Cumhuriyet Üniversitesi Fen Fakültesi
Fen Bilimleri Dergisi (CFD), Cilt:36, No: 3 Özel Sayı (2015)
ISSN: 1300-1949

Cumhuriyet University Faculty of Science
Science Journal (CSJ), Vol. 36, No: 3 Special Issue (2015)
ISSN: 1300-1949

Concepts of Digital Topology

Afsaneh REZAIE

Mathematics Department of Ferdowsi University of Mashhad

Received: 01.02.2015; Accepted: 05.05.2015

Abstract. In image processing and computer graphics an object in the plane or 3-space is often approximated digitaly by a set of pixels or voxels. Digital topology studies properties of pixels or voxels that correspond to topological properties of the original object. In this paper, we discuss about digital space and digital picture from Rosenfeld's aspect of view and introduce regular and strongly normal digital picture space. Using these introductions, we impose restrictions on adjacency relation between points to establish some important theorem in digital space like as the Jordan Curve Theorem. Also, one can explore digital fundamental group in regular digital picture space but in this paper we do not deal with it. At the end, we express that the Jordan Curve Theorem in the strongly normal digital picture space is verified.

Keywords: digital picture space, picture space, black point, white point, adjacency relation, regular digital picture space

1. INTRODUCTION

The purpose of this paper is to introduce digital geometry. In digital geometry definitions and theorems are given form classical geometry and converted to its digital concepts. In practice, this conversion does not happen automatically and takes a great effort and exact analysis. One can transform a certain theorem from classical geometry to a digital counterpart, while for all theorems it may not be practical. Some researchers like Boxer have tried to convert classical topology to concept of digital topology, but they were not completely successful. Rosenfeld has a great role in presenting the concepts in different part of digital geometry. Thus, in this paper, we discuss about digital topology form Rosenfeld's aspect of view to present an exact definitions of classic topology's concepts in digital topology.

2. PRELIMINARIES

One can start the study of digital geometry by identifying the digital geometry on the Z^2 and Z^3, possibly Z^n, together with an adjacency relation. A binary digital picture space is a triple (V, β, ω), where V is the set of grid points in a 2-d or 3-d grid and each of β and ω is a set of closed straight line segments joining pairs of points in V. V is a set of grid points in a 2-d or 3-d grid if it satisfies following conditions:

*Corresponding author. *Email address*: Afc.math.17844@gmail.com

Special Issue: The Second National Conference on Applied Research in Science and Technology

1. V is an infinite set of points in E^2 (2-d case) or E^3 (3-d case).
2. V has no accumulation point.
3. There exists a positive constant D such that every point in E^2 or respectively E^3 is within distance D of a point in V.

We will often abbreviate a " digital picture space " to DPS.

A binary digital picture is a quadruple (V, β, ω, B), where (V, β, ω) is a DPS and B is a subset of V. Points of the DPS, i.e. points in V, are also referred to as points of the picture. Points in B are called black points of the picture and points in V-B are called white points of the picture. An important notion of digital topology is that of adjacency between black points and white points. On a DPS (V, β, ω) these adjacency relations are defined by the line segments in the sets β and ω. A line segment in β is called a β-adjacency. Similarly, a line segment in ω is called a ω-adjacency.

One of the important theorems in classical mathematic, is the Jordan Curve Theorem. This theorem explains the fact that which is axiom in intuitive. It means that every simple closed curve separates the rest of the space in exactly two components, but it is too difficult to prove it by geometry arguments. In this paper, we want to provide some conditions which the Jordan Curve Theorem be verified in digital form. Because of that, we present some definitions of regular digital picture space and strongly normal digital picture space.

A DPS (V, β, ω) is said to be regular if it satisfies both of the following conditions:

1. No β-adjacency or ω-adjacency passes through any point in V other than its endpoints,
2. No β-adjacency meets a ω-adjacency with which it does not share an end point.

