Generalized omni-Lie algebras

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Abstract

We introduce the notion of generalized omni-Lie algebras from omni-Lie algebras constructed by Weinstein. We prove that there is a one-to-one correspondence between Dirac structures of a generalized omni-Lie algebra and Lie structures on its linear space.

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1. Introduction

The notion of omni-Lie algebras was introduced by Weinstein[7], which is the linearization of the Courant bracket. Let $V$ be a linear space, and an omni-Lie algebra is the direct sum space $gl(V) \oplus V$ with a skew-symmetric bracket operation $\langle \cdot, \cdot \rangle$ and a non-degenerate symmetric bilinear pairing $\langle \cdot, \cdot \rangle$ given by

$$\langle A + x, B + y \rangle = \frac{1}{2}(Ay - Bx),$$

and

$$\langle A + x, B + y \rangle = \frac{1}{2}(Ay + Bx).$$

An omni-Lie algebra is not a Lie algebra, but its Dirac structures are Lie algebras. Actually, an omni-Lie algebra is a Lie 2-algebra since Roytenberg and Weinstein proved that every Courant algebroid gives rise to a Lie 2-algebra[5]. Recently, omni-Lie algebras were generalized to omni-Lie superalgebras, omni-Lie color algebras and omni-Lie algebroids[1,8]. In[2], they generalized omni-Lie algebras from a linear space to a linear bundle $E$ in order to characterize all possible Lie algebroid structures on $E$. Dirac structures were also studied from several aspects[2,3,6].

In this paper, we introduce the notion of a generalized omni-Lie algebra, which is the $(\delta, \alpha)$ omni-Lie algebra and discuss special situations when $\delta, \alpha$ are fixed values. Then we study Dirac structures of the generalized omni-Lie algebra in order to characterize all Lie algebra structures on the linear space and prove that there is a one-to-one correspondence between Dirac structures of the generalized omni-Lie algebra $(\Omega, \langle \cdot, \cdot \rangle)$ and Lie algebra structures on subspaces of $V$ if $\delta = \frac{1}{2}$. Moreover, we prove that a generalized omni-Lie algebra is a Leibniz algebra.

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2. Generalized omni-Lie algebras

Let $V$ be a linear space over a field $F$. The set of all linear transformations on $V$ is a Lie algebra denoted by $gl(V)$, given by $[A, B] = AB - BA$, for any $A, B \in gl(V)$.

**Definition 2.1.** A generalized omni-Lie algebra is the linear space $\Omega = gl(V) \oplus V$ with a skew-symmetric bilinear bracket operation $\langle \cdot, \cdot \rangle$ and a non-degenerate symmetric bilinear pairing $\langle \cdot, \cdot \rangle$, for any $A, B \in gl(V), x, y \in V$, and $\delta, \alpha \in F$,

$$[A + x, B + y] = [A, B] + \delta(Ay - Bx),$$  \hspace{1cm} (2.1)

and

$$\langle A + x, B + y \rangle = \alpha(AY + Bx).$$  \hspace{1cm} (2.2)

We call $(\Omega, \langle \cdot, \cdot \rangle, \langle \cdot, \cdot \rangle)$ a generalized omni-Lie algebra.

**Proposition 2.2.** Let $J$ denote the Jacobiator for the bracket $\langle \cdot, \cdot \rangle$ of $\Omega$, then for any $e_1 = A + x, e_2 = B + y, e_3 = C + z \in \Omega$,

$$J(e_1, e_2, e_3) = \langle \langle e_1, e_2 \rangle, e_3 \rangle + \langle \langle e_2, e_3 \rangle, e_1 \rangle + \langle \langle e_3, e_1 \rangle, e_2 \rangle = \delta(Ay - Bx) + \delta(Bz - Cy) + \delta(Cx - Az),$$

\((i)\) If $\delta = 0, 1$, $(\Omega, \langle \cdot, \cdot \rangle)$ is a Lie algebra.

\((ii)\) If $\alpha = \delta = 1$, $(\Omega, \langle \cdot, \cdot \rangle)$ is an omni-Lie algebra.

