Cumhuriyet Üniversitesi Fen Fakültesi
Fen Bilimleri Dergisi (CFD), Cilt:36, No: 3 Özel Sayı (2015)
ISSN: 1300-1949

Cumhuriyet University Faculty of Science
Science Journal (CSJ), Vol. 36, No: 3 Special Issue (2015)
ISSN: 1300-1949

A study on certain inequalities for p-valent functions

Roya GHASEMKHANİ

Department Of Mathematics, Faculty of Science, University Of Jiroft, Jiroft, Iran

Received: 01.02.2015; Accepted: 05.05.2015

Abstract. In this paper, we derive conditions on the parameters a, b, c so that the function $z^p F(a,b;c;z)$ is p-valent starlike and p-valent convex functions in $\mathcal{U}$, where $F(a,b;c;z)$ denotes the classical hypergeometric function. We also consider an integral operator related to the hypergeometric function.

Keywords: starlike; convex; p-valent; hypergeometric function.

1. INTRODUCTION

Let $\mathcal{A}(p)$ be the class of functions f of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n\, z^n, (p \in \mathbb{N} = \{1,2,3, \dots\})$$ (**Error! No text of specified style in document.**-1)

which are analytic and p-valent in the open unit disk $\mathcal{U} = \{z \in \mathbb{C}: |z| < 1\}$.

A function $f(z) \in \mathcal{A}(p)$ is called p-valent starlike of order β $(0 \le \beta < p), p \in \mathbb{N}$ if and only if

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > \beta, z \in \mathcal{U}.$$ (**Error! No text of specified style in document.**-2)

This function class is denoted by $\mathcal{S}^*(p,\beta)$. Also a function $f(z) \in \mathcal{A}(p)$ is said to be p-valent convex of order β $(0 \le \beta < p), p \in \mathbb{N}$ if and only if

$$Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \beta, z \in \mathcal{U}.$$ (**Error! No text of specified style in document.**-3)

This class is denoted by $\mathcal{K}(p,\beta)$. Note that $\mathcal{S}^*(1,\beta) \equiv \mathcal{S}^*(\beta)$ and $\mathcal{K}(1,\beta) \equiv \mathcal{K}(\beta)$ are, respectively, the usual classes of starlike and convex functions of order β. Furthermore $\mathcal{S}^*(p,0) \equiv \mathcal{S}^*(p)$ and $\mathcal{K}(p,0) \equiv \mathcal{K}(p)$

*Corresponding author. *Email address*: *Roya.Ghasemkhani@gmail.com*

Special Issue: The Second National Conference on Applied Research in Science and Technology

are, respectively p-valent starlike and p-valent convex functions

From (1.2) and (1.3), we have

$$f \in \mathcal{K}(p,\beta) \Leftrightarrow \frac{zf'}{p} \in \mathcal{S}^*(p,\beta).$$

A function $f(z) \in \mathcal{A}(p)$ is said to be in $\mathcal{US}_p(\beta)$, the class of uniformly p-valent starlike functions of order β if

it satisfies the condition

$$Re\left\{\frac{zf'(z)}{f(z)} - \beta\right\} \geq \left|\frac{zf'(z)}{f(z)} - p\right|, z \in \mathcal{U}. \quad \textbf{(Error! No text of specified style in document.}\text{-4)}$$

Also a function $f(z) \in \mathcal{A}(p)$ is said to be in $\mathcal{UCV}_p(\beta)$, the class of uniformly p-valent convex functions of order β if it satisfies the condition

$$Re\left\{1 + \frac{zf''(z)}{f'(z)} - \beta\right\} \geq \left|\frac{zf''(z)}{f'^{(z)}} - (p-1)\right|, z \in \mathcal{U}. \quad \textbf{(Error! No text of specified style in document.}\text{-5)}$$

Uniformly p-valent starlike and p-valent convex functions were first introduced [1] when $p = 1, \beta = 0$ and [2] when $p \geq 1, p \in \mathbb{N}$ and then studies by various authors.

