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f-Asymptotically $\mathcal{I}_2^{\sigma}$-Equivalence of Double Sequences of Sets

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Asymptotic Equivalence; $\mathcal{I}_2$-Convergence; Invariant Convergence; Wijsman Convergence; Modulus Function.

Abstract

In this study, first, we present the concepts of strongly asymptotically $\mathcal{I}_2^{\sigma}$-equivalence, f-asymptotically $\mathcal{I}_2^{\sigma}$-equivalence, strongly f-asymptotically $\mathcal{I}_2^{\sigma}$-equivalence for double sequences of sets. Then, we investigated some properties and relationships among this new concepts. After, we present asymptotically $\mathcal{I}_2^{\sigma}$-statistical equivalence for double sequences of sets. Also we investigate relationships between asymptotically $\mathcal{I}_2^{\sigma}$-statistical equivalence and strongly f-asymptotically $\mathcal{I}_2^{\sigma}$-equivalence.

Çift Küme Dizilerinin f-Asimptotik $\mathcal{I}_2^{\sigma}$-Denkliği

Anahtar kelimeler
Asimptotik Denklik; $\mathcal{I}_2$-Yakınsaklık; Invariant Yakınsaklık; Wijsman Yakınsaklık; Modülüs Fonksiyonu.

Öz

Bu çalışmada, ilk olarak, çift küme dizilerinin kuvvetli asimptotik $\mathcal{I}_2^{\sigma}$-denkliği, f-asimptotik $\mathcal{I}_2^{\sigma}$-denkliği, kuvvetli f-asimptotik $\mathcal{I}_2^{\sigma}$-denkliği kavramları tanımlandı. Daha sonra bu kavramlar arasındaki ilişkiler ve bazı özellikler incelendi. İkinci olarak, yine çift küme dizileri için asimptotik $\mathcal{I}_2^{\sigma}$-istatistiksel denklik kavramı tanımlandı. Ayrıca, asimptotik $\mathcal{I}_2^{\sigma}$-istatistiksel denklik kavramı ve kuvvetli f-asimptotik $\mathcal{I}_2^{\sigma}$-denkliği kavramı arasındaki ilişkiler incelendi.

1. Introduction and Definitions

Statistical convergence and ideal convergence of real numbers, which are of great importance in the theory of summability, are studied by many mathematicians. Fast (1951) and Schoenberg (1959), independently, introduced the concept of statistical convergence and many authors studied these concepts. Mursaleen and Edely (2009) extended this concept to the double sequences. Recently, the statistical convergence has been extended to ideal convergence of real numbers and some important properties about ideal convergence have been investigated by many mathematicians.

Kostyrko et al. (2000) defined $\mathcal{I}$ of subset of $\mathbb{N}$ (natural numbers) and investigated $\mathcal{I}$-convergence with some properties and proved theorems about $\mathcal{I}$-convergence. The idea of $\mathcal{I}_2$-convergence and some properties of this convergence were studied by Das et al. (2008). Nuray and Rhoades (2012) defined the idea of statistical convergence of set sequence and investigated some theorems about this notion and important properties. Kişi and Nuray (2013) defined Wijsman $\mathcal{I}$-convergence of sequence of sets and also examined some theorems about it. After, several authors extended the convergence of real numbers sequences to convergence of sequences of sets and investigated it's characteristic in summability.

Several authors have studied invariant convergent sequences [see, Mursaleen (1983), Nuray and Savaş (1994), Pancaroğlu and Nuray (2013a, 2013b, 2014), Raimi (1963), Savaş (1989a, 1989b), Savaş and Nuray (1993), Schaefer (1972) and Ulusu et al. (2018)]. Nuray et al. (2011) defined the notions of invariant uniform density of subsets E of $\mathbb{N}$, $\mathcal{I}_\sigma$-convergence and investigated relationships between $\mathcal{I}_\sigma$-convergence and σ-convergence also $\mathcal{I}_\sigma$-convergence and $[V_\sigma]_p$-convergence. Tortop and Dündar (2018) introduced $\mathcal{I}_2$-invariant convergence of double set sequences. Akın studied Wijsman lacunary $\mathcal{I}_2$-invariant convergence of double sequences of sets.

