Cumhuriyet Üniversitesi Fen Fakültesi
Fen Bilimleri Dergisi (CFD), Cilt:36, No: 3 Özel Sayı (2015)
ISSN: 1300-1949

Cumhuriyet University Faculty of Science
Science Journal (CSJ), Vol. 36, No: 3 Special Issue (2015)
ISSN: 1300-1949

Cross Efficiency Matrix in Non-Radial Models

Samira SAFARIANI GILAN[1,2], Mohammad Reza MOZAFFARI[3,*]

[1] 1.MA student, Department of Management, Fars science and Fars science and Research Branch, Islamic Azad University, Marvdasht, Iran

[2]Department of Mnagement, Marvdasht Branch, Islamic Azad University, Marvdasht, Iran.

[3]Assistant Professor, Department of Mathematics, Shiraz Branch, Islamic Azad University, Shiraz, Iran

Received: 01.02.2015; Accepted: 05.05.2015

Abstract. In data envelopment analysis each decision making unit is analyzed in best situation and in CCR models for the unit under analysis the best weights are considered for inputs and outputs. So, if the decision making unit in its best situation is less than one, it is inefficient, otherwise it is efficient. Therefore, the present article is an attempt to achieve the cross efficiency matrix by using SBM and then rank the decision making units by using the cross efficiency matrix. Overall, selecting the optimal portfolio is done by cross efficiency matrix. In order to find the efficient firms the mean, column variance and covariance of data of cross efficiency scale are utilized. Finally, the cross efficiency matrix of 22 firms with two inputs and one output has been calculated using non-radial models.

Keywords: Data Envelopment Analysis, Cross Efficiency Matrix, Non-radial Models

1. INTRODUCTION

Data envelopment analysis (DEA) is a mathematical tool for evaluating the performance of decision-making units in an organization. DEA was first suggested by Farrell for the limitation of multiple inputs of an output. Then, Charnes et al. in their well-known article CCR for evaluating DMU with multiple inputs proposed multiple output. Since DEA was first introduced in 1987, it is widely used in areas such as banking, government services and education. In specific, multi-criteria decision making is an active area in which DEA can provide a useful base for various problem solving approaches (Stewart, 1996)[9]. In DEA decision making units are equivalent to multi options in multi criteria decision making. DEA can be described as multi-criteria performance measurement model when it is utilized as a multi-criteria decision-making method. The functional areas such as classification of inventory, supplier selection and project selection are among the areas in which DMU is widely used as a multi-criteria performance measurement model. Besides, selection of a portfolio in specific can be considered a multi-criteria decision making in which for the inclusion of stocks in a portfolio the multiple criteria of the performance are used in comparison with options such as the product, stock, projects, therefore it can be said in selection of a portfolio data envelopment analysis is a practical approach. DEA provides a logical and systematic method for selecting a set of weights for the multiple criteria in which the optimal weights are determined by solving mathematical programs. In DEA an efficient score is determined for each decision making unit and the decision making units can be ordered according to their efficiency scores. A drawback of DEA

*Corresponding author. *Email address: mozzafari854@yahoo.com*

Special Issue: The Second National Conference on Applied Research in Science and Technology

is its great flexibility in selecting the optimal weights for input and output factors. Green and Doyle (1994)[2] called the decision making unit which can achieve the full efficiency through selecting very high weights in some factors and very low weights for other factors as maverick. This problem leads to the illogical ranking by DEA. To overcome this problem approaches such as: the Cone-ratio model (Charnes et al., 1990) and the Assurance-Region model (Thompson et al., 1990)[8] imposes restrictions on the weights, the super efficiency model (Anderson and Peterson, 1993)[1] draw a greater distinction between efficient decision making units and the cross efficiency evaluation model utilizes the pair evaluation mechanism which based on the cross efficiency evaluation of maverick enjoys the less chance for obtaining high evaluation; therefore, considering the desirable characteristics of cross efficiency evaluation it is more common in many areas. Examples of ranking methods include: AP method proposed by Anderson and Peterson in 1993 which despite the failure, in some cases, other methods were presented. The other two methods for ranking efficient units presented by Jahanshahloo, Mehrabian and Alirezaee [6] and the more complete method by Saati et al. [7]. Another set of methods for ranking are based on the constraining weights. In this paper, it is tried to obtain the cross-efficiency matrix using the enhanced non-radial Russell model and the SBM model presented by Pastor (2001) and Tone (2001) [4], respectively. Then, using statistical methods DMUs are ranked.

