpha)a_2^2-2(1+2\alpha)a_3=\frac{1}{2}\left(d_2-\frac{d_1^2}{2}\right)k\tau_k+\frac{d_1^2}{4}(k^2+2)\tau_k^2. \tag{26}$$

From (23) and (25), we have

$$c_1=-d_1, \tag{27}$$

and

$$2a_2^2=\frac{(c_1^2+d_1^2)}{4(1+\alpha)^2}k^2\tau_k^2. \tag{28}$$

Now, by summing (24) and (26), we obtain

$$2(1+\alpha)a_2^2 = \frac{1}{2}(c_2+d_2)k\tau_k - \frac{1}{4}(c_1^2+d_1^2)k\tau_k + \frac{1}{4}(c_1^2+d_1^2)(k^2+2)\tau_k^2. \tag{29}$$

By putting (28) in (29), we have

$$2(1+\alpha)\left[(-2(1+\alpha)-\alpha k^2)\tau_k + (1+\alpha)k\right]a_2^2 = \frac{1}{2}(c_2+d_2)k^3\tau_k^2. \tag{30}$$

Therefore, using Lemma 5 we obtain

$$|a_2| \le \frac{k\sqrt{k}|\tau_k|}{\sqrt{(1+\alpha)^2 k - (1+\alpha)(2(1+\alpha)+\alpha k^2)\tau_k}}. \tag{31}$$

Now, so as to find the bound on $|a_3|$, let's subtract from (24) and (26). So, we find

$$4(1+2\alpha)a_3 - 4(1+2\alpha)a_2^2 = \frac{1}{2}\left(c_2-d_2\right)k\tau_k. \tag{32}$$

Hence, we get

$$4(1+2\alpha)|a_3| \le 2k|\tau_k| + 4(1+2\alpha)|a_2|^2.$$

Then, in view of (31), we obtain

$$|a_3| \le \frac{k|\tau_k|\left\{(1+\alpha)^2 k - \left[2(1+\alpha)^2 + (\alpha^2+5\alpha+2)k^2\right]\tau_k\right\}}{2(1+2\alpha)(1+\alpha)\left[(1+\alpha)k - (2(1+\alpha)+\alpha k^2)\tau_k\right]}.$$

□

If we can take the parameter $\alpha = 0$ and $\alpha = 1$ in the above theorem, we have the following the initial Taylor coefficients $|a_2|$ and $|a_3|$ for the function classes $\mathcal{SL}_\Sigma^k(\tilde{p}_k(z))$ and $\mathcal{KSL}_\Sigma^k(\tilde{p}_k(z))$, respectively.

Corollary 10. *Let f given by (1) be in the class $\mathcal{SL}_\Sigma^k(\tilde{p}_k(z))$. Then*

$$|a_2| \le \frac{k\sqrt{k}|\tau_k|}{\sqrt{k-2\tau_k}}$$

and

$$|a_3| \le \frac{k|\tau_k|\left\{k - 2(k^2+1)\tau_k\right\}}{2(k-2\tau_k)}.$$

Corollary 11. *Let f given by (1) be in the class $\mathcal{KSL}_\Sigma^k(\tilde{p}_k(z))$. Then*

$$|a_2| \le \frac{k\sqrt{k}|\tau_k|}{\sqrt{4k-2(4+k^2)\tau_k}}$$

and

$$|a_3| \le \frac{k|\tau_k|\left\{k - 2(k^2+1)\tau_k\right\}}{3(2k-(4+k^2)\tau_k)}.$$

If we can take the parameter $k = 1$ in the above corollaries, we have the following the initial Taylor coefficients $|a_2|$ and $|a_3|$ for the function classes $\mathcal{SL}_\Sigma(\tilde{p}(z))$ and $\mathcal{KSL}_\Sigma(\tilde{p}(z))$, respectively, which were obtained in [6] by Güney et.al.