**Proof.** By a direct calculation, we get

$$J(e_1, e_2, e_3) = \langle \langle e_1, e_2 \rangle, e_3 \rangle + \langle \langle e_2, e_3 \rangle, e_1 \rangle + \langle \langle e_3, e_1 \rangle, e_2 \rangle = \delta(Ay - Bx) + \delta(Bz - Cy) + \delta(Cx - Az),$$

If $\delta = 0, 1$, $J$ satisfies the Jacobi identity, so $(\Omega, \langle \cdot, \cdot \rangle)$ is a Lie algebra. Especially if $\delta = 1$, $(\Omega, \langle \cdot, \cdot \rangle)$ is a semidirect product of $gl(V)$ and $V$. \(\square\)

**Proposition 2.3.** Let $J$ denote the Jacobiator for the bracket $\langle \cdot, \cdot \rangle$ of $\Omega$, for any $e_1 = A + x, e_2 = B + y, e_3 = C + z \in \Omega$, we set

$$T(e_1, e_2, e_3) = \langle \langle e_1, e_2 \rangle, e_3 \rangle + \langle \langle e_2, e_3 \rangle, e_1 \rangle + \langle \langle e_3, e_1 \rangle, e_2 \rangle,$$

then we have

$$T'(e_1, e_2, e_3) = J(e_1, e_2, e_3).$$  \hspace{1cm} (2.3)

**Proof.** We have proved that for any $e_1 = A + x, e_2 = B + y, e_3 = C + z \in \Omega$,

$$J(e_1, e_2, e_3) = (\delta - \delta^2)([A, B]z + [B, C]x + [C, A]y).$$

By Definition 2.1, we get

$$T(e_1, e_2, e_3) = \langle \langle e_1, e_2 \rangle, e_3 \rangle + \langle \langle e_2, e_3 \rangle, e_1 \rangle + \langle \langle e_3, e_1 \rangle, e_2 \rangle = \delta(Ay - Bx) + \delta(Bz - Cy) + \delta(Cx - Az),$$
Thus, Eq. (2.3) holds.

Let $\omega$ be a bilinear operation on $V$, and define the adjoint operator $\text{ad}_\omega : V \to gl(V)$ by

$$\text{ad}_\omega(x)(y) := \omega(x, y), \forall x, y \in V,$$

then the graph of the adjoint operator

$$\mathcal{F}_\omega = \{ \text{ad}_\omega(x) + x, \forall x \in V \}$$

is a subspace of $\Omega$. $\mathcal{F}_\omega^\perp$ denote the orthogonal complement of $\mathcal{F}_\omega$ with respect to the bilinear form (2.2) of $\Omega$.

**Proposition 2.4.** If $\delta = \frac{1}{2}$, $(V, \omega)$ is a Lie algebra if and only if $\mathcal{F}_\omega$ is maximal isotropic, i.e.,

$$\mathcal{F}_\omega = \mathcal{F}_\omega^\perp,$$

and is closed under the bracket $[\cdot, \cdot]$.

**Proof.** First, for any $\text{ad}_\omega(x) + x, \text{ad}_\omega(y) + y \in \mathcal{F}_\omega$,

$$\langle \text{ad}_\omega(x) + x, \text{ad}_\omega(y) + y \rangle = \alpha(\text{ad}_\omega(x)(y) + \text{ad}_\omega(y)(x)) = \alpha(\omega(x, y) + \omega(y, x)),$$

which means that $\omega(\cdot, \cdot)$ is skew-symmetric if and only if its graph is isotropic and $\mathcal{F}_\omega = \mathcal{F}_\omega^\perp$. Then if $\delta = \frac{1}{2}$ and $\omega(\cdot, \cdot)$ is skew-symmetric, let us check

$$[\text{ad}_\omega(x) + x, \text{ad}_\omega(y) + y]$$

$$= [\text{ad}_\omega(x), \text{ad}_\omega(y)] + \frac{1}{2}(\text{ad}_\omega(x)(y) - \text{ad}_\omega(y)(x))$$

$$= [\text{ad}_\omega(x), \text{ad}_\omega(y)] + \frac{1}{2}(\omega(x, y) - \omega(y, x))$$