In the second section the basic concepts of DEA are explained briefly and in the third section the proposed approach of cross efficiency matrix using the enhanced Russell model and SBM model is presented. In the fourth section as a numerical example considering 22 firms with two inputs and one output taken from the article by Lim et al. (2014) [5] the proposed approach is examined and finally conclusions are presented.

2. Review of DEA

Suppose that n decision making units with consuming m inputs $x_j = (x_{1j},\dots,x_{mj})$ produce S outputs $y_j = (y_{1j},\dots,y_{sj})$. For evaluating DMU$_0$ under constant returns to scale the CCR model is used as follows: [2]

$$
\begin{aligned}
Min\ \ & \theta - \varepsilon\left(\sum_{i=1}^{m} s_i^- + \sum_{r=1}^{s} s_r^+\right) \\
s.t\ \ & \sum_{j=1}^{n} \lambda_j x_{ij} + s_i^- = \theta x_{i0} && i = 1,\dots,m \\
& \sum_{j=1}^{n} \lambda_j y_{rj} - s_r^+ = y_{r0} && r = 1,\dots,s \\
& \lambda_j \geq 0 && j = 1,\dots,s \\
& s_i^- \geq 1 \\
& s_r^+ \geq 1
\end{aligned}
\tag{1}
$$

Model (1) is radial as DMU$_0$ is evaluated in both phases, first the amount of θ^* is calculated and then the maximum increase of input and output slack variables s_i^{+*} and s_r^{-*} is obtained.

3. Cross efficiency matrix in non-radial models

Considering n decision making units and m input and s outputs the modified and enhanced Russell model for evaluation of DMU_0 with the input X_0 and output Y_0 is proposed as follows. [3]

$$
\begin{aligned}
& Min\ \frac{1}{m}\sum_{i=1}^{m}\theta_i - \frac{1}{s}\sum_{r=1}^{s}\varphi_r \\
& s.t \\
& \qquad \sum_{j=1}^{n}\lambda x_{ij} \leq \theta_i x_{i0} \qquad i = 1,\dots,m \\
& \qquad \sum_{j=1}^{n}\lambda y_{rj} \geq \varphi_r y_{r0} \qquad r = 1,\dots,s \\
& \theta_i \leq 1 \quad i = 1,\dots m \\
& \varphi_r \geq 1 \quad r = 1,\dots s \\
& \lambda \geq 0 \quad j = 1,\dots n
\end{aligned}
\tag{2}
$$

The duality of model (2) considering the dual variables V_i and U_r corresponding to input and output constrains and φ_i and β_r is suggested as follows:

$$
\begin{aligned}
s.t & \sum_{r=1}^{s}u_r y_{rj} - \sum_{i=1}^{m}v_r x_{ij} \leq 0 \quad j = 1,\dots \\
& V_i x_{i0} - \varphi_i = \frac{1}{m} \qquad i = 1,\dots,m \\
& -u_r y_{r0} + \beta_r = -\frac{1}{s} \qquad r = 1,\dots,s \\
& \quad v_i \geq 0 \quad i = 1,\dots,m \\
& \quad u_r \geq 0 \quad r = 1,\dots,s \\
& \quad \varphi_i \geq 0 \quad i = 1,\dots,m \\
& \quad \beta_r \geq 0 \quad r = 1,\dots,s
\end{aligned}
\tag{3}
$$

$$
\theta_{0j} = \frac{\sum_{i=1}^{m} u_r^{*} y_{rj}}{\sum^{n} v_i^{*} x_{ij}} \tag{4}
$$

As v_i^{*} and u_r^{*} are the definite solution of model (3), the cross efficiency matrix using the equation (4) is as follows:

Table1. cross efficiency matrix of model (3)

DMU_1	$\theta_{1\ 1}$	$\theta_{1\ 2}$	…	$\theta_{1\ 1}$
DMU_0	$\theta_{0\ 1}$	$\theta_{0\ 2}$	…	$\theta_{0\ n}$
DMU_n	$\theta_{n\ 1}$	$\theta_{n\ 2}$	…	$\theta_{n\ n}$