Also denote $\mathcal{T}(p)$, the subclass of $\mathcal{A}(p)$ consisting of functions of the form

$$f(z) = z^p - \sum_{n=p+1}^{\infty} a_n\, z^n, \mathrm{a_n} \geq 0, p \in \mathbb{N}. \quad \textbf{(Error! No text of specified style in document.}\text{-6)}$$

We denote by $\mathcal{T}^*(\mathrm{p}, \beta)$, $\mathcal{C}(\mathrm{p}, \beta)$, $\mathcal{T}^*{}_{\mathrm{p}}(\beta)$ and $\mathcal{C}_{\mathrm{p}}(\beta)$ the classes obtained by taking intersections, respectively, of the classes $\mathcal{S}^*(\mathrm{p}, \beta)$ and $\mathcal{K}(\mathrm{p}, \beta)$with $\mathcal{T}(\mathrm{p})$; that is

$$\mathcal{T}^*(\mathrm{p}, \beta) = \mathcal{S}^*(\mathrm{p}, \beta) \cap \mathcal{T}(\mathrm{p})$$

$$\mathcal{C}(\mathrm{p}, \beta) = \mathcal{K}(\mathrm{p}, \beta) \cap \mathcal{T}(\mathrm{p})$$

$$\mathcal{T}^*{}_{\mathrm{p}}(\beta) = \mathcal{US}_{\mathrm{p}}(\beta) \cap \mathcal{T}(\mathrm{p})$$

$$\mathcal{C}_{\mathrm{p}}(\beta) = \mathcal{UCV}_{\mathrm{p}}(\beta) \cap \mathcal{T}(\mathrm{p})$$

The classes $\mathcal{T}^*(\mathrm{p}, \beta)$ and $\mathcal{C}(\mathrm{p}, \beta)$ were introduced by Owa (1985)[4]. In particular, the classes

$\mathcal{T}^*(1, \beta) = \mathcal{T}^*(\beta)$ and $\mathcal{C}(1, \beta) = \mathcal{C}(\beta)$ when p=1, were studied by Silverman (1975)[3].

Note that $f \in \mathcal{T}^*(\mathrm{p},\beta) \Leftrightarrow f \in \mathcal{T}^*{}_{\mathrm{p}}(\beta), f \in \mathcal{UCV}_{\mathrm{p}}(\beta) \Leftrightarrow f \in \mathcal{K}(\mathrm{p},\beta)$ and $f \in \mathcal{S}^*(\mathrm{p},\beta) \Leftrightarrow f \in \mathcal{US}_{\mathrm{p}}(\beta)$.

Let F(a,b;c;z) be the Gaussian hypergeometric function defined by

$$\mathrm{F}(\mathrm{a},\mathrm{b};\mathrm{c};\mathrm{z}) = \sum_{\mathrm{n}=0}^{\infty} \frac{(\mathrm{a})_{\mathrm{n}}(\mathrm{b})_{\mathrm{n}}}{(\mathrm{c})_{\mathrm{n}}(1)_{\mathrm{n}}} \mathrm{z}^{\mathrm{n}}$$

(**Error! No text of specified style in document.**-7)

Where $c \neq 0,-1,-2,\ldots$ and $(a)_n$ is the pochhammer symbol defined by

$$(\mathrm{a})_{\mathrm{n}} = \begin{cases} 1 & \mathrm{n}=0 \\ \mathrm{a}(\mathrm{a}+1)(\mathrm{a}+2)\ldots(\mathrm{a}+\mathrm{n}-1) & \mathrm{n}\epsilon\mathbb{N} \end{cases}$$

We note that $\mathrm{F}(\mathrm{a},\mathrm{b};\mathrm{c};1)$ converges for $Re(c-a-b)>0$ and is related to the Gamma function by

$$\mathrm{F}(\mathrm{a},\mathrm{b};\mathrm{c};1) = \frac{\Gamma(\mathrm{c})\Gamma(\mathrm{c}-\mathrm{a}-\mathrm{b})}{\Gamma(\mathrm{c}-\mathrm{a})\Gamma(\mathrm{c}-\mathrm{b})}$$

(**Error! No text of specified style in document.**-8)

Corresponding to the function F(a,b;c;z), we define $h_p\,(\mathrm{a},\mathrm{b};\mathrm{c};\mathrm{z}) = z^p\mathrm{F}(\mathrm{a},\mathrm{b};\mathrm{c};\mathrm{z})$.