Asymptotically equivalent and some properties of equivalence are studied by several authors [see, Kişi et al. (2015), Pancaroğlu et al. (2013), Patterson (2003), Savaş (2013), Ulusu and Nuray (2013)]. Ulusu and Gülle introduced the concept of asymptotically $\mathcal{I}_\sigma$-equivalence of sequences of sets. Recently, Dündar et al. studied on asymptotically ideal invariant equivalence of double sequences.

Several authors define some new concepts and give inclusion theorems using a modulus function f [see, Khan and Khan (2013), Kılınç and Solak (2014), Maddox (1986), Nakano (1953), Pehlivan and Fisher(1995)]. Kumar and Sharma (2012) studied $\mathcal{I}_\theta$-equivalent sequences using a modulus function f. Kişi et al. (2015) introduced f-asymptotically $\mathcal{I}_\theta$-equivalent set sequences. Akın and Dündar (2018) and Akın et al. (2018) give definitions of f-asymptotically $\mathcal{I}_\sigma$ and $\mathcal{I}_{\sigma\theta}$-statistical equivalence of set sequences.

Now, we recall the basic concepts and some definitions and notations (see, [Baronti and Papini (1986), Beer (1985, 1994), Das et al. (2008), Dündar et al. (2016, 2017), Fast (1951), Kostyrko et al. (2000), Lorentz (1948), Marouf (1993), Mursaleen (1983), Nakano (1953), Nuray et al. (2011, 2016), Pancaroğlu and Nuray (2014), Akın and Dündar (2018), Pehlivan and Fisher (1995), Raimi (1963), Tortop and Dündar, Ulusu and Dündar (2014) and Wijsman (1964, 1966)]).

Let $u=(u_k)$ and $v=(v_k)$ be two non-negative sequences. If $\lim_k \frac{u_k}{v_k}=1$, then they are said to be asymptotically equivalent (denoted by $u \sim v$).

Let (Y,ρ) be a metric space, $y \in Y$ and E be any non-empty subset of Y, we define the distance from y to E by

$$d(y,E)=\inf_{e\in E}\rho(y,e).$$

Let σ be a mapping of the positive integers into itself. A continuous linear functional φ on ℓ_∞, the space of real bounded sequences, is said to be an invariant mean or a σ mean if and only if

1. $\phi(u) \geq 0$, when the sequence $u=(u_j)$ has $u_j \geq 0$ for all j,
2. $\phi(i)=1$, where $i=(1,1,1\ldots)$,
3. $\phi(\mathrm{u}_{\sigma(j)})=\phi(u)$, for all $u \in \ell_\infty$.

The mapping ϕ is supposed to be one-to-one and such that $\sigma^m(j)\neq j$ for all positive integers j and m, where $\sigma^m(j)$ denotes the mth iterate of the mapping σ at j. Hence, ϕ extends the limit functional on c, the space of convergent sequences, in the sense that $\phi(u)=\lim u$ for all $u \in c$. If σ is a translation mapping that is $\sigma(j)=j+1$, the σ mean is often called a Banach limit.

Let (Y,ρ) be a metric space and E, F, E_i and F_i $(i=1,2,\ldots)$ be non-empty closed subsets of Y.

Let $L \in \mathbb{R}$. Then, we define $d(y;E_i,F_i)$ as follows:

$$d(y;E_i,F_i)=\begin{cases}\dfrac{d(y,E_i)}{d(y,F_i)}, & y \notin E_i \cup F_i,\\ L, & y \in E_i \cup F_i.\end{cases}$$

Let $E_i, F_i \subseteq Y$. If for each $y \in Y$,

$$\lim_n \frac{1}{n}\sum_{i=1}^{n}\left|d(y;E_{\sigma^i(m)},F_{\sigma^i(m)})-L\right|=0,$$

uniformly in m, then, the sequences $\{E_i\}$ and $\{F_i\}$ are strongly asymptotically invariant equivalent of multiple L, (denoted by $E_i \overset{[WV]_\sigma^L}{\sim} F_i$) and if $L=1$, simply strongly asymptotically invariant equivalent.