The cross efficiency matrix of model (3) is a square matrix of order n.
The modified SBM model is suggested as follows:

$$
\begin{aligned}
& Max \left(\frac{1}{m} \sum_{i=1}^{m} \frac{s_i^-}{x_{i0}} + \frac{1}{s} \sum_{r=1}^{s} \frac{s_r^+}{y_{r0}} \right) \\
& s.t \quad \sum_{j=1}^{n} \lambda x_{ij} + s_i^- = x_{i0} \qquad i = 1,\dots,m \\
& \qquad \sum_{j=1}^{n} \lambda y_{rj} - s_r^+ = y_{r0} \qquad r = 1,\dots,s \\
& \qquad \lambda \ge 0,\ s_i^- \ge 0 \ s_r^+ \ge 0 \quad i = 1,\dots,m \\
& \qquad\qquad\qquad\qquad r = 1,\dots,s \\
& \qquad\qquad\qquad\qquad j = 1,\dots,n
\end{aligned}
\tag{5}
$$

Duality of model (5) with taking into the account the variables of dual u_r and v_i is as follows:

$$
\begin{aligned}
& Min \sum_{i=1}^{m} v_i x_{i0} + \sum_{r=1}^{s} u_r y_{r0} \\
& s.t \sum_{r=1}^{s} u_r y_{rj} + \sum_{i=1}^{m} v_r x_{ij} \ge 0 \qquad j = 1,\dots,n \\
& v_i \ge \frac{1}{m} \left[\frac{1}{x_{i0}} \right] \qquad i = 1,\dots,m \\
& - u_r \ge \frac{1}{s} \left[\frac{1}{y_{r0}} \right] \qquad r = 1,\dots,s
\end{aligned}
\tag{6}
$$

Overall, θ_{u^*} is obtained like equation (4) but the solutions of u_r^* and v_i^* is obtained from model (6) and the cross efficiency matrix is like the table.

4. Numerical Example

Consider the data of 22 firms with two inputs and one output taken from Lim et al. (2014) [4]:

Table 2. Input and Output Data

DMU	i_1	i_2	o_1
1	5	4	1
2	6	5	1
3	4	5	1
4	8	5	1
5	5	6	1
6	8	3	1
7	4.4	4.4	1
8	2.6	8	1
9	3.4	8	1
10	3.6	4.4	1
11	2	7	1
12	3	7	1
13	3	5.6	1
14	2.6	5	1
15	4	4	1
16	5	3.2	1
17	6	4	1
18	4	3.5	1
19	7	3	1
20	6	2.5	1
21	8	2	1

22	9	2	1

Utilizing proposed model (6) the cross efficiency matrix which is a matrix of order $n \times n$ is achieved as follows:

Table 2. The Cross Efficiency Matrix

DMU	1	2	3	4	5	6	7	8	9	10	11
1	0.84	0.69	0.8	0.6	0.65	0.76	0.84	0.61	0.58	0.9	0.71
2	0.85	0.69	0.79	0.57	0.65	0.79	0.83	0.59	0.57	0.89	0.69
3	0.83	0.68	0.84	0.57	0.69	0.67	0.85	0.72	0.67	0.94	0.85
4	0.85	0.69	0.79	0.61	0.65	0.79	0.83	0.59	0.57	0.89	0.69
5	0.83	0.68	0.84	0.57	0.69	0.67	0.85	0.72	0.67	0.94	0.85
6	0.76	0.62	0.67	0.57	0.55	0.67	0.73	0.46	0.45	0.75	0.53
7	0.83	0.68	0.84	0.57	0.69	0.8	0.85	0.72	0.67	0.94	0.85
8	0.66	0.55	0.75	0.43	0.6	0.67	0.72	0.82	0.71	0.83	1
9	0.66	0.55	0.75	0.43	0.6	0.46	0.72	0.82	0.71	0.83	1
10	0.83	0.68	0.84	0.57	0.69	0.46	0.85	0.72	0.67	0.94	0.85
11	0.65	0.54	0.74	0.42	0.6	0.67	0.71	0.82	0.7	0.82	1
12	0.66	0.55	0.75	0.43	0.6	0.45	0.72	0.82	0.71	0.83	1
13	0.78	0.64	0.81	0.53	0.66	0.46	0.81	0.75	0.68	0.91	0.89
14	0.73	0.6	0.79	0.49	0.64	0.6	0.78	0.77	0.69	0.88	0.92
15	0.83	0.68	0.84	0.57	0.69	0.54	0.85	0.72	0.67	0.94	0.85
16	0.85	0.69	0.79	0.61	0.65	0.67	0.83	0.59	0.57	0.89	0.69
17	0.85	0.69	0.79	0.61	0.65	0.79	0.83	0.59	0.57	0.89	0.69
18	0.84	0.68	0.82	0.58	0.67	0.79	0.85	0.68	0.64	0.93	0.8
19	0.85	0.69	0.79	0.61	0.65	0.7	0.83	0.59	0.57	0.89	0.69
20	0.82	0.67	0.75	0.6	0.62	0.79	0.8	0.55	0.53	0.85	0.64
21	0.76	0.62	0.67	0.57	0.55	0.8	0.73	0.46	0.45	0.75	0.53
22	0.72	0.58	0.62	0.55	0.51	0.78	0.68	0.42	0.41	0.7	0.49

