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Research Article

Almost L-Dunford-Pettis sets in Banach lattices and its applications

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Abstract

We introduce and study the notion of almost L-Dunford-Pettis sets in Banach lattices and we give some characterizations of it in terms of sequences. As an application, we establish new properties of almost Dunford-Pettis completely continuous operators. Finally, by introducing the concept of aL-Dunford-Pettis property in Banach lattices, we investigate the weak compactness of almost Dunford-Pettis completely continuous operator.

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1. Introduction and notation

A norm bounded subset A of a Banach space X is said to be Dunford-Pettis set, if every weakly null sequence (f_n) in X' converges uniformly to zero on A, that is, $\lim_{n\to\infty} \sup_{x\in A} f_n(x) = 0$. Recall from [6] that a norm bounded subset A of a topological dual Banach space X' is an L-Dunford-Pettis if every weakly null sequence (x_n), which is a Dunford-Pettis subset of X converges uniformly to zero on A, that is $\lim_{n\to\infty} \sup_{f\in A} f(x_n) = 0$.

A Banach space X has
- the relatively compact Dunford-Pettis property (DPrcP for short) if every weakly null sequence, which is a Dunford-Pettis set in X, is norm null [7].
- the L-Dunford-Pettis property if every L-Dunford-Pettis set in X' is relatively weakly compact [6].

A Banach lattice E has the positive relatively compact Dunford-Pettis property (PDPrcP for short) if every disjoint weakly null sequence, which is a Dunford-Pettis set in X, is norm null [4]. Note that if a Banach lattice E has the DPrcP then, it has PDPrcP but the converse is not true in general (see Example 3.4 of [4]).

An operator T from a Banach space X into a another Banach space Y is called Dunford-Pettis completely continuous (DPcc for short) if each weakly null sequence (x_n), which is a Dunford-Pettis set in X, we have $\|T(x_n)\|_Y \to 0$, as $n\to\infty$ [7]. Recall from [4] that an operator T from a Banach lattice E into a Banach space Y is called almost Dunford-Pettis completely continuous (aDPcc for short) if each disjoint weakly null sequence (x_n), which is a Dunford-Pettis set in E, we have $\|T(x_n)\|_Y \to 0$, as $n\to\infty$.

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Dunford-Pettis sets definition is given firstly by K.T. Andrews [2] as a norm bounded subset A of a Banach space X is a Dunford-Pettis set whenever every weakly compact operator from X to an arbitrary Banach space Y carries A to a norm totally bounded set. Then Andrew characterized the Dunford-Pettis sets by using sequences (f_n) in X'. Recently in [3], Bouras considered the disjoint version of the Dunford-Pettis sets and introduced the almost Dunford-Pettis sets in Banach lattices. Following Bouras, a bounded subset A of a Banach lattice E is said to be an almost Dunford-Pettis set if every disjoint weakly null sequence (f_n) in E' converges uniformly to zero on A. In this paper, using the disjoint sequence techniques we consider the disjoint version of L-Dunford-Pettis sets, that we call almost L-Dunford-Pettis sets in Banach lattices (Definition 2.1). In addition, we introduce the aL-Dunford-Pettis property which is shared by those Banach lattice whose every almost L-Dunford-Pettis subset of his topological dual is relatively weakly compact (Definition 4.1).

The article is organized as follows. In Section 2 we establish some characterizations of almost L-Dunford-Pettis set in terms of sequences (Proposition 2.2), and we show that each order interval in a dual Banach lattice is an almost L-Dunford-Pettis set (Proposition 2.4). Also, we give some equivalent condition for $T'(A)$ to be almost L-Dunford-Pettis set where A is a norm bounded solid subset of E and $T : E \to F$ is an order bounded operator between two Banach lattices (Theorem 2.7). In Section 3, using the notion of almost L-Dunford-Pettis set, we give characterizations of aDPcc operator and PDPrcP (Theorem 3.1 and Corollary 3.2). After that, we characterize Banach lattice E such that each almost L-Dunford-Pettis set of E' is L-Dunford-Pettis (Theorem 3.12), and we derive some sufficient conditions such that the PDPrcP coincide with the DPrcP (Corollary 3.13). In Section 4, we prove that a Banach lattice E has the aL-Dunford-Pettis property if and only if each aDPcc operator from a Banach lattice E into any Banach space Y is weakly compact (Theorem 4.2), and we deduce an important result about the reflexive space (Corollary 4.3).