$$= [\text{ad}_\omega(x), \text{ad}_\omega(y)] + \omega(x, y).$$

Hence, the bracket is closed if and only if

$$[\text{ad}_\omega(x), \text{ad}_\omega(y)] = \text{ad}_\omega(\omega(x, y)),$$

it follows that for any $z \in V$,

$$[\text{ad}_\omega(x), \text{ad}_\omega(y)](z) - \text{ad}_\omega(\omega(x, y))(z)$$

$$= \text{ad}_\omega(x)\text{ad}_\omega(y)(z) - \text{ad}_\omega(y)\text{ad}_\omega(x)(z) - \text{ad}_\omega(\omega(x, y))(z)$$

$$= \text{ad}_\omega(x)\omega(y, z) - \text{ad}_\omega(y)\omega(x, z) - \omega(\omega(x, y), z)$$

$$= \omega(x, \omega(y, z)) - \omega(y, \omega(x, z)) - \omega(\omega(x, y), z),$$

it is clear that the bracket is closed if and only if the Jacobi identity of $\omega(\cdot, \cdot)$ on $V$ is satisfied. Thus, the proof is completed.

**Definition 2.5.** Let $L$ be a maximal isotropic subspace of $\Omega = gl(V) \oplus V$ and closed under the bracket $[\cdot, \cdot]$, then we call $L$ a Dirac structure of the generalized omni-Lie algebra $(\Omega, [\cdot, \cdot], \langle \cdot, \cdot \rangle)$.

**Remark 2.6.** By Proposition 2.3, for a Dirac structure $L$, we can get

$$T'(e_1, e_2, e_3) = J(e_1, e_2, e_3) = 0, \forall e_i \in L, i = 1, 2, 3,$$

then a Dirac structure $(L, [\cdot, \cdot])$ is a Lie algebra.
According to the Definition 2.5, we can rewrite Proposition 2.4 that “if $\delta = \frac{1}{2}$, $(V, \omega)$ is a Lie algebra if and only if $\mathcal{F}_\omega$ is a Dirac structure of the generalized omni-Lie algebra $(\Omega, [\cdot, \cdot], \langle \cdot, \cdot \rangle)$.”

Then we want to know the concrete form of Dirac structures, for a maximal isotropic subspace $L$, let $D = L \cap gl(V)$, define $D^0$ to be the kernel of $D$,

$$D^0 := \{ x \in V \mid X(x) = 0, \forall X \in D \} \subseteq V,$$

$$(D^0)^0 := \{ X \in gl(V) \mid X(x) = 0, \forall x \in D^0 \} \subseteq gl(V) = D.$$

**Lemma 2.7.** A subspace $L$ is maximal isotropic if and only if

$$L = D \oplus \mathcal{F}_\pi|_{D^0} = \{ X + \pi(x) + x \mid \forall X \in D, x \in D^0 \}, \quad (2.4)$$

where $\pi : V \to gl(V)$ is a skew-symmetric map.

**Proof.** First, suppose that $L$ is given by (2.4), for any $X + \pi(x) + x, Y + \pi(y) + y \in L$,

$$X + \pi(x) + x, Y + \pi(y) + y = \alpha(X(y) + \pi(x)(y) + Y(x) + \pi(y)(x))$$

$$= \alpha(\pi(x)(y) + \pi(y)(x))$$

$$= 0.$$

Thus, $L$ is isotropic, then we prove that $L$ is maximal isotropic. For all $B + z \in L^\perp$, we have

$$0 = \langle X, B + z \rangle = \alpha X(z), \forall X \in D,$$

so $z \in D^0$,

$$0 = \langle X + \pi(x) + x, B + z \rangle$$

$$= \alpha(X(z) + \pi(x)(z) + B(x))$$

$$= \alpha(B - \pi(z))(x), \forall X + \pi(x) + x \in L,$$

let $Z := B - \pi(z) \in D$,

$$B + z = Z + \pi(z) + z \in L^\perp = L,$$

therefore, $L$ is maximal isotropic. The converse part is straightforward, so we omit the details. \qed

**Lemma 2.8.** Let $(D, \pi)$ be given above for a maximal isotropic subspace $L \subset \Omega$. Then $L$ is a Dirac structure if and only if the following conditions are satisfied:

(i) $D$ is a subalgebra of $gl(V)$;

(ii) $\pi(\pi(x, y)) - [\pi(x), \pi(y)] \in D, \forall x, y \in D^0$;

(iii) $\pi(x, y) \in D^0, \forall x, y \in D^0$.