Clearly $z^pF(a,b;c;z)$ has the series representation of the form ((Error**! No text of specified style in document.**-1) where $a_n = \frac{(a)_{n-\mathrm{p}}(b)_{n-\mathrm{p}}}{(c)_{n-\mathrm{p}}(1)_{n-\mathrm{p}}}$, hence we have

$$h_p\,(\mathrm{a},\mathrm{b};\mathrm{c};\mathrm{z}) = z^p + \sum_{n=p+1}^{\infty} \frac{(a)_{n-p}(b)_{n-p}}{(c)_{n-p}(1)_{n-p}} z^n.$$

2. MAIN RESULT

In this section, we determine necessary and sufficient conditions for $h_p\,(\mathrm{a},\mathrm{b};\mathrm{c};\mathrm{z}) = z^pF(a,b;c;z)$ to be in $\mathcal{T}^*(\mathrm{p},\beta)$ *and* $\mathcal{C}(\mathrm{p},\beta)$. Furthermore, we consider an integral operator related to the hypergeometric function.

To prove the main result, we need the following lemmas.

Lemma 2.1 [4]

(i) A function $f(z)$ of the form ((Error**! No text of specified style in document.**-1) is in $\mathcal{US}_p(\beta)$ if it satisfies the condition

$$\sum_{n=p+1}^{\infty} (n-\beta)\ |a_n|\ \leq p-\beta. \qquad \text{(Error! No text of specified style in document.-9)}$$

(ii) A function $f(z)$ of the form ((Error! **No text of specified style in document.**-1) is in $\mathcal{UCV}_p(\beta)$ if it satisfies the condition

$$\sum_{n=p+1}^{\infty} \frac{n}{p}(n-\beta)\ |a_n|\ \leq p-\beta. \qquad \text{(Error! No text of specified style in document.-10)}$$

Lemma 2.2[4]

(i) A function $f(z)$ of the form ((Error! **No text of specified style in document.**-1) is in $\mathcal{T}^*(p,\beta)$ if and only if it satisfies

$$\sum_{n=p+1}^{\infty} (n-\beta)\ a_n \leq p-\beta. \qquad \text{(Error! No text of specified style in document.-11)}$$

(ii) A function $f(z)$ of the form ((Error! **No text of specified style in document.**-1) is in $\mathcal{C}(p,\beta)$ if and only if it satisfies

$$\sum_{n=p+1}^{\infty} \frac{n}{p}(p-\beta)\ a_n\ \leq p-\beta. \qquad \text{(Error! No text of specified style in document.-12)}$$

Lemma 2.3[7] A function $f(z)$ of the form ((Error! **No text of specified style in document.**-1) is p-valent in $\mathcal{U}$ if $\sum_{n=1}^{\infty}(n+p)\ a_{n+p} \leq p$.

Theorem 2.4 If $a,b>0$ and $c>a+b+1$, then $h_p\ (a,b;c;z)=z^pF(a,b;c;z)$ to be in $\mathcal{US}_p(\beta)$ if

$$\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\left\{1+\frac{ab}{(p-\beta)(c-a-b-1)}\right\} \leq 2. \qquad \text{(Error! No text of specified style in document.-13)}$$

Proof. Since

$$h_p\ (a,b;c;z)=z^pF(a,b;c;z)=\ z^p+\sum_{n=p+1}^{\infty}\frac{(a)_{n-p}(b)_{n-p}}{(c)_{n-p}(1)_{n-p}}z^n, \qquad \text{(Error! No text of specified style in}$$

document.-14)