Corollary 12. *Let f given by (1) be in the class $\mathcal{SL}_\Sigma(\tilde{p}(z))$. Then*

$$|a_2| \leq \frac{|\tau|}{\sqrt{1-2\tau}}$$

and

$$|a_3| \leq \frac{|\tau|(1-4\tau)}{2(1-2\tau)}.$$

Corollary 13. *Let f given by (1) be in the class $\mathcal{KSL}_\Sigma(\tilde{p}(z))$. Then*

$$|a_2| \leq \frac{|\tau|}{\sqrt{4-10\tau}}$$

and

$$|a_3| \leq \frac{|\tau|(1-4\tau)}{3(2-5\tau)}.$$

3. Bi-Univalent function class $\mathcal{SLG}^k_{\gamma,\Sigma}(\widetilde{p}_k(z))$

In this section, we define a new class $\mathcal{SLG}^k_{\gamma,\Sigma}(\widetilde{p}_k(z))$ of $\gamma-$ bi-starlike functions associated with shell-like domain.

Definition 14. *Let $0 \leq \gamma \leq 1$, and k be any positive real number. A function $f \in \Sigma$ of the form (1) is said to be in the class $SLG^k_{\gamma,\Sigma}(\widetilde{p}_k(z))$ if the following subordination hold:*

$$\left(\frac{zf'(z)}{f(z)}\right)^{\gamma}\left(1+\frac{zf''(z)}{f'(z)}\right)^{1-\gamma} \prec \tilde{p}_k(z) \tag{1}$$

and

$$\left(\frac{wg'(w)}{g(w)}\right)^{\gamma}\left(1+\frac{wg''(w)}{g'(w)}\right)^{1-\gamma} \prec \tilde{p}_k(w), \tag{2}$$

where the function $\widetilde{p}_k$ is defined in (3) and $z, w \in D$.

Remark 15. *Taking $\gamma = 1$, we get $\mathcal{SLG}^k_{1,\Sigma}(\widetilde{p}_k(z)) \equiv \mathcal{SL}^k_\Sigma(\tilde{p}_k(z))$ the class as given in Definition 7 satisfying the conditions given in (13) and (14).*

Remark 16. *Taking $\gamma = 0$, we get $\mathcal{SLG}^k_{0,\Sigma}(\widetilde{p}_k(z)) \equiv \mathcal{KSL}^k_\Sigma(\tilde{p}_k(z))$ the class as given in Definition 8 satisfying the conditions given in (15) and (16).*

Theorem 17. *Let f given by (1) be in the class $SLG^k_{\gamma,\Sigma}(\widetilde{p}_k(z))$. Then*

$$|a_2| \leq \frac{k\sqrt{2k}|\tau_k|}{\sqrt{2(2-\gamma)^2k-(4(2-\gamma)^2+(\gamma^2-5\gamma+4)k^2)\tau_k}}$$

and

$$|a_3| \leq \frac{k|\tau_k|\left[2(2-\gamma)^2k-(4(2-\gamma)^2+(\gamma^2-13\gamma+16)k^2)\tau_k\right]}{2(3-2\gamma)\left[2k(2-\gamma)^2-(4(2-\gamma)^2+(\gamma^2-5\gamma+4)k^2)\tau_k\right]}.$$

Proof. Let $f \in \mathcal{SLG}^k_{\gamma,\Sigma}(\widetilde{p}_k(z))$ and $g = f^{-1}$ given by (2). Considering (1) and (2), we have

$$\left(\frac{zf'(z)}{f(z)}\right)^{\gamma}\left(1+\frac{zf''(z)}{f'(z)}\right)^{1-\gamma} = \tilde{p}_k(u(z)) \tag{3}$$

and

$$\left(\frac{wg'(w)}{g(w)}\right)^{\gamma}\left(1+\frac{wg''(w)}{g'(w)}\right)^{1-\gamma} = \tilde{p}_k(v(w)), \tag{4}$$

where the function $\widetilde{p}_k$ is defined in (3), $z, w \in \mathbb{D}$ and g is given by (2). We also have

$$\begin{aligned} &\left(\frac{zf'(z)}{f(z)}\right)^{\gamma}\left(1+\frac{zf''(z)}{f'(z)}\right)^{1-\gamma} \qquad (5)\\ = \quad & 1+(2-\gamma)a_2z+\left(2(3-2\gamma)a_3+\frac{1}{2}[(\gamma-2)^2-3(4-3\gamma)]a_2^2\right)z^2+\dots \end{aligned}$$

and

$$\begin{aligned} &\left(\frac{wg'(w)}{g(w)}\right)^{\gamma}\left(1+\frac{wg''(w)}{g'(w)}\right)^{1-\gamma} \qquad (6)\\ = \quad & 1-(2-\gamma)a_2w+\left([8(1-\gamma)+\frac{1}{2}\gamma(\gamma+5)]a_2^2-2(3-2\gamma)a_3\right)w^2+\dots. \end{aligned}$$