Such a pair $(D, \pi)$ is called a characteristic pair of a Dirac structure $L$.

**Proof.** By Definition 2.5, $L$ is a Dirac structure if and only if $L$ is closed with respect to the bracket (2.1). First, for any $X + \pi(x) + x, Y + \pi(y) + y \in L$, by straightforward calculation, we get

$$[[X + \pi(x) + x, Y + \pi(y) + y]]$$

$$= [X + \pi(x), Y + \pi(y)] + \delta(\pi(x)(y) - \pi(y)(x))$$

$$= [X, Y] + [X, \pi(y)] + [\pi(x), Y] + [\pi(x), \pi(y)] + 2\delta \pi(x, y).$$

If $\delta = \frac{1}{2}$, $L$ is closed under the bracket (2.1) if and only if $\pi(x, y) \in D^0, \pi(\pi(x, y)) - [\pi(x), \pi(y)] \in D, \forall x, y \in D^0$. Moreover for any $X, Y \in D, x, y, z \in D^0$, we have

$$[X, Y](z) = X Y(z) - Y X(z) = 0,$$

$$[X, \pi(y)](z) = X \pi(y)(z) - \pi(y) X(z) = 0,$$

$$[\pi(x), Y](z) = \pi(x) Y(z) - Y \pi(x)(z) = 0.$$
so \([X, Y], [X, \pi(y)], [\pi(x), Y] \in D\), that is to say, \(D\) is a subalgebra of \(gl(V)\).

**Theorem 2.9.** There is a one-to-one correspondence between Dirac structures of the generalized omni-Lie algebra \((\Omega, \llbracket \cdot, \cdot \rrbracket, \langle \cdot, \cdot \rangle)\) and Lie algebra structures on subspaces of \(V\) if \(\delta = \frac{1}{2}\).

**Proof.** First, by Lemmas 2.7, 2.8, if \(L\) is a Dirac structure, then \(L = D \oplus \mathcal{F}_\pi|D_0\) and satisfies three conditions in Lemma 2.8. Define operation \([\cdot, \cdot]_{D^0}\) on \(D^0 \subseteq V\) by

\[
[x, y]_{D^0} := \pi(x, y) \in D^0, \forall x, y \in D^0,
\]

\([\cdot, \cdot]_{D^0}\) is a skew-symmetric operation because \(\pi\) is a skew-symmetric map. Then, we check the Jacobi identity, for any \(x, y, z \in D^0\),

\[
[[x, y]_{D^0}, z]_{D^0} = \pi([x, y]_{D^0})(z)
\]

Thus, \((D^0, [\cdot, \cdot]_{D^0})\) is a Lie algebra.

Conversely, \(W\) is a subspace of \(V\), for any Lie algebra \((W, [\cdot, \cdot]_W)\), and define \(D\) by

\[
D := W^0 = \{X \in gl(V) \mid X(x) = 0, \forall x \in W\},
\]

\(D^0 = (W^0)^0 = W\).

Let \(\text{ad} : W \to gl(W)\) represents the limitation of \(\pi : V \to gl(V)\) on \(W\),

\[
\text{ad}_x(y) = [x, y]|_W,
\]

then we get a maximal isotropic subspace

\[
L = D \oplus \mathcal{F}_\pi|W.
\]

Next is to prove that \(L\) is closed under the bracket \([\cdot, \cdot]\), if \(\delta = \frac{1}{2}\), for \(X + \text{ad}_x + x, Y + \text{ad}_y + y \in L\),

\[
\llbracket X + \text{ad}_x + x, Y + \text{ad}_y + y \rrbracket = [X + \text{ad}_x + x, Y + \text{ad}_y + y] + \frac{1}{2}((X + \text{ad}_x)(y) - (Y + \text{ad}_y)(x)) + \frac{1}{2}(\text{ad}_x(y) - \text{ad}_y(x))
\]