Table 3. Continuation from DMU No. 12 to DMU No. 22

DMU	12	13	14	15	16	17	18	19	20	21	22
1	0.67	0.79	0.9	0.92	0.95	0.78	1	0.82	0.97	0.88	0.8
2	0.65	0.77	0.87	0.92	0.96	0.79	1	0.85	1	0.92	0.85
3	0.76	0.88	1	0.94	0.91	0.75	1	0.74	0.87	0.74	0.67
4	0.65	0.77	0.87	0.92	0.96	0.79	1	0.85	1	0.92	0.85
5	0.76	0.88	1	0.94	0.91	0.75	1	0.74	0.87	0.74	0.67
6	0.52	0.68	0.71	0.8	0.9	0.73	0.89	0.84	1	1	0.94
7	0.76	0.88	1	0.94	0.91	0.75	1	0.74	0.87	0.74	0.67
8	0.8	0.88	1	0.79	0.69	0.57	0.81	0.52	0.61	0.48	0.43
9	0.8	0.88	1	0.79	0.69	0.57	0.81	0.52	0.61	0.48	0.48
10	0.76	0.88	1	0.94	0.91	0.75	1	0.74	0.87	0.74	0.74
11	0.8	0.87	0.99	0.78	0.68	0.56	0.8	0.51	0.6	0.47	0.47
12	0.8	0.88	1	0.79	0.69	0.57	0.81	0.52	0.61	0.48	0.48
13	0.77	0.88	1	0.89	0.84	0.69	0.94	0.66	0.78	0.64	0.64
14	0.78	0.88	1	0.86	0.78	0.64	0.89	0.61	0.71	0.58	0.58
15	0.76	0.88	1	0.94	0.91	0.75	1	0.74	0.87	0.74	0.74
16	0.65	0.77	0.87	0.92	0.96	0.79	1	0.85	1	0.92	0.92
17	0.65	0.77	0.87	0.92	0.96	0.79	1	0.85	1	0.92	0.92
18	0.73	0.85	0.96	0.93	0.92	0.76	1	0.77	0.9	0.78	0.78
19	0.65	0.77	0.87	0.92	0.96	0.79	1	0.85	1	0.92	0.92
20	0.61	0.73	0.82	0.88	0.95	0.77	0.97	0.85	1	0.94	0.94
21	0.52	0.68	0.71	0.8	0.9	0.73	0.89	0.84	1	1	1
22	0.47	0.58	0.65	0.75	0.86	0.69	0.84	0.82	0.97	1	1

For instance, the cross efficiency matrix for DMU_3 and DMU_9 is 0.68 and 0.72, respectively.

Table 3. The mean, variance and covariance of the 22 aforementioned firms

Firm	1	2	3	4	5	6	7	8	9	10	11
mean	0.78545455	0.64272727	0.7747619	0.54571429	0.63333333	0.66761905	0.79285714	0.66285714	0.61333333	0.86809524	0.78571429
variance	0.00534026	0.00298347	0.00362532	0.00416797	0.00254913	0.01568225	0.00345476	0.01512835	0.00832035	0.00467121	0.02496126
covariance	0.005097521	0.002983471	0.003460537	0.003978512	0.002433264	0.014969421	0.003297727	0.014440702	0.007942149	0.004458884	0.023826653

Table 3. The mean, variance and covariance of the 22 aforementioned firms

Firm	12	13	14	15	16	17	18	19	20	21	22
mean	0.69761905	0.81142857	0.91380952	0.87428571	0.86904762	0.71333333	0.93571429	0.73380952	0.86380952	0.76904762	0.74714286
variance	0.00995758	0.00741407	0.01207035	0.00449091	0.00989697	0.0065671	0.006279	0.01559935	0.02216472	0.03389199	0.03306169
covariance	0.009504959	0.007077066	0.004286777	0.004286777	0.009447107	0.006268595	0.005993595	0.014890289	0.021157231	0.032351446	0.031558884

According to the cross efficiency matrix of Table 3, the mean, variance and covariance of firm 3, for example, are 0.7747619, 0.00362532 and 0.003460537, respectively.