Equating the coefficients in (5) and (6), with (7)-(10), respectively, we get,

$$(2-\gamma)a_2 = \frac{c_1k\tau_k}{2} \tag{7}$$

$$2(3-2\gamma)a_3+\frac{1}{2}[(\gamma-2)^2-3(4-3\gamma)]a_2^2 = \frac{1}{2}\left(c_2-\frac{c_1^2}{2}\right)k\tau_k+\frac{c_1^2}{4}(k^2+2)\tau_k^2, \tag{8}$$

and

$$-(2-\gamma)a_2 = \frac{d_1k\tau_k}{2} \tag{9}$$

$$-2(3-2\gamma)a_3+[8(1-\gamma)+\frac{1}{2}\gamma(\gamma+5)]a_2^2 = \frac{1}{2}\left(d_2-\frac{d_1^2}{2}\right)k\tau_k+\frac{d_1^2}{4}(k^2+2)\tau_k^2 \tag{10}$$

From (7) and (9), we have

$$a_2 = \frac{c_1k\tau_k}{2(2-\gamma)} = -\frac{d_1k\tau_k}{2(2-\gamma)},$$

which implies

$$c_1 = -d_1$$

and

$$a_2^2 = \frac{(c_1^2 + d_1^2)k^2\tau_k^2}{8(2-\gamma)^2}.$$

Now, by summing (8) and (10), we obtain

$$(\gamma^2 - 3\gamma + 4)a_2^2 = \frac{1}{2}(c_2 + d_2)k\tau_k - \frac{1}{4}(c_1^2 + d_1^2)k\tau_k + \frac{1}{4}(c_1^2 + d_1^2)(k^2+2)\tau_k^2.$$

Proceeding similarly as in the earlier proof of Theorem 9 and using Lemma 5, we obtain

$$|a_2| \le \frac{k\sqrt{2k}|\tau_k|}{\sqrt{2(2-\gamma)^2 k - (4(2-\gamma)^2 + (\gamma^2 - 5\gamma + 4)k^2)\tau_k}}. \tag{11}$$

Now, so as to find the bound on $|a_3|$, let's subtract from (8) and (10). So, we find

$$4(3-2\gamma)a_3 - 4(3-2\gamma)a_2^2 = \frac{1}{2}\left(c_2 - d_2\right)k\tau_k.$$

Hence, we get

$$4(3-2\gamma)|a_3| \le 2k|\tau_k| + 4(3-2\gamma)|a_2|^2.$$

Then, in view of (11), we obtain

$$|a_3| \le \frac{k|\tau_k|\left[2(2-\gamma)^2 k - (4(2-\gamma)^2 + (\gamma^2 - 13\gamma + 16)k^2)\tau_k\right]}{2(3-2\gamma)\left[2k(2-\gamma)^2 - (4(2-\gamma)^2 + (\gamma^2 - 5\gamma + 4)k^2)\tau_k\right]}.$$

□

Remark 18. *By taking $\gamma = 1$ and $\gamma = 0$ in the above theorem, we have the initial Taylor coefficients $|a_2|$ and $|a_3|$ for the function classes $\mathcal{SL}_\Sigma^k(\tilde{p}_k(z))$ and $\mathcal{KSL}_\Sigma^k(\tilde{p}_k(z))$, as stated in Corollary 10 and Corollary 11 respectively. Further note that by taking $k = 1$ we have the initial Taylor coefficients $|a_2|$ and $|a_3|$ for the function classes $\mathcal{SL}_\Sigma(\tilde{p}(z))$ and $\mathcal{KSL}_\Sigma(\tilde{p}(z))$, as stated in Corollary 12 and Corollary 13 respectively.*