By the condition (2.1), we need only show that

$$\sum_{n=p+1}^{\infty} (n-\beta)\,\frac{(a)_{n-p}(b)_{n-p}}{(c)_{n-p}(1)_{n-p}} \le p-\beta.$$

(**Error! No text of specified style in document.**-15)

Now

$$\sum_{n=p+1}^{\infty} (n-\beta)\,\frac{(a)_{n-p}(b)_{n-p}}{(c)_{n-p}(1)_{n-p}} = \sum_{n=p+1}^{\infty} (n-p)\,\frac{(a)_{n-p}(b)_{n-p}}{(c)_{n-p}(1)_{n-p}} + (p-\beta)\sum_{n=p+1}^{\infty} \frac{(a)_{n-p}(b)_{n-p}}{(c)_{n-p}(1)_{n-p}}$$

$$= \sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_{n-1}} + (p-\beta)\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n}.$$

(**Error! No text of specified style in document.**-16)

Noting that $(\theta)_n = \theta(\theta+1)_{n-1}$ and applying (1.7), (1.8) we may express (2.8) as

$$\frac{ab}{c}\sum_{n=1}^{\infty}\frac{(a+1)_{n-1}(b+1)_{n-1}}{(c+1)_{n-1}(1)_{n-1}} + (p-\beta)\left(\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n} - 1\right)$$

$$= \frac{ab}{c}\frac{\Gamma(c+1)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)} + (p\text{-}\beta)\left(\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} - 1\right)$$

$$= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\left\{\frac{ab}{c-a-b-1} + (p\text{-}\beta)\right\} - (p\text{-}\beta).$$

By the condition (2.7) and above expression, we have:

$$\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\left\{\frac{ab}{c-a-b-1} + (p\text{-}\beta)\right\} - (p\text{-}\beta) \le p-\beta.$$

(**Error! No text of specified style in document.**-17)

Hence condition (2.5) holds.■

Remark 2.5 In Theorem 2.4, $\beta = 0$ and $p = 1$ gives the sufficient condition for $zF(a, b; c; z)$ to be in the class $\mathcal{S}_{\mathcal{P}}$ of uniformly starlike functions in $\mathcal{U}$.

In the next theorem, we find constraints on a, b and c that lead to necessary and sufficient conditions for $h_p\,(a, b; c; z)$ to be in the class $\mathcal{T}^*(p, \beta)$.

Theorem 2.6

(i) If $a,b > -1$ and $c > 0$ and $ab < 0$, then h_p (a, b; c; z) to be in $\mathcal{T}^*(p,\beta)\left(\mathcal{T}^*{}_p(\beta)\right)$ if and only if

$c \geq a+b+1-\frac{ab}{p-\beta}$.

(ii) If $a,b > 0$ and $c > a+b+1$, then $F_p(a,b;c;z) = z^p(2 - F(a,b;c;z))$ is in $\mathcal{T}^*(p,\beta)\left(\mathcal{T}^*{}_p(\beta)\right)$ if and only if

$$\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\left\{1+\frac{ab}{(p-\beta)(c-a-b-1)}\right\} \leq 2.$$ (**Error! No text of specified style in document.**-18)

Proof. (i) Since

$$h_p\,(a,b;c;z) = z^p + \sum_{n=p+1}^{\infty} \frac{(a)_{n-p}(b)_{n-p}}{(c)_{n-p}(1)_{n-p}} z^n = z^p + \frac{ab}{c}\sum_{n=p+1}^{\infty} \frac{(a+1)_{n-p-1}(b}{(c+1)_{n-p-1}}$$
$$= z^p - \left|\frac{ab}{c}\right| \sum_{n=p+1}^{\infty} \frac{(a+1)_{n-p-1}(b+1)_{n-p-1}}{(c+1)_{n-p-1}(1)_{n-p}} z^n,$$ (**Error! No text of specified style in document.**-19)