*Corresponding author. *Email address: mozzafari854@yahoo.com*

Special Issue: The Second National Conference on Applied Research in Science and Technology

http://dergi.cumhuriyet.edu.tr/cumuscij ©2015 Faculty of Science, Cumhuriyet University

Table 4. Ranking of 22 firms based on mean, variance and covariance

firm	variance	Ranking based on variance	Ranking based on variance	covariance	Ranking based on covariace
1	0.0051	15	10	0.0050975	14
2	0.0031	21	19	0.0029835	21
3	0.0036	19	11	0.0034605	19
4	0.0042	18	22	0.0039785	18
5	0.0025	22	20	0.0024333	22
6	0.0157	5	17	0.0149694	5
7	0.0035	20	8	0.0032977	20
8	0.0151	7	18	0.0144407	7
9	0.0083	11	21	0.0079421	10
10	0.0047	16	5	0.0044589	15
11	0.025	3	9	0.0238267	3
12	0.01	9	16	0.009505	8
13	0.0074	12	7	0.0070771	11
14	0.0121	8	2	0.0042868	16
15	0.0045	17	3	0.0042868	16
16	0.0099	10	4	0.0094471	9
17	0.0066	13	15	0.0062686	12
18	0.0063	14	1	0.0059936	13
19	0.0156	6	14	0.0148903	6
20	0.0222	4	6	0.0211572	4
21	0.0339	1	12	0.0323514	1
22	0.0331	2	13	0.0315589	2

According to the cross efficiency matrix the ranking of firms has been done based on the mean, variance and covariance. For instance, based on the mean, firms 4, 9, 5 achieved the first three ranks and based on the covariance firms 21, 22, 11 achieved the first three ranks.

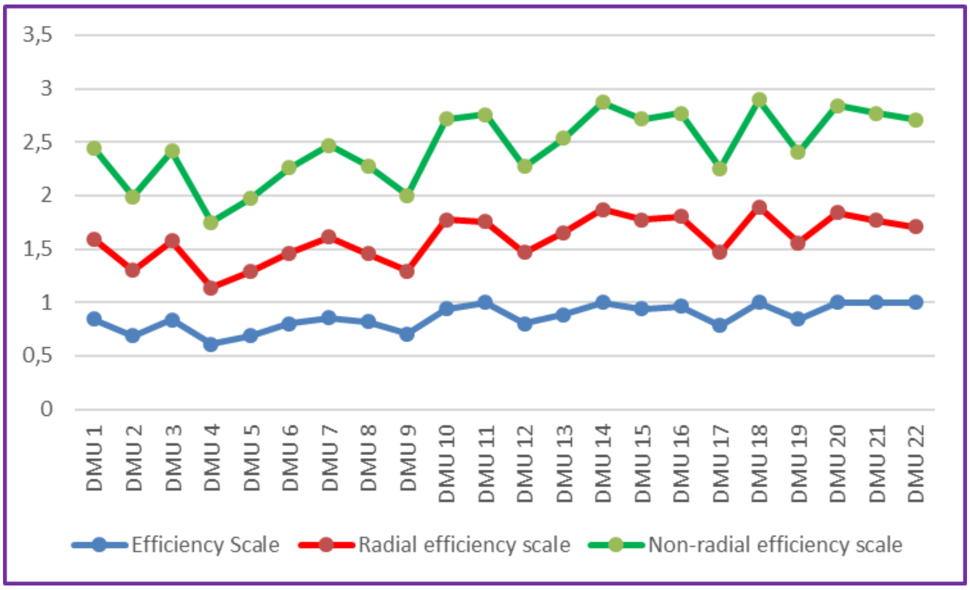

Figure 1. The Regional Diagram of 22 Firms.

*Corresponding author. *Email address: mozzafari854@yahoo.com*

Special Issue: The Second National Conference on Applied Research in Science and Technology

http://dergi.cumhuriyet.edu.tr/cumuscij ©2015 Faculty of Science, Cumhuriyet University

In the plotted figure the horizontal axis is related to the data of 22 firms according to non-radial efficiency scale shown by green color and its approximate range is around 1.5-2.5.

The blue line is related to the efficiency scale which is approximately between 0.5 and 1 and finally on the border 1 up to 1.5 the radial cross efficiency scale shown by red color can be observed.