$\mathcal{I} \subseteq 2^{\mathbb{N}}$ which is a family of subsets of $\mathbb{N}$ is called an ideal, if the followings hold:

$(i)\ \emptyset \in \mathcal{I}$, (ii) For each $E, F \in \mathcal{I}$, $E \cup F \in \mathcal{I}$, (iii) For each $E \in \mathcal{I}$ and each $F \subseteq E$, we have $F \in \mathcal{I}$.

Let $\mathcal{I} \subseteq 2^{\mathbb{N}}$ be an ideal. $\mathcal{I} \subseteq 2^{\mathbb{N}}$ is called non-trivial if $\mathbb{N} \notin \mathcal{I}$. Also, for non-trivial ideal and for each $n \in \mathbb{N}$ if $\{n\} \in \mathcal{I}$, then $\mathcal{I} \subseteq 2^{\mathbb{N}}$ is admissible ideal.

After that, we consider that $\mathcal{I}$ is an admissible ideal.

Let $K \subseteq \mathbb{N}$ and

$$s_m = \min_n |K \cap \{\sigma(n), \sigma^2(n), \dots, \sigma^m(n)\}|$$

and

$$S_m = \max_n |K \cap \{\sigma(n), \sigma^2(n), \dots, \sigma^m(n)\}|.$$

If the limits

$$\underline{V}(K) = \lim_{m\to\infty} \frac{s_m}{m} \text{ and } \overline{V}(K) = \lim_{m\to\infty} \frac{S_m}{m}$$

exists then, they are called a lower σ-uniform density and an upper σ-uniform density of the set K, respectively. If $\underline{V}(K) = \overline{V}(K)$, then $V(K) = \underline{V}(K) = \overline{V}(K)$ is called the σ-uniform density of K.

Denote by $\mathcal{I}_\sigma$ the class of all $K \subseteq \mathbb{N}$ with $V(K) = 0$. It is clearly that $\mathcal{I}_\sigma$ is admissible ideal.

If for every $\gamma > 0$, $A_\gamma = \{i : |x_i - L| \geq \gamma\}$ belongs to $\mathcal{I}_\sigma$, i.e., $V(A_\gamma) = 0$ then, the sequence $u = (u_i)$ is said to be $\mathcal{I}_\sigma$-convergent to L. It is denoted by $\mathcal{I}_\sigma - \lim u_i = L$.

Let $\{E_i\}$ and $\{F_i\}$ be two sequences. If for every $\gamma > 0$ and for each $y \in Y$,

$$A_{\gamma,y}^{\sim} = \{i : |d(y; E_i, F_i) - L| \geq \gamma\}$$

belongs to $\mathcal{I}_\sigma$, that is, $V(A_{\gamma,y}^{\sim}) = 0$ then, the sequences $\{E_i\}$ and $\{F_i\}$ are asymptotically $\mathcal{I}$-invariant equivalent or asymptotically $\mathcal{I}_\sigma$-equivalent of multiple L. In this instance, we write $E_i \overset{W_{\mathcal{I}_\sigma}^L}{\sim} F_i$ and if $L = 1$, simply asymptotically $\mathcal{I}$-invariant equivalent.

If following conditions hold for the function $f: [0, \infty) \to [0, \infty)$, then it is called a modulus function:

1. $f(u) = 0$ if and only if $u = 0$,
2. $f(u + v) \leq f(u) + f(v)$,
3. f is nondecreasing,
4. f is continuous from the right at 0.

This after, we let f as a modulus function.

The modulus function f may be unbounded (for example $f(u) = u^q$, $0 < q < 1$) or bounded (for example $f(u) = \frac{u}{u+1}$).

Let $\{E_i\}$ and $\{F_i\}$ be two sequences. If for every $\gamma > 0$ and for each $y \in Y$,

$$\left\{n \in \mathbb{N}: \frac{1}{n}\sum_{i=1}^{n} |d(y; E_i, F_i) - L| \geq \gamma\right\} \in \mathcal{I}_\sigma,$$

then, $\{E_i\}$ and $\{F_i\}$ are strongly asymptotically $\mathcal{I}$-invariant equivalent of multiple L (denoted by $E_i \overset{[W_{\mathcal{I}_\sigma}^L]}{\sim} F_i$) and if $L = 1$, simply strongly asymptotically $\mathcal{I}_\sigma$-equivalent.