To state our results, we need to fix some notations and recall some definitions. A Banach lattice is a Banach space $(E, \|\cdot\|)$ such that E is a vector lattice and its norm satisfies the following property: for each $x, y \in E$ such that $|x| \leq |y|$, we have $\|x\| \leq \|y\|$. If E is a Banach lattice, its topological dual E', endowed with the dual norm, is also a Banach lattice. The sequence (x_n) of a Banach lattice E is disjoint if $|x_n| \wedge |x_m| = 0, n \neq m$ (we denote by $x_n \perp x_m$).

Recall that a nonzero element x of a vector lattice G is discrete if the order ideal generated by x equals the subspace generated by x. The vector lattice G is discrete, if it admits a complete disjoint system of discrete elements. The lattice operations of a Banach lattice E are weakly sequentially continuous, whenever $x_n \to 0$ for $\sigma(E, E')$ as $n \to \infty$ imply $|x_n| \to 0$ for $\sigma(E, E')$, as $n \to \infty$. We will use the term operator $T : X \longrightarrow Y$ between two Banach space to mean a bounded linear mapping, its dual operator T' is defined from Y' into X' by $T'(f)(x) = f(T(x))$ for each $f \in Y'$ and for each $x \in X$. We refer the reader to [1] for unexplained terminology of Banach lattice theory and operators.

2. Almost L-Dunford-Pettis set in a topological dual of Banach lattice

We start this work by a definition of almost L-Dunford-Pettis set, which is a disjoint version of L-Dunford-Pettis set.

Definition 2.1. Let E be a Banach lattice. A norm bounded subset A of E' is called an almost L-Dunford-Pettis set, if every disjoint weakly null sequence (x_n), which is a DP set in E converge uniformly to zero on A, that is, $\lim_{n\to\infty} \sup_{f\in A} |f(x_n)| = 0$.

Now, for a norm bounded subset of a topological dual Banach lattice, we give a characterization of an almost L-Dunford-Pettis sets.

Proposition 2.2. *Let E be a Banach lattice and let A be a norm bounded subset of E'. The following statements are equivalent:*

(1) *A is an almost L-Dunford-Pettis set in E'.*
(2) *For every sequence (f_n) in A and every disjoint weakly null sequence (x_n), which is a Dunford-Pettis set in E, we have $f_n(x_n) \to 0$ as $n \to \infty$.*

Proof. (2) $\Rightarrow$ (1) Assume by way of contradiction that A is not an almost L-Dunford-Pettis set in E'. Then, there exists a disjoint weakly null sequence (x_n), which is a Dunford-Pettis subset of E such that $\sup_{f\in A}|f(x_n)| > \epsilon > 0$ for some $\epsilon > 0$ and each n. Hence, for every n there exists some f_n in A such that $|f_n(x_n)| > \epsilon$, which is impossible from our hypothesis (2). This prove that A is an almost L-Dunford-Pettis set in E'.

(1) $\Rightarrow$ (2) Let (f_n) be a sequence in A and (x_n) be a disjoint weakly null sequence, which is a Dunford-Pettis set in E. Since

$$|f_n(x_n)| \leq \sup_{f\in A}|f(x_n)|,$$

for every n, and A is an almost L-Dunford-Pettis set in E' then, $f_n(x_n) \to 0$ as $n \to \infty$. This completes the proof. □

As a consequence of Proposition 2.2, we obtain the following result.

Proposition 2.3. *Let E be a Banach lattice and let (f_n) be a norm bounded sequence in E'. The following statements are equivalent:*

(1) *The subset $\{f_n, n \in N\}$ is an almost L-Dunford-Pettis set in E'.*
(2) *For every disjoint