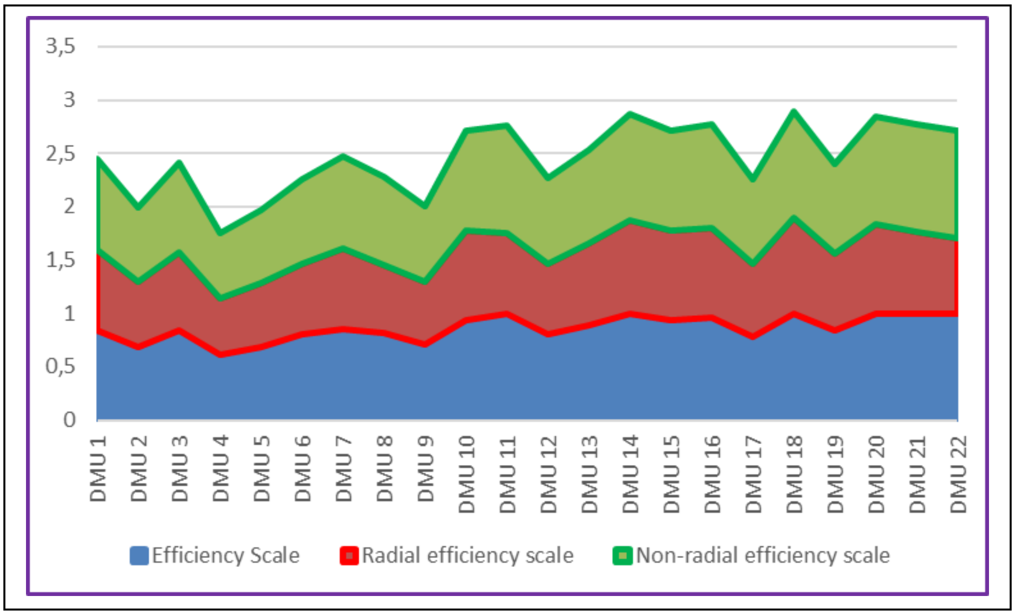

Figure 2. The Regional Diagram of 22 Firms

In Figure 2, the efficiency scale, radial cross efficiency scale and non-radial efficiency scale are shown by blue, orange and gray, respectively, and they can be observed on the approximate borders of 0.5 to 1, 1 to 1.5 and 1.5 to 2.5.

5. CONCLUSION

In the DEA cross-efficiency matrix is calculated by the fractional model which is basically obtained based on the envelopment and radial model. In this article, first, the cross-efficiency matrix of order n was obtained and then by using the mean and the variance and the covariance the ranking of each firm was determined. Finally, the optimal portfolio is decided from 22 firms. For future work, weight control and methods such as entropy and TOPSIS are recommended.

REFERENCES

[1] Andersen, P., Petersen, N. C., (1993). A procedure for ranking efficient units in data envelopment analysis. Management science, 39(10):1261-4.

[2] Doyle JR and R Green (1994). “Efficiency and Cross-Efficiency in DEA: Derivations, Meanings and Uses.” JORS 45(5): 567-578.

[3] F. Hosseinzadeh Lotfi, G. R. Jahanshahloo and M. Esmaeili,(2007). Classification of Decision Making Units with Interval Data Using SBM Model. Department of Mathematics, Azad University Research & Science Branch, Tehran, Iran.

[4] J.T. Pastor, J.L. Ruiz, I. SirventCorresponding author contact information. An enhanced DEA Russell graph efficiency measure. European Journal of Operational Research. Volume 115, Issue 3, 16 June 1999.
[5] Lim, S., Oh, K.W., Zhu, J., Use of DEA Cross-Efficiency Evaluation in Portfolio Selection:An application to Korean Stock Market, European Journal of Operational Research (2013), 10.1016/j.ejor.2013.12.002
[6] Mehrabian, S., Alirezaee, M. R., Jahanshahloo, G. R., (1999). A complete efficiency ranking decision making units in data envelopment analysis", Computational optimization and applications, 4, 261-266.
[7] Saati, S., Zarafat Angizl, M., Memariani, A., Jahanshahloo, G. R., (1999). A model for ranking decision making units in data envelopment analysis, Riceria operative, 31(97).
[8] Thomson, D.J., 1990. Time series analysis of Holocene climate data. Philosophical Transactions of the Royal Society of London, Series A 330, 601-616.
[9] Stewart, T. and others. (1996). Intelletual capital. London. Sage.