If for every $\gamma > 0$ and for each $y \in Y$,

$$\{i \in \mathbb{N}: f(|d(y; E_i, F_i) - L|) \geq \gamma\} \in \mathcal{I}_\sigma$$

then, we say that the sequences $\{E_i\}$ and $\{F_i\}$ are said to be f-asymptotically $\mathcal{I}$-invariant equivalent of multiple L (denoted by $E_i \overset{W_{\mathcal{I}_\sigma}^L(f)}{\sim} F_i$) and if $L = 1$ simply f-asymptotically $\mathcal{I}$-invariant equivalent.

Let $\{E_i\}$ and $\{F_i\}$ be two sequences. If for every $\gamma > 0$ and for each $y \in Y$,

$$\left\{n \in \mathbb{N}: \frac{1}{n}\sum_{i=1}^{n} f(|d(y; E_i, F_i) - L|) \geq \gamma\right\} \in \mathcal{I}_\sigma$$

then, we say that the sequences $\{E_i\}$ and $\{F_i\}$ are said to be strongly f-asymptotically $\mathcal{I}$-invariant equivalent of multiple L (denoted by $E_i \overset{[W_{\mathcal{I}_\sigma}^L(f)]}{\sim} F_i$) and if $L = 1$, simply strongly f-asymptotically $\mathcal{I}$-invariant equivalent.

Let $\{E_i\}$ and $\{F_i\}$ be two sequences. If for every $\gamma > 0$ and for each $y \in Y$,

$$\left\{n \in \mathbb{N}: \frac{1}{n}|\{i \leq n: |d(y; E_i, F_i) - L| \geq \gamma\}| \geq \gamma\right\} \in \mathcal{I}_\sigma$$

then, we say that the sequences $\{E_i\}$ and $\{F_i\}$ are asymptotically $\mathcal{I}$-invariant statistical equivalent of multiple L (denoted by $E_i \overset{W_{\mathcal{I}_\sigma}^L(S)}{\sim} F_i$) and if $L = 1$, simply asymptotically $\mathcal{I}$-invariant statistical equivalent.

Let $\mathcal{J}_2$ be a nontrivial ideal of $\mathbb{N}\times\mathbb{N}$. It is called strongly admissible ideal if $\{k\}\times\mathbb{N}$ and $\mathbb{N}\times\{k\}$ belong to $\mathcal{J}_2$ for each $k\in N$. This after, we let $\mathcal{J}_2$ as a strongly admissible ideal in $\mathbb{N}\times\mathbb{N}$.

If we let a ideal as a strongly admissible ideal then, it is clear that it is admissible also.

Let $\mathcal{J}_2^0=\{E\subset\mathbb{N}\times\mathbb{N}:(\exists\ i(E)\in\mathbb{N})(r,s\geq i(E)\Rightarrow(r,s)\notin E)\}$.

It is clear that $\mathcal{J}_2^0$ is a strongly admissible ideal. Also, it is evidently $\mathcal{J}_2$ is strongly admissible if and only if $\mathcal{J}_2^0\subset\mathcal{J}_2$.

Let (Y,ρ) be a metric space and $y=(y_{ij})$ be a sequence in Y. If for any $\gamma>0$,

$$A(\gamma)=\{(i,j)\in\mathbb{N}\times\mathbb{N}:\rho(y_{ij},L)\geq\gamma\}\in\mathcal{J}_2$$

then, it is said to be $\mathcal{J}_2$-convergent to L. In this instance, y is $\mathcal{J}_2$-convergent and we write

$$\mathcal{J}_2-\lim_{i,j\to\infty}y_{ij}=L.$$

Let $E\subseteq\mathbb{N}\times\mathbb{N}$ and

$$s_{mk}:\min_{i,j}|\mathrm{E}\cap\{(\sigma(i),\sigma(j)),(\sigma^2(i),\sigma^2(j)),\ldots,(\sigma^m(i),\sigma^k(j))\}|$$

and

$$S_{mk}:\max_{i,j}|E\cap\{(\sigma(i),\sigma(j)),(\sigma^2(i),\sigma^2(j)),\ldots,(\sigma^m(i),\sigma^k(j))\}|.$$

If the limits

$$\underline{V_2}(E):=\lim_{m,k\to\infty}\frac{s_{mk}}{mk},\quad \overline{V_2}(E):=\lim_{m,k\to\infty}\frac{S_{mk}}{mk}$$

exists then $\underline{V_2}(E)$ is called a lower and $\overline{V_2}(E)$ is called an upper σ-uniform density of the set E, respectively. If $\underline{V_2}(E)=\overline{V_2}(E)$ holds then, $V_2(E)=\underline{V_2}(E)=\overline{V_2}(E)$ is called the σ-uniform density of E.