According to the condition (2.3), we must show that

$$\sum_{n=p+1}^{\infty} (n-\beta)\,\frac{(a+1)_{n-p-1}(b+1)_{n-p-1}}{(c+1)_{n-p-1}(1)_{n-p}} \leq \left|\frac{c}{ab}\right| p-\beta.$$ (**Error! No text of specified style in document.**-20)

Noting that $(\theta)_n = \theta(\theta+1)_{n-1}$ and then applying (**Error! Reference source not found.**) and (**Error! Reference source not found.**), we have

$$\sum_{n=p+1}^{\infty} (n-\beta)\,\frac{(a+1)_{n-p-1}(b+1)_{n-p-1}}{(c+1)_{n-p-1}(1)_{n-p}} = \sum_{n=0}^{\infty} (n+1+p-\beta)\,\frac{(a+1)_n(b}{(c+1)_n(1}$$
$$= \sum_{n=0}^{\infty} (n+1)\,\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+1}} + (p-\beta)\sum_{n=0}^{\infty} \frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+1}}$$
$$= \sum_{n=0}^{\infty} \frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_n} + (p-\beta)\sum_{n=1}^{\infty} \frac{(a+1)_{n-1}(b+1)_{n-1}}{(c+1)_{n-1}(1)_n}$$
$$= \sum_{n=0}^{\infty} \frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_n} + (p-\beta)\,\frac{c}{ab}\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n}$$
$$= \frac{\Gamma(c+1)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)} + (p-\beta)\,\frac{c}{ab}\left(\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}-1\right)$$ (**Error! No text of specified style in document.**-21)

Hence, (2.12) is equivalent to

$$\frac{\Gamma(c+1)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)}\left\{1+(p-\beta)\frac{c-a-b-1}{\mathrm{ab}}\right\} \le (p-\beta)\left(\left|\frac{c}{ab}\right|+\frac{c}{ab}\right)$$ **(Error! No text of specified style in document.-22)**

The above expression is valid if and only if

$$1+(p-\beta)\frac{c-a-b-1}{\mathrm{ab}} \le 0,$$

which is equivalent to $c \ge a+b+1-\frac{ab}{p-\beta}$ and the proof is complete. ■

(ii) Since F_p (a, b; c; z) $= z^p - \sum_{n=p+1}^{\infty} \frac{(a)_{n-p}(b)_{n-p}}{(c)_{n-p}(1)_{n-p}} z^n$, the necessary of (2-13) for F_p to be in $\mathcal{T}^*(p,\beta)\ \left(\mathcal{T}^*{}_p(\beta)\right)$ follows from condition (2.4).■

Theorem 2.7 If $a, b > 0$ and $c > a+b+2$, then h_p (a, b; c; z) to be in $\mathcal{UCV}_p(\beta)$ if

$$\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\left\{1+\left(\frac{2p+1-\beta}{p(p-\beta)}\right)\left(\frac{\mathrm{ab}}{c-a-b-1}\right)+\frac{(a)_2(b)_2}{p(p-\beta)(c\text{-}a\text{-}b\text{-}2)_2}\right\} \le$$ **(Error! No text of specified style in document.-23)**

Proof. In view of (ii) of lemma (2.1), we need only show that

$$\sum_{n=p+1}^{\infty} n(n-\beta)\,\frac{(a)_{n-p}(b)_{n-p}}{(c)_{n-p}(1)_{n-p}} \le p(p-\beta)$$ **(Error! No text of specified style in document.-24)**