For any \(X, Y \in D\) and \(x, y \in W\),

\[
[X, Y](x) = XY(x) - YX(x) = 0,
\]

which means \([X, Y] \in D\), \(D\) is a subalgebra of \(gl(V)\).

\[
[X, \text{ad}_x](y) = X([x, y]|_W) - [x, X(y)] = 0,
\]

\[
[\text{ad}_x, Y](y) = [x, Y(y)] - Y([x, y]|_W) = 0,
\]

so \([X, \text{ad}_y], [\text{ad}_x, Y] \in D\).

Since \([\cdot, \cdot]_W\) satisfies the Jacobi identity, we obtain

\[
[\text{ad}_x, \text{ad}_y] = \text{ad}|_{[x, y]|_W},
\]

\([X + \pi(x) + x, Y + \pi(y) + y] \in D \oplus \mathcal{F}_\pi|W\).

Thus, \(L\) is a Dirac structure. \(\square\)
Let $\Lambda$ denotes the family of all Lie structures on the subspaces of $V$, and $\Gamma$ denotes the family of all Dirac structures of the generalized omni-Lie algebra $\Omega$, then according to Theorem 2.9, there exists a bijective

$$\Psi: \Lambda \to \Gamma,$$

and an embedding

$$\varphi_W: W \to L, \forall W \in \Lambda, L \in \Gamma.$$

**Definition 2.10.** [4] Let $L$ be a linear space over a field $F$ together with a bilinear operation $\circ: L \times L \to L$ satisfying

$$((x \circ y) \circ z) = (x \circ (y \circ z)) - (y \circ (x \circ z)), \forall x, y, z \in L,$$

then we call $(L, \circ)$ a Leibniz algebra.

**Proposition 2.11.** $(\Omega, \circ)$ is a Leibniz algebra.

**Proof.** We check if the Leibniz identity is satisfied, for any $e_1 = A + x, e_2 = B + y, e_3 = C + z \in \Omega$,

$$(e_1 \circ e_2) \circ e_3 - e_1 \circ (e_2 \circ e_3) + e_2 \circ (e_1 \circ e_3)$$

$$= ([A, B] + \delta Ay) \circ (C + z) - (A + x) \circ ([B, C] + \delta Bz) + (B + y) \circ ([A, C] + \delta Az)$$

$$= [[A, B], C] - [A, [B, C]] + [B, [A, C]] + \delta (ABz - BAz) - \delta ABz + \delta BAz$$

$$= 0.$$

By Definition 2.10, it holds. \[\square\]

**Proposition 2.12.** Let $V$ be a Lie algebra. $D$ is a derivation of $V$ that satisfies

$$D[x, y] = [Dx, y] + [x, Dy], \forall x, y \in V$$

if and only if $\mathcal{F}_\omega$ is an invariant subspace of $D$ under the operation “$\circ$” if $\delta = 1$, i.e.,

$$D \circ \mathcal{F}_\omega \subseteq \mathcal{F}_\omega.$$

**Proof.** If $\delta = 1$, for $\text{ad}_\omega(x) + x \in \mathcal{F}_\omega, y \in V$,

$$D \circ (\text{ad}_\omega(x) + x) = [D, \text{ad}_\omega(x)] + Dx.$$

The right side belongs to $\mathcal{F}_\omega$ if and only if

$$[D, \text{ad}_\omega(x)] = \text{ad}_\omega(Dx),$$

for convenience, we denote $\omega(x, y) := [x, y]$,

$$[D, \text{ad}_\omega(x)](y) - \text{ad}_\omega(Dx)(y)$$

$$= D\text{ad}_\omega(x)(y) - \text{ad}_\omega(x)D(y) - \text{ad}_\omega(Dx)(y)$$

$$= D[x, y] - [x, Dy] - [Dx, y].$$

Thus, $D$ is a derivation of $V$ if and only if $D \circ \mathcal{F}_\omega \subseteq \mathcal{F}_\omega$. \[\square\]

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