Denote by $\mathcal{J}_2^\sigma$ the class of all $E\subseteq\mathbb{N}\times\mathbb{N}$ with $V_2(E)=0$.

This after, let (Y,ρ) be a separable metric space and $E_{\mathrm{ij}},F_{ij},E,F$ be any nonempty closed subsets of Y.

If for each $y\in Y$,

$$\lim_{m,k\to\infty}\frac{1}{mk}\sum_{i,j=1,1}^{m,k}d\big(y,E_{\sigma^i(s),\sigma^j(t)}\big)=d(y,E),$$

uniformly in *s,t* then, the double sequence $\{E_{ij}\}$ is said to be invariant convergent to E in Y.

If for every $\gamma>0$,

$$A(\gamma,y)=\{(i,j):|d(y,E_{ij})-d(y,E)|\geq\gamma\}\in\mathcal{J}_2^\sigma$$

that is, $V_2(A(\gamma,y))=0$, then, the double sequence $\{E_{ij}\}$ is said to be Wijsman $\mathcal{J}_2$-invariant convergent or $\mathcal{J}_{W_2}^\sigma$-convergent to E, In this instance, we write $E_{ij}\to E(\mathcal{J}_{W_2}^\sigma)$ and by $\mathcal{J}_{W_2}^\sigma$ we will denote the set of all Wijsman $\mathcal{J}_2^\sigma$-convergent double sequences of sets.

For non-empty closed subsets E_{ij}, F_{ij} of Y define $d(y;E_{ij},F_{ij})$ as follows:

$$d(y;E_{ij},F_{ij})=\begin{cases}\dfrac{d(y,E_{ij})}{d(y,F_{ij})}, & y\notin E_{ij}\cup F_{ij}\\ L, & y\in E_{ij}\cup F_{ij}.\end{cases}$$

Lemma 1. *[*Pehlivan and Fisher, 1995*]*

Let $0<\gamma<1$. *Thus, for each* $u\geq\gamma$, $f(u)\leq 2f(1)\gamma^{-1}u$.

2. *f-Asymptotically $\mathcal{J}_2^\sigma$-Equivalence of Double Sequences of Sets*

Definition 2.1 If for every $\gamma>0$ and each $y\in Y$,

$$\left\{(m,k):\in\mathbb{N}\times\mathbb{N}:\frac{1}{mk}\sum_{i,j=1,1}^{m,k}|d(y;E_{ij},F_{ij})-L|\geq\gamma\right\}\in\mathcal{J}_2^\sigma$$

then, double sequences $\{E_{ij}\}$ and $\{F_{ij}\}$ are said to be strongly asymptotically $\mathcal{J}_2$-invariant equivalent of multiple L (denoted by $E_{ij}\overset{[W_{\mathcal{J}_2^\sigma}^L]}{\sim}F_{ij}$) and if $L=1$, simply strongly asymptotically $\mathcal{J}_2^\sigma$-equivalent.

Definition 2.2 If for every $\gamma>0$ and each $y\in Y$,

$$\{(i,j)\in\mathbb{N}\times\mathbb{N}:f(|d(y;E_{ij},F_{ij})-L|)\geq\gamma\}\in\mathcal{J}_2^\sigma$$

then, the double sequences $\{E_{ij}\}$ and $\{F_{ij}\}$ are said to be f-asymptotically $\mathcal{J}_2$-invariant equivalent of multiple L (denoted by $E_{ij}\overset{W_{\mathcal{J}_2^\sigma}^L(f)}{\sim}F_{ij}$) and if $L=1$, simply f-asymptotically $\mathcal{J}_2^\sigma$-equivalent.