Now

$$\sum_{n=p+1}^{\infty} n(n-\beta)\,\frac{(a)_{n-p}(b)_{n-p}}{(c)_{n-p}(1)_{n-p}} = \sum_{n=0}^{\infty}\left[(n+1+p)(n+1+p-\beta)\right]\frac{(a)_{n+1}(b)_{n}}{(c)_{n+1}(1)_{n}}$$

$$= \sum_{n=0}^{\infty}(n+1)^2\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}} + (2p-\beta)\sum_{n=0}^{\infty}(n+1)\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}} + p(p-\beta)\sum_{n=0}^{\infty}\frac{(a}{(c}$$

$$= \sum_{n=0}^{\infty}(n+1)\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n}} + (2p-\beta)\sum_{n=0}^{\infty}\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n}} + p(p-\beta)\sum_{n=0}^{\infty}\frac{(a)_{n+1}(b)_{n}}{(c)_{n+1}(1)_{n}}$$ **(Error! No text of specified style in document.-25)**

$$= \sum_{n=1}^{\infty}\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n-1}} + (2p+1-\beta)\sum_{n=0}^{\infty}\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n}} + p(p-\beta)\sum_{n=0}^{\infty}\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}}$$

$$= \sum_{n=0}^{\infty} \frac{(a)_{n+2}(b)_{n+2}}{(c)_{n+2}(1)_n} + (2p+1-\beta)\sum_{n=0}^{\infty} \frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_n} + p(p-\beta)\sum_{n=1}^{\infty} \frac{(a)_n(b)_n}{(c)_n(1)_n}$$

Using $(a)_{n+k} = (a)_n(a+k)_n$ we may write the above expression as

$$\frac{(a)_2(b)_2}{(c)_2}\frac{\Gamma(c+2)\Gamma(c-a-b-2)}{\Gamma(c-a)\Gamma(c-b)} + (2p+1-\beta)\frac{ab}{c}\frac{\Gamma(c+1)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)} + p(p-\beta)\left(\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} - 1\right).$$

The last expression is bounded above by p(p-β) if and only if (2.15) holds.■

Remark 2.8 In Theorem 2.7, $\beta = 0$ *and* $p = 1$ reduces to a necessary and sufficient condition for $zF(a, b; c; z)$ to be in the class $\mathcal{UCV}$ of uniformly convex functions in $\mathcal{U}$.

Theorem 2.9

(i) If $a, b > -1, ab < 0$ and $c > a+b+2$, then h_p (a, b; c; z) is in $\mathcal{C}(p, \beta)\left(\mathcal{C}_p(\beta)\right)$ if and only if

$$(a)_2(b)_2+(1+2p-\beta)ab(c-a-b-2)+p(p-\beta)(c-a-b-2)_2 \geq 0.$$ (**Error! No text of specified style in document.**-26)

(ii) If $a, b > 0$ and $c > a+b+2$, then $F_p(a, b; c; z) = z^p(2 - F(a, b; c; z))$ is in $\mathcal{C}(p, \beta)\left(\mathcal{C}_p(\beta)\right)$ if and only if

$$\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\left\{1 + \left(\frac{2p+1-\beta}{p(p-\beta)}\right)\left(\frac{ab}{c-a-b-1}\right) + \frac{(a)_2(b)_2}{p(p-\beta)(c-a-b-2)_2}\right\} \leq$$ (**Error! No text of specified style in document.**-27)

Proof. (i) Since h_p (a, b; c; z) has the form (2.11), we see from (ii) of lemma 2.2 that our conclusion is equivalent to

$$\sum_{n=p+1}^{\infty} n\,(p-\beta)\,\frac{(a+1)_{n-p-1}(b+1)_{n-p-1}}{(c+1)_{n-p-1}(1)_{n-p}} \leq \left|\frac{c}{ab}\right| p(p-\beta).$$ (**Error! No text of specified style in document.**-28)