A DPS (V, β, ω) is strongly normal if it is regular and also satisfies all of the following conditions:

1. $V = Z^2$ (the 2-d case) or $V = Z^3$ (the 3-d case).
2. In the 2-d case every 4-adjacency and in the 3-d case every 6-adjacency is both a β-adjacency and a ω-adjacency.
3. All β-adjacencies and ω-adjacencies are 8-adjacencies in the 2-d case and 26-adjacencies in the 3-d case.
4. In any given unite lattice square either both diagonals are β-adjacencies or both diagonals are ω-adjacencies or one of diagonals is both a β-adjacency and a ω-adjacency.
5. Every picture P on DPS (V, β, ω) has the property that whenever a black component of P is either β-adjacent or ω-adjacent to a white component of P, the black component is in the 2-d case 4-adjacent and in the 3-d case 6-adjacent to the white component.

3. JORDAN CURVE THEOREM

In this section, we first remind the classical Jordan Curve Theorem and then try to make some conditions for the Jordan Curve Theorem in digital space. Let C be a Jordan curve in the S^2. Then its complement, S^2-C, consist of exactly two connected components. One of these components is bounded (the interior) and the other is unbounded (the exterior) and the curve C is the boundary of each component.

Note that, the start of investigating in digital topology is simple, but the way of using from different adjacency relations for black and white points is important.

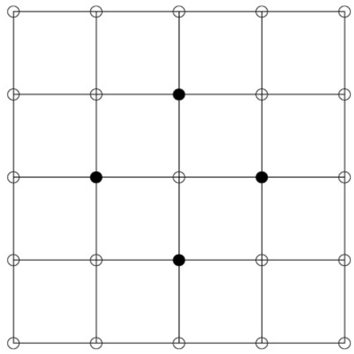

Figure 1. Connectivity paradox.

For example, in figure1, let the digital picture space be (Z^2, 4, 4). Hence, the black points are " totally disconnected " but they still separate the central white point from the other white points. On the other hand, in this figure, if the digital picture space be (Z^2, 8, 8), then the black points forma Jordan curve but they do not separate the white points. Using 4-adjacency for the white points and 8-adjacency for black, or vice versa can eliminate this problem. As we had in the definition of regular digital picture space, by supposing the digital picture spaces of (Z^2, 4,4) and (Z^2, 8, 8), we observe that these two spaces are not regular and the Jordan curve theorem is not verified in them. But by giving the digital picture spaces (Z^2, 4, 8) and (Z^2, 8, 4), the Jordan curve theorem is verified, because digital picture spaces (Z^2, 4, 8) and (Z^2, 8, 4) are regular and also have strongly normal digital picture space conditions.

Similarly, in three dimensions, 6-adjacency is used for the white points and 18- or 26-adjacency for the black, or vice versa.

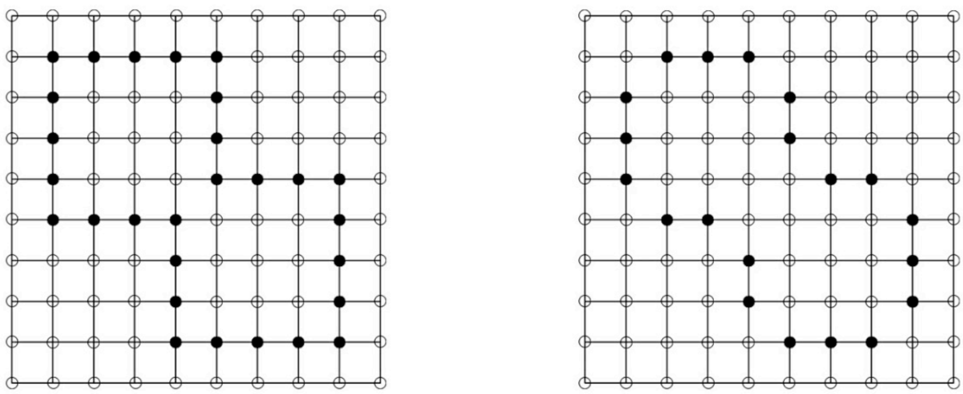

Figure 2. Left a simple 4-curve, right a simple 8-curve.

