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VE-DEGREES IN SOME GRAPH PRODUCTS

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ABSTRACT

Let G be a graph and v be a vertex of G. The *ve*-degree of the vertex v defined as the number of different edges incident to the vertices of the open neighborhood of v. In this study we investigate the *ve*-degrees in Cartesian, direct and strong products of two graphs.

1. INTRODUCTION

We consider only connected and simple graphs throughout this paper. Let G be a graph with the vertex set $V(G)$, the edge set $E(G)$ and $v \in V(G)$. The degree of a vertex $v \in V(G)$, $deg(v)$, equals the number of edges incident to v that is the cardinality of the set $N(v) = \{u|uv \in E(G)\}$. This set is named as "the open neighborhood of v". For the vertex v, n_v denotes the number of triangles which contain the vertex v. Let A and B be two non-empty sets. Then the Cartesian product of these sets is the set $A\Box B = \{(a,b)|a \in A \,\text{and}\, b \in B\}$. The Cartesian, direct and strong products of two graphs G and H have the vertex set $V(G)\Box V(H)$. The edge set of the Cartesian product of G and H is $E(G\Box H)= \{(a,b)(c,d)|a = c$ and $bd \in E(H)$ or $b = d$ and $ac \in E(G)\}$. The edge set of the direct product of G and H is $E(G \times H) = \{(a,b)(c,d)|ac \in E(G) \,\text{and}\, bd \in E(H)\}$. And the edge set of the strong product of G and H is $E(GH)= \{(a,b)(c,d)|a = c$ and $bd \in E(H)$ or $b = d$ and $ac \in E(G)$ or $ac \in E(G)$ and $bd \in E(H)\}$. Note that $E(GH) = E(G\Box H) \cup E(G \times H)$ and $E(G\Box H) \cap E(G \times H) = \emptyset$. The following Lemma 1 gives the degrees in above mentioned graph products.

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Lemma 1. *Let G and H be two graphs. Then;*
(a) $deg_{G\Box H}(a,b) = deg(a) + deg(b)$,
(b) $deg_{G\times H}(a,b) = deg(a)deg(b)$,
(c) $deg_{GH}(a,b) = deg(a) + deg(b) + deg(a)deg(b)$.

Very recently Chellali *et al* have published very seminal study about the novel two degree concepts: *ve*-degrees and *ev*-degrees in graph theory [**1**]. The authors defined these novel degree concepts in relation to the vertex-edge domination and the edge-vertex domination parameters[**2, 3, 4**]. And also, the authors defined the *ve*-regularity, the *ev*-regularity and investigated basic mathematical properties of *ev* and *ve* regularities of graphs [**1**]. The *ev*-degree and *ve*-degree topological indices have been defined and their basic mathematical properties have been investigated in [**5, 6**]. In this paper we investigate the *ve*-degrees in Cartesian, direct and strong product of two graphs.

2. THE VE-DEGREES IN GRAPH PRODUCTS

We firstly give some basic facts related to *ve*-degrees.

Definition 2. *[**1**] Let G be a connected graph and $v \in V(G)$. The ve-degree of the vertex v, $deg_{ve}(v)$, equals the number of different edges that incident to any vertex from the closed (or open) neighborhood of v.*

Definition 3. *[**1**] Let G be a connected graph and $e = uv \in E(G)$. The ev-degree of the edge e, $deg_{ev}(e)$, equals the number of vertices of the union of the closed (or open) neighborhoods of u and v.*

Lemma 4. *[**5**] Let G be a connected graph and $v \in V(G)$, then;*

$$deg_{ve}(v) = \sum_{u\in N(v)} deg(u) - n_v$$

where n_v denotes the number of triangles contain the vertex v.

And now, we begin to compute the *ve*-degrees in Cartesian product of two graphs.

Lemma 5. *Let G and H be two graphs. Then both G and H contain no triangle if and only if $G\Box H$ contains no triangle*

Proof. Let accept that G and H be two triangle-free graphs and $G\Box H$ contains a triangle, say $(a,u)(b,u)(c,u)$. Then absolutely abc must be a triangle in G. This is a contradiction. The same argument holds for the supposition that $(v,a)(v,b)(v,c)$ a triangle in $G\Box H$. The other part of the proof comes from the fact that if $G\Box H$ contains a triangle then either G or H must be contain a triangle. $\Box$