Now

$$\sum_{n=p+1}^{\infty} n\,(p-\beta)\,\frac{(a+1)_{n-p-1}(b+1)_{n-p-1}}{(c+1)_{n-p-1}(1)_{n-p}}$$

$$=\sum_{n=0}^{\infty} (n+1+p)(n+1+p-\beta)\,\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+1}}$$

$$=\sum_{n=0}^{\infty} \left[(n+1)^2+(2p-\beta)(n+1)+p(p-\beta)\right]\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+1}}$$

$$=\sum_{n=0}^{\infty}(n+1)^2\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+1}} + (2p-\beta)\sum_{n=0}^{\infty}(n+1)\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+1}}+p(p-\beta)\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+1}}$$

$$=\sum_{n=0}^{\infty}(n+1)\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_n} + (2p-\beta)\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_n}+p(p-\beta)\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+1}}$$

$$=\frac{(a+1)(b+1)}{(c+1)}\sum_{n=0}^{\infty}\frac{(a+2)_n(b+2)_n}{(c+2)_n(1)_n}+(2p+1-\beta)\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_n} + p(p-\beta)\frac{c}{ab}\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n}$$

$$=\frac{\Gamma(c+1)\Gamma(c-a-b-2)}{\Gamma(c-a)\Gamma(c-b)}\times$$

$$\left((a+1)(b+1)+(2p+1-\beta)(c-a-b-2)+\frac{p(p-\beta)}{ab}(c-a-b-2)_2\right) -\frac{p(p-\beta)c}{ab}.$$

This is less than or equal to $\left|\frac{c}{ab}\right|p(p\text{-}\beta)$ if and only if

$$(a+1)(b+1)+(2p+1-\beta)(c-a-b-2)+\frac{p(p-\beta)}{ab}(c-a-b-2)_2\le 0,$$

Which is equivalent to (2.18). ■

(iii) From Theorem (2.7) it follows.■

3. AN INTEGRAL OPERATOR

In this section, we illustrate some results obtained by a particular integral operator $G_p(\mathrm{a,b;c;z})$ acting on $F(\mathrm{a,b;c;z})$ as follows

$$G_p(\mathrm{a,b;c;z}) = p\int_0^{\mathrm{z}} t^{p-1}\mathrm{F(a,b;c;t)dt}$$

$$= z^p + \sum_{n=1}^{\infty}\left(\frac{p}{n+p}\right)\frac{(a)_n(b)_n}{(c)_n(1)_n}\, z^{n+p}.$$

Theorem 3.1

(i) If $a,b>0$ and $c>a+b$, then $G_p(\mathrm{a,b;c;z})$ to be in $\mathcal{S}^*(\mathrm{p})$ if

$$\frac{\Gamma(\mathrm{c})\Gamma(\mathrm{c-a-b})}{\Gamma(\mathrm{c})\Gamma(\mathrm{c-b})} \le 2.$$ **(Error! No text of specified style in document.**-29)

(ii) If $a,b>-1, ab<0$ and $c>0$, then $G_p(\mathrm{a,b;c;z})$ is in $\mathcal{T}(p)$ or $\mathcal{A}(p)$ if and only if $c>max\{a,b\}$.

Proof. (i) Since

$$G(a,b;c;z) = z^p + \sum_{n=1}^{\infty}\left(\frac{p}{n+p}\right)\frac{(a)_n(b)_n}{(c)_n(1)_n}\, z^{n+p},$$

According to the Lemma (2.3), we must show that

$$\sum_{n=1}^{\infty}(n+p)\left(\frac{p}{n+p}\right)\frac{(a)_n(b)_n}{(c)_n(1)_n} \le \mathrm{p}.$$ **(Error! No text of specified style in document.**-30)

We see that

$$\sum_{n=1}^{\infty}(n+p)\left(\frac{p}{n+p}\right)\frac{(a)_n(b)_n}{(c)_n(1)_n} = p\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n} = p\left(\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n} - 1\right)$$ **(Error! No text of specified style in document.**-31)

The last expression is bounded above by p if and only if condition (3.1) holds. ■

(ii) Since

$$G(a,b;c;z) = z^p - \frac{|ab|}{c}\sum_{n=p+1}^{\infty}\frac{p}{n}\,\frac{(a+1)_{n-p-1}(b+1)_{n-p-1}}{(c+1)_{n-p-1}(1)_{n-p}}\, z^n,$$