Definition 2.3 If for every $\gamma>0$ and each $y\in Y$,

$$\left\{(m,k)\in\mathbb{N}\times\mathbb{N}:\frac{1}{mk}\sum_{i,j=1,1}^{mk} f(|d(y;E_{ij},F_{ij})-L|)\geq\gamma\right\}\in\mathcal{I}_2^\sigma$$

then, the double sequences $\{E_{ij}\}$ and $\{F_{ij}\}$ are said to be strongly f-asymptotically $\mathcal{I}_2^\sigma$-equivalent of multiple L (denoted by $E_{ij}\overset{[W_{\mathcal{I}_2^\sigma}^L(f)]}{\sim}F_{ij}$) and if $L=1$, simply strongly f-asymptotically $\mathcal{I}_2^\sigma$-equivalent.

Theorem 2.1 Let $0<\delta<1$ and $\gamma>0$ such that $f(z)<\gamma$ for $0\leq z\leq\delta$. Then, we have

$$E_{ij}\overset{[W_{\mathcal{I}_2^\sigma}^L]}{\sim}F_{ij}\Rightarrow E_{ij}\overset{[W_{\mathcal{I}_2^\sigma}^L(f)]}{\sim}F_{ij}.$$

Proof. Let $E_{kj}\overset{[W_{\mathcal{I}_2^\sigma}^L]}{\sim}F_{kj}$ and $\gamma>0$. Select $0<\delta<1$ such that $f(z)<\gamma$ for $0\leq z\leq\delta$. Then, for each $x\in X$ and for $s,t=1,2,...$, we have

$$\frac{1}{mk}\sum_{i,j=1}^{m,k} f(\left|d(y;E_{\sigma^i(s)\sigma^j(t)},F_{\sigma^i(s)\sigma^j(t)})-L\right|)$$

$$=\frac{1}{mk}\sum_{\substack{i,j=1,1\\ \left|d(y;E_{\sigma^i(s)\sigma^j(t)},F_{\sigma^i(s)\sigma^j(t)})-L\right|\leq\delta}}^{m,k} f(|d(y;E_{\sigma^i(s)\sigma^j(t)},F_{\sigma^i(s)\sigma^j(t)})-L|)$$

$$+\frac{1}{mk}\sum_{\substack{i,j=1,1\\ \left|d(y;E_{\sigma^i(s)\sigma^j(t)},F_{\sigma^i(s)\sigma^j(t)})-L\right|>\delta}}^{m,k} f(|d(y;E_{\sigma^i(s)\sigma^j(t)},B_{\sigma^i(s)\sigma^j(t)})-L|)$$

and so by Lemma 1

$$\frac{1}{mk}\sum_{i,j=1,1}^{m,k} f(\left|d(y;E_{\sigma^i(s)\sigma^j(t)},F_{\sigma^i(s)\sigma^j(t)})-L\right|)$$

$$<\gamma+(\frac{2f(1)}{\delta})\frac{1}{mk}\sum_{i,j=1,1}^{m,k}\left|d(y;E_{\sigma^i(s)\sigma^j(t)},F_{\sigma^i(s)\sigma^j(t)})-L\right|,$$

uniformly in s,t. Thus, for every $\varepsilon>0$ and for each $y\in Y$,

$$\left\{(m,k):\frac{1}{mk}\sum_{i,j=1,1}^{m,k} f(|d(y;E_{\sigma^i(s)\sigma^j(t)},F_{\sigma^i(s)\sigma^j(t)})-L|)\geq\varepsilon\right\}$$

$$\subseteq\left\{(m,k):\frac{1}{mk}\sum_{i,j=1,1}^{m,k}|d(y;E_{\sigma^i(s)\sigma^j(t)},F_{\sigma^i(s)\sigma^j(t)})-L|\geq\frac{(\varepsilon-\gamma)\delta}{2f(1)}\right\},$$

uniformly in s,t. Since $E_{ij}\overset{[W_{\mathcal{I}_2^\sigma}^L]}{\sim}F_{ij}$ then, it is clear that the later set belongs to $\mathcal{I}_2^\sigma$ and thus, the first set belongs to $\mathcal{I}_2^\sigma$. This proves that $E_{ij}\overset{[W_{\mathcal{I}_2^\sigma}^L(f)]}{\sim}F_{ij}$.