Lemma 6. *Let G and H be two connected graphs and $(a,b) \in G\Box H$. Then;*

$$n_{(a,b)} = n_a + n_b$$

Proof. Let auv be one of the triangle which contains the vertex a in G. Then $(a,b)(u,b)(v,b)$ must be a triangle in $G\Box H$ from the definition of Cartesian product of graphs. Similarly let brt be one of the triangle which contains the vertex b in H. Then $(a,b)(a,r)(a,t)$ must be a triangle in $G\Box H$. The same argument holds for the all triangles which contain the vertex a and the vertex b in G and H, respectively. From the Lemma 5 the desired result acquired. □

Proposition 7. *Let $(a,b) \in V(G\Box H)$. Then;*

$$\sum_{(c,d)\in N((a,b))} deg(c,d) = 2deg(a)deg(b) + deg_{ve}a + deg_{ve}b + n_a + n_b.$$

Proof. Let $N(a) = \{v_1, v_2, ..., v_n\}$ and $N(b) = \{u_1, u_2, ..., u_m\}$. From the definition of the Cartesian product of graphs, we can write that;

$$\sum_{(c,d)\in N((a,b))} deg(c,d) = \sum_{(a=c),d\in N(b)} deg(a,d) + \sum_{(b=d),c\in N(a)} deg(c,b)$$

$= deg(a,u_1) + deg(a,u_2) + ... + deg(a,u_m) + deg(b,v_1) + deg(b,v_2) + ... + deg(b,v_n)$
$= deg(a)deg(b) + \sum_{d\in N(b)} deg(d) + deg(b)deg(a) + \sum_{c\in N(a)} deg(c)$.
From the Lemma 4 we know that $\sum_{u\in N(v)} deg(u) = deg_{ve}(v) + n_v$. Therefore, from the last equality we get that;
$= 2deg(a)deg(b) + deg_{ve}a + deg_{ve}b + n_a + n_b$. □

Corollary 8. *Let $(a,b) \in V(G\Box H)$. Then;*

$$deg_{ve}(a,b) = deg_{ve}a + deg_{ve}b + 2deg(a)deg(b).$$

Proof. The proof comes from the Lemma 4, Lemma 6 and Proposition 7. □

And now we can investigate the ve-degrees in direct(tensor, kronocker) product of two arbitrary graphs. We know that the direct product of two graphs is connected as far as at least one of the two graphs contains an odd cycle. But here, we focus degree not connectivity.

Lemma 9. *Let G and H be two connected simple graphs. If both G and H contain triangles then $G \times H$ contains triangles.*

Proof. Clearly if both G and H contain no triangle then $G \times H$ must not contain no triangle. Let accept that G contain a triangle and H contain no triangle. Then we can said from the definition of the direct product of graphs that $G \times H$ contains no triangle. Let accept that both G and H contain exactly one triangle, say uvw and brt respectively. Then $(ub)(vr)(wt)$ and $(ub)(vt)(wr)$ are the corresponding triangles in $G \times H$. □

Lemma 10. *Let G and H be two connected graphs and $(a,b) \in G \times H$. Then;*

$$n_{(a,b)} = 2n_a n_b$$

Proof. Let $n_a = k$ and $n_b = l$ for the positive natural numbers $k, l > 0$. And let $u_1v_1w_1$, $u_2v_2w_2$, ..., $u_kv_kw_k$ be the triangles in G and $b_1r_1t_1$, $b_2r_2t_2$, ..., $b_lr_lt_l$ be the triangles in H. Then the pairs of triangles $(u_1b_1)(v_1r_1)(w_1t_1)$ and $(u_1b_1)(v_1t_1)(w_1r_1)$, $(u_1b_2)(v_1r_2)(w_1t_2)$ and $(u_1b_2)(v_1t_2)(w_1r_2)$,..., $(u_1b_l)(v_1r_l)(w_1t_l)$ and $(u_1b_l)(v_1t_l)(w_1r_l)$ are corresponding triangles of the triangle $u_1v_1w_1$. That is exactly there are $2l$ triangles in $G \times H$ for the the triangle $u_1v_1w_1$. The same argument holds for the other triangles of G. Therefore, there are exactly $2kl$ triangles in $G \times H$. □