According to the Lemma (3.2), we must show that

$$\sum_{\mathrm{n=p+1}}^{\infty} \frac{(a+1)_{n-p-1}(b+1)_{n-p-1}}{(c+1)_{n-p-1}(1)_{n-p}} \leq \frac{c}{|ab|},$$

or, equivalently,

$$\sum_{\mathrm{n=0}}^{\infty} \frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+1}} = \frac{c}{ab}\sum_{n=1}^{\infty} \frac{(a)_n(b)_n}{(c)_n(1)_n} \leq \frac{c}{|ab|}.$$

But this is equivalent to

$$\frac{\Gamma(\mathrm{c})\Gamma(\mathrm{c-a-b})}{\Gamma(\mathrm{c})\Gamma(\mathrm{c-b})} - 1 \geq -1,$$

which is true if and only if $c > max\{a,b\}$. This complets the proof of Theorem 3.1(ii). ■

Since $\frac{z}{p}G'_p = h_p, \frac{z}{p}G''_p = h'_p - \frac{1}{p}\, G'_p$, and so $1 + \frac{zG''_p}{G_p} = \frac{zh'_p}{h_p}$.

$$G_p(\mathrm{a,b;c;z}) \in \mathcal{K}(p,\beta)\left(\mathcal{UCV}_p(\beta)\right) \Leftrightarrow \frac{z}{p}G'_p(\mathrm{a,b;c;z}) = h_p\ (\mathrm{a,b;c;z}) \in \mathcal{S}^*(p,\beta)\left(\mathcal{US}_p(\beta)\right).$$

Thus any p-valent starlike about h_p leads to a p-valent convex about G_p. Hence we obtain the following analogues to theorems (2.4) and (2.6).

Theorem 3.2

(i) If $a,b > 0$ and $c > a+b+1$, then $G_p(\mathrm{a,b;c;z})$ to be in $\mathcal{UCV}_p(\beta)$ if

$$\frac{\Gamma(\mathrm{c})\Gamma(\mathrm{c-a-b})}{\Gamma(\mathrm{c-a})\Gamma(\mathrm{c-b})}\left\{1 + \frac{\mathrm{ab}}{(p-\beta)\mathrm{c-a-b-1}}\right\} \leq 2.$$

(ii)if $a,b > -1, ab < 0$ and $c > a+b+2$, then $G_p(\mathrm{a,b;c;z})$ is in $C(p,\beta)\left(\ \mathcal{C}_p(\beta)\right)$ if and only if

$$\mathrm{c} \geq \mathrm{a+b+1} - \frac{\mathrm{ab}}{\mathrm{p}-\beta}.$$

REFERENCES

[1] Goodman, A.W. (1991), “On uniformly starlike functions, ”J. Math. Anal. Appl., 155, 364–370.
[2] Al-kharsani H.A. and Al-Hajiry, S.S. (2006), “Subordination results for the family of uniformly convex p-valent functions,” J. Inequal. Pure and Appl. Math., 7(1), Art. 20. [online: http: //jipam.vu.edu.au/article.php?
[3] Silverman, H. (1975), “Univalent functions with negative coefficients,” Proc. Amer. Math. Soc. 51, pp. 109-116.

[4] Owa, S. (1985), “On certain classes of p-valent functions with negative coefficients,” Bull. Belg. Math. Soc. Simon stevin, 59, pp. 385-402.
[5] Al-kharsani H.A. and Al-Hajiry, S.S. (2008), “a note on certain inequalities for p-valent functions,” Journal of inequalities in pure and applied mathmatics, vol 9(2008), Issue 3, Article 90, 6pp.
[6] El-ashwah, R.M., Aouf, M.K. and Moustafa, A. O.(2010), “Starlike and convexity properties for p-valent hypergeometric functions, ” Acta Math. Univ. Comenianae, Vol. LXXIX, 1, pp. 55-64.
[7] Chen M.-P., Multivalent functions with negative coe_cients in the unit disc, Tamkang J. Math. 17(3) (1986), 127{137.