4. Fekete-Szegö inequalities for the above function classes

Due to Zaprawa [16], we will give Fekete-Szegö inequalities for the above function classes in this section. The first theorem is the solution of the Fekete-Szegö problem in $\mathcal{SLM}_{\alpha,\Sigma}^k(\widetilde{p}_k(z))$ and it looks like the following:

Theorem 19. *Let f given by (1) be in the class $\mathcal{SLM}_{\alpha,\Sigma}^k(\widetilde{p}_k(z))$ and $\mu \in \mathbb{R}$. Then we have*

$$|a_3 - \mu a_2^2| \le \begin{cases} \frac{k|\tau_k|}{2(1+2\alpha)}, & |\mu - 1| \le \frac{4(1+\alpha)\left[(1+\alpha)k - (2(1+\alpha)+\alpha k^2)\tau_k\right]}{8(1+2\alpha)k^2|\tau_k|}, \\ \frac{|1-\mu|k^3\tau_k^2}{(1+\alpha)[(1+\alpha)k - (2(1+\alpha)+\alpha k^2)\tau_k]}, & |\mu - 1| \ge \frac{4(1+\alpha)\left[(1+\alpha)k - (2(1+\alpha)+\alpha k^2)\tau_k\right]}{8(1+2\alpha)k^2|\tau_k|}. \end{cases}$$

Proof. From (30) and (32)we obtain

$$a_3-\mu a_2^2=(1-\mu)\frac{k^3\tau_k^2(c_2+d_2)}{4(1+\alpha)\left[(1+\alpha)k-(2(1+\alpha)+\alpha k^2)\tau_k\right]}+\frac{k\tau_k(c_2-d_2)}{8(1+2\alpha)} \quad (1)$$

$$=\left(\frac{(1-\mu)k^3\tau_k^2}{4(1+\alpha)\left[(1+\alpha)k-(2(1+\alpha)+\alpha k^2)\tau_k\right]}+\frac{k\tau_k}{8(1+2\alpha)}\right)c_2$$

$$+\left(\frac{(1-\mu)k^3\tau_k^2}{4(1+\alpha)\left[(1+\alpha)k-(2(1+\alpha)+\alpha k^2)\tau_k\right]}-\frac{k\tau_k}{8(1+2\alpha)}\right)d_2.$$

So we have

$$a_3-\mu a_2^2=\left(h(\mu)-\frac{k|\tau_k|}{8(1+2\alpha)}\right)c_2+\left(h(\mu)+\frac{k|\tau_k|}{8(1+2\alpha)}\right)d_2, \quad (2)$$

where

$$h(\mu)=\frac{(1-\mu)k^3\tau_k^2}{4(1+\alpha)\left[(1+\alpha)k-(2(1+\alpha)+\alpha k^2)\tau_k\right]}. \quad (3)$$

Then, by taking modulus of (2), we conclude that

$$|a_3-\mu a_2^2|\leq\begin{cases}\frac{k|\tau_k|}{2(1+2\alpha)}, & 0\leq|h(\mu)|\leq\frac{k|\tau_k|}{8(1+2\alpha)},\\ 4|h(\mu)|, & |h(\mu)|\geq\frac{k|\tau_k|}{8(1+2\alpha)}.\end{cases}$$

Taking $\mu=1$, we have the following corollary.

Corollary 20. *If $f\in\mathcal{SLM}^k_{\alpha,\Sigma}(\widetilde{p}_k(z))$, then*

$$|a_3-a_2^2|\leq\frac{k|\tau_k|}{2(1+2\alpha)}. \quad (4)$$

The second theorem is the solution of the Fekete-Szegö problem in $\mathcal{SLG}^k_{\gamma,\Sigma}(\widetilde{p}_k(z))$ and it looks like the following:

Theorem 21. *Let f given by (1) be in the class $\mathcal{SLG}^k_{\gamma,\Sigma}(\widetilde{p}_k(z))$ and $\mu\in\mathbb{R}$. Then we have*

$$|a_3-\mu a_2^2|\leq\begin{cases}\frac{k|\tau_k|}{2(3-2\gamma)}, & |\mu-1|\leq\frac{2(2-\gamma)^2k-(4(2-\gamma)^2+(\gamma^2-5\gamma+4)k^2)\tau_k}{4(3-2\gamma)k^2|\tau_k|},\\ \frac{2|1-\mu|k^3\tau_k^2}{2(2-\gamma)^2k-(4(2-\gamma)^2+(\gamma^2-5\gamma+4)k^2)\tau_k}, & |\mu-1|\geq\frac{2(2-\gamma)^2k-(4(2-\gamma)^2+(\gamma^2-5\gamma+4)k^2)\tau_k}{4(3-2\gamma)k^2|\tau_k|}.\end{cases}$$

Taking $\mu=1$, we have the following corollary.

Corollary 22. *If $f\in\mathcal{SLG}^k_{\gamma,\Sigma}(\widetilde{p}_k(z))$, then*

$$|a_3-a_2^2|\leq\frac{k|\tau_k|}{2(3-2\gamma)}. \quad (5)$$

If we can take the parameter $\alpha = 0$ and $\alpha = 1$ in the Theorem 19 or $\gamma = 1$ and $\gamma = 0$ in the Theorem 21, we have the following the Fekete-Szegö inequalities for the function classes $\mathcal{SL}_{\Sigma}^{k}(\tilde{p}_k(z))$ and $\mathcal{KSL}_{\Sigma}^{k}(\tilde{p}_k(z))$, respectively.

Corollary 23. *Let f given by* (1) *be in the class $\mathcal{SL}_{\Sigma}^{k}(\tilde{p}_k(z))$ and $\mu \in \mathbb{R}$. Then we have*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{k|\tau_k|}{2}, & |\mu - 1| \leq \frac{k-2\tau_k}{2k^2|\tau_k|}, \\ \frac{|1-\mu|k^3\tau_k^2}{k-2\tau_k}, & |\mu - 1| \geq \frac{k-2\tau_k}{2k^2|\tau_k|}. \end{cases}$$

Corollary 24. *Let f given by* (1) *be in the class $\mathcal{KSL}_{\Sigma}^{k}(\tilde{p}_k(z))$ and $\mu \in \mathbb{R}$. Then we have*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{k|\tau_k|}{6}, & |\mu - 1| \leq \frac{2k-(k^2+4)\tau_k}{3k^2|\tau_k|}, \\ \frac{|1-\mu|k^3\tau_k^2}{2(2k-(k^2+4)\tau_k)}, & |\mu - 1| \geq \frac{2k-(k^2+4)\tau_k}{3k^2|\tau_k|}. \end{cases}$$

5. Concluding Remarks and Observations

In our present investigation, we have introduced new classes $\mathcal{SLM}_{\alpha,\Sigma}^{k}(\widetilde{p}_k(z))$ and $\mathcal{SLG}_{\gamma,\Sigma}^{k}(\widetilde{p}_k(z))$ of bi-univalent functions in the open unit disk U. For the initial Taylor- Maclaurin coefficients of functions belonging to these classes, we have studied the problem of finding the upper bound associated with the Fekete-Szegö inequality. We have also considered several results which are closely related to our investigation in this paper.

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Current address: H. Özlem Güney: Dicle University, Faculty of Science, Department of Mathematics, Diyarbakır- Turkey.
E-mail address: `ozlemg@dicle.edu.tr`
ORCID Address: `http://orcid.org/0000-0002-3010-7795`
Current address: G. Murugusundaramoorthy: School of Advanced Sciences, VIT University, Vellore -632014, India.
E-mail address: `gmsmoorthy@yahoo.com`
ORCID Address: `http://orcid.org/0000-0001-8285-6619`
Current address: J. Sokół: University of Rzeszów, Faculty of Mathematics and Natural Sciences, ul. Prof. Pigonia 1, 35-310 Rzeszów, Poland.
E-mail address: `jsokol@ur.edu.pl`
ORCID Address: `http://orcid.org/0000-0003-1204-2286`