Another example is shown in figure2. The left side is a simple 4-curve which makes three components and the right side is a simple 8-curve which makes one component. According to the Jordan Curve Theorem in digital form, there exist simple curves under the 4-adjacency which subdivide the rest of the plane into more than two components and under the 8-adjacency, there exist simple closed curves which does not subdivide the rest of the plane at all. To solve this problem, Rosenfeld has suggested that to consider a " mixed " adjacency: the 8-adjacency for the black points and the 4-adjacency for the white points or vice versa. As it is shown in above examples, by applying some conditions to the digital space we can utter the Jordan Curve Theorem in the digital space. The Jordan theorem in the digital space comes as follows:

Let $P = (Z^2, \beta, \omega, B)$ be a picture on a strongly normal DPS, where B is a black simple closed curve of P that is not contained in any unit lattice square. Then P has just two white components, and each point in B is adjacent to both of them.

4. P-WALK AND P-LOOPS; THE DIGITAL FUNDAMENTAL GROUP

As it can be clearly seen, using different relations is of great importance in holding theorem. The role of black and white points is also of great concern in digital topology. In so doing, in order to show it, digital fundamental group is defined, then we provide an example showing that the white point in space plays an important role in calculating the digital fundamental group.

A P-walk is a curve $\gamma : [0,1] \rightarrow E^{n,}$ where n=2 or 3 according as P is 2-d or 3-d, such that $\gamma(0)$ and $\gamma(1)$ are black points of P, and there exists a positive integer k such that for all nonnegative integer $i < k$:

(1) $\gamma (i / k)$ is a black points, and
(2) $\gamma (i / k)$ is equal or adjacent to $\gamma ((i + 1) / k)$, and
(3) γ is linear on the closed interval $[i /k, (i +1) / k]$.

A P-walk from a point p to itself is called a P-loop, and is said to be based at p; we also call p the base point of the P-loop.

Now let P be a picture on an n-dimensional DPS, where n = 2 or 3. Two P-loops white the same base point are called equivalent if they are fixed base point homotopic in $E^n - W$, where W is the union of all white points of P if n = 2, and the union of all white adjacencies of P if n = 3. This is of course an equvalence relation. We write $[\lambda]_p$ for the equivalence class consisting of all P-loops which have the same base point as λ and which are equivalent to λ. If the P-loops λ and λ' have the same base point, then define $[\lambda]_p . [\lambda']_p$ to be the equivalence class $[\lambda . \lambda']_p$.

Definition . Let P be a picture on a regular DPS. The digital fundamental groupm of P white base point p, denoted by $\pi (P, p)$, is the group of all equivalence classes $[\lambda]_p$ where λ is a P-loop based at p, under the . operation .

Example .In figure3 let (Z^2, 8, 4, B) be a digital picture space and suppose B = {p_0 = (0,0), p_1 = (1,1), p_2 = (2,0), p_3 = (1,-1)}. The point (1,0) is a white point and two P-loops p_0 and $p_0p_1p_2p_3p_0$ are not digital homotopic. Then, digital fundamental group π (P, p_0) is not trivial group.

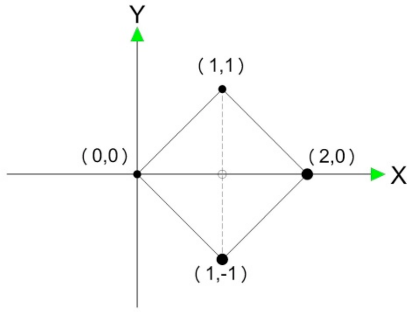

Figure 3.

5. CONCLUSIONS

Using the adjacency relations in digital topology is so important because the way of using these relations may contradict some theorems. As we observed, for converting the classical concepts into digital concepts we should impose restriction on digital spaces. For instate, to express the Jordan Curve Theorem in digital space we have to impose two restrictions on the set of β and ω to convert the space into strongly normal digital space. Thus, the Jordan Curve Theorem will be verified in strongly normal digital spaces.

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