Proposition 11. *Let $(a,b) \in V(G \times H)$. Then;*

$$\sum_{(c,d)\in N((a,b))} deg(c,d) = (deg_{ve}a + n_a)(deg_{ve}b + n_b)$$

Proof. Let $N(a) = \{v_1, v_2, ..., v_n\}$ and $N(b) = \{u_1, u_2, ..., u_m\}$. From the definition of the direct product of graphs, we can write that;

$$\sum_{(c,d)\in N((a,b))} deg(c,d) = \sum_{(c,d)\in N((a,b))} deg(c)deg(d)$$

$= (deg(v_1) + deg(v_2) + ... + deg(v_n))(deg(u_1) + deg(u_2) + ... + deg(u_m))$
From the Lemma 4 we know that $\sum_{u\in N(v)} deg(u) = deg_{ve}(v) + n_v$. Therefore, from the last equality we get that;
$= (deg_{ve}(a) + n_a)(deg_{ve}b + n_b)$. □

Corollary 12. *Let $(a,b) \in V(G \times H)$. Then;*

$$deg_{ve}(a,b) = (deg_{ve}a + n_a)(deg_{ve}b + n_b) - 2n_a n_b$$

Proof. The proof comes from the Lemma 4, Lemma 10 and Proposition 11. □

We begin to investigate ve-degrees in strong product of two graphs.

Lemma 13. *Let G and H be two connected graphs and $(a,b) \in GH$. Then;*

$$n_{(a,b)} = n_a(1 + |E(H)|) + n_b(1 + |E(G)|) + 2n_a n_b + 2deg(a)deg(b)$$

Proof. The triangles belong to $G\Box H$ and $G \times H$ are in the GH. There are new two kind of triangles in GH. The first group consists of two Cartesian (product) edges and one direct (product) edge. And the second group consists of one Cartesian (product) edge and two direct (product) edges.
(a) *2 Cartesian 1 direct triangles*: Let $(a,b) \in GH$, $N(a) = \{v_1, v_2, ..., v_n\}$ and $N(b) = \{u_1, u_2, ..., u_m\}$. There are m four cycles such as; ab, au_1, v_1b, v_1u_1, ab, au_2, v_1b, v_1u_2,..., ab, au_m, v_1b, v_1u_m for the Cartesian product of the edges av_1 and bu_i for the $1 \le i \le m$. And the same argument holds for Cartesian product of all the

edges av_i and bu_j. Therefore, there are $nm = deg(a)deg(b)$ four cycles in $G\Box H$ including the vertex (a, b). Note that every four cycle consists of only the Cartesian product edges in GH has two triangles with two cartesian edges and one direct edges. For example there are two triangles, namely ab, v_1u_1, v_1b and ab, au_1, v_1b in the four cycle ab, au_1, v_1b, v_1u_1 in GH. Thus, there are $2deg(a)deg(b) = 2nm$ triangles with two Cartesian edges and one direct edge in GH including the vertex (a, b).

(b) *1 Cartesian 2 direct triangles*: Let $(a, b) \in GH$. If $n_a = n_b = 0$ then there is no any triangle with two direct edge and one Cartesian edge in GH. Let accept that $n_a = 1$, av_1v_2 be triangle in G and bu_1 be an edge of H. Note that the triangle ab, v_1u_1, v_2u_1 is the only triangle, which consists of two direct edge and one Cartesian edge in GH, contains the vertex (a, b). Therefore, there are $|E(H)|$ triangle, which consists of two direct edge and one Cartesian edge in GH, contains the vertex (a, b). Clearly if $n_a = k \geq 1$ then there are $k|E(H)|$ triangle, which consists of two direct edge and one Cartesian edge in GH, contain the vertex (a, b). The same argument of the vertex a holds for the vertex b. Thus, the proof is completed from these facts and Lemma 6 and Lemma 10. □

Proposition 14. *Let $(a, b) \in V(GH)$. Then;*
$\sum_{(c,d)\in N((a,b))} deg(c, d) = 2deg(a)deg(b) + deg_{ve}a + deg_{ve}b + (deg_{ve}a + n_a)(deg_{ve}b + n_b) + n_a + n_b$

Proof. We know that $E(GH) = E(G\Box H) \cup E(G \times H)$. From this and Propositions 7 and 11, we get the desired result. □

Corollary 15. *Let $(a, b) \in V(GH)$. Then;*
$deg_{ve}(a, b) = deg_{ve}a + deg_{ve}b + (deg_{ve}a + n_a)(deg_{ve}b + n_b) - 2n_an_b - n_a|E(H)| - n_b|E(G)|$

Proof. The proof comes from the Lemma 4, Lemma 13 and Proposition 14. □

3. CONCLUSION

There are many open problems related to ve-degrees for further studies. It can be interesting to compute the exact values of ve-degrees in some other graph operations.

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