Theorem 2.2 Let $z\in Y$. If $\lim_{z\to\infty}\frac{f(z)}{z}=\alpha>0$, then

$$E_{ij}\overset{[W_{\mathcal{I}_2^\sigma}^L]}{\sim}F_{ij}\Leftrightarrow E_{ij}\overset{[W_{\mathcal{I}_2^\sigma}^L(f)]}{\sim}F_{ij}.$$

Proof. The necessity is obvious from the Theorem 2.1.

If $\lim_{z\to\infty}\frac{f(z)}{z}=\alpha>0$, then we have $f(z)\geq\alpha z$ for all $z\geq 0$. Assume that $E_{ij}\overset{[W_{\mathcal{I}_2^\sigma}^L(f)]}{\sim}F_{ij}$. Since for each $y\in Y$ and for $s,t=1,2,\ldots$ we have

$$\frac{1}{mk}\sum_{i,j=1,1}^{m,k} f(\left|d(y;E_{\sigma^i(s)\sigma^j(t)},F_{\sigma^i(s)\sigma^j(t)})-L\right|)$$

$$\geq\frac{1}{mk}\sum_{i,j=1,1}^{m,k}\alpha(\left|d(y;E_{\sigma^i(s)\sigma^j(t)},F_{\sigma^i(s)\sigma^j(t)})-L\right|)$$

$$=\alpha\left(\frac{1}{mk}\sum_{i,j=1,1}^{m,k}\left|d(y;E_{\sigma^i(s)\sigma^j(t)},F_{\sigma^k(s)\sigma^j(t)})-L\right|\right),$$

and so, for every $\gamma>0$

$$\left\{(m,k):\frac{1}{mk}\sum_{i,j=1,1}^{m,k}|d(y;E_{\sigma^i(s)\sigma^j(t)},F_{\sigma^i(s)\sigma^j(t)})-L|\geq\gamma\right\}$$

$$\subseteq\left\{(m,k):\frac{1}{mk}\sum_{i,j=1,1}^{m,k} f(|d(y;E_{\sigma^i(s)\sigma^j(t)},F_{\sigma^i(s)\sigma^j(t)})-L|)\geq\alpha\gamma\right\},$$

uniformly in s,t. Since $E_{kj}\overset{[W_{\mathcal{I}_2^\sigma}^L(f)]}{\sim}F_{kj}$, then, later set belongs to $\mathcal{I}_2^\sigma$. This proves that

$$E_{ij}\overset{[W_{\mathcal{I}_2^\sigma}^L]}{\sim}F_{ij}\Leftrightarrow E_{ij}\overset{[W_{\mathcal{I}_2^\sigma}^L(f)]}{\sim}F_{ij}.$$

Definition 2.4 If for every $\gamma>0$, $\delta>0$ and for each $x\in X$,

$$\left\{(m,k):\frac{1}{mk}|\{i\leq m,j\leq k:|d(y;E_{ij},E_{ij})-L|\geq\gamma\}|\geq\delta\right\}\in\mathcal{I}_2^\sigma$$

then, the double sequences $\{E_{ij}\}$ and $\{F_{ij}\}$ are said to be asymptotically $\mathcal{I}_2$-invariant statistical equivalent of multiple L (denoted by $E_{ij}\overset{W_{\mathcal{I}_2^\sigma}^L(S)}{\sim}F_{ij}$) and if $L=1$, simply asymptotically $\mathcal{I}_2$-invariant statistical equivalent.

Theorem 2.3 For each $y\in Y$, following holds:

$$E_{kj}\overset{[W_{\mathcal{I}_2^\sigma}^L(f)]}{\sim}F_{kj}\Rightarrow E_{kj}\overset{W_{\mathcal{I}_2^\sigma}^L(S)}{\sim}F_{kj}.$$

Proof. Assume that $E_{ij} \overset{[W^L_{\mathcal{I}_2^\sigma}(f)]}{\sim} F_{ij}$ and $\gamma > 0$ be given. Since for each $y \in Y$ and for $s,t = 1,2,\ldots$

$$\frac{1}{mk}\sum_{i,j=1,1}^{m,k} f(|d(y;E_{\sigma^i(s)\sigma^j(t)},F_{\sigma^i(s)\sigma^j(t)})-L|)$$

$$\geq \frac{1}{mk}\sum_{\substack{i,j=1,1\\ |d(y;E_{\sigma^i(s)\sigma^j(t)},F_{\sigma^i(s)\sigma^j(t)})-L|\geq\gamma}}^{m,k} f(|d(y;E_{\sigma^i(s)\sigma^j(t)},B_{\sigma^i(s)\sigma^j(t)})-L|)$$

$$\geq f(\gamma).\frac{1}{mk}|\{i\leq m, j\leq k: |d(y;E_{\sigma^i(s)\sigma^j(t)},F_{\sigma^i(s)\sigma^j(t)})-L|\geq\gamma\}|,$$

it follows that for every $\delta > 0$ and for each $y \in Y$,

$$\{(m,k): \frac{1}{mk}|\{i\leq m, j\leq k: |d(y;E_{\sigma^i(s)\sigma^j(t)},F_{\sigma^i(s)\sigma^j(t)})-L|\geq\gamma\}|\geq\frac{\delta}{f(\gamma)}\}$$

$$\subseteq\left\{(m,k): \frac{1}{mk}\sum_{i,j=1,1}^{m,k} f(|d(y;E_{\sigma^i(s)\sigma^j(t)},F_{\sigma^i(s)\sigma^j(t)})-L|)\geq\delta\right\},$$

uniformly in s,t. Since $E_{ij} \overset{[W^L_{\mathcal{I}_2^\sigma}(f)]}{\sim} F_{ij}$, then it is clear the later set belongs to $\mathcal{I}_2^\sigma$. Then, the first set belongs to $\mathcal{I}_2^\sigma$ and so, $E_{ij} \overset{W^L_{\mathcal{I}_2^\sigma}(S)}{\sim} F_{ij}$.

Theorem 2.4 If f is bounded, then for each $y \in Y$,

$$\mathrm{E_{ij}} \overset{[W^L_{\mathcal{I}_2^\sigma}(f)]}{\sim} F_{ij} \Leftrightarrow E_{ij} \overset{W^L_{\mathcal{I}_{\sigma_2}}(S)}{\sim} F_{ij}.$$

Proof. Suppose that f is bounded and let $E_{ij} \overset{W^L_{\mathcal{I}_2^\sigma}(S)}{\sim} F_{ij}$. Because f is bounded then, there exists a real number $T > 0$ such that $\sup f(z) \leq T$ for all $y \geq 0$. More using the truth, for $s,t = 1,2,\ldots$ we have

$$\frac{1}{mk}\sum_{i,j=1,1}^{m,k} f(|d(y;E_{\sigma^i(s)\sigma^j(t)},F_{\sigma^i(s)\sigma^j(t)})-L|)$$

$$=\frac{1}{mk}\sum_{\substack{i,j=1,1\\ |d(y;E_{\sigma^i(s)\sigma^j(t)},F_{\sigma^i(s)\sigma^j(t)})-L|\geq\gamma}}^{m,k} f(|d(y;E_{\sigma^i(s)\sigma^j(t)},F_{\sigma^i(s)\sigma^j(t)})-L|)$$

$$+\frac{1}{mk}\sum_{\substack{i,j=1,1\\ |d(y;E_{\sigma^i(s)\sigma^j(t)},F_{\sigma^i(s)\sigma^j(t)})-L|<\gamma}}^{m,k} f(|d(y;E_{\sigma^i(s)\sigma^j(t)},F_{\sigma^i(s)\sigma^j(t)})-L|)$$

$$\leq\frac{\boldsymbol{T}}{\boldsymbol{mk}}|\{\boldsymbol{i}\leq\boldsymbol{m},\boldsymbol{j}\leq\boldsymbol{k}: |\boldsymbol{d}(\boldsymbol{y};\boldsymbol{E}_{\sigma^i(s)\sigma^j(t)},\boldsymbol{B}_{\sigma^i(s)\sigma^j(t)})-\boldsymbol{L}|\geq\boldsymbol{\gamma}\}|+\boldsymbol{f}(\boldsymbol{\gamma}),$$

uniformly in s,t. This proves that $E_{ij} \overset{[W^L_{\mathcal{I}_2^\sigma}(f)]}{\sim} F_{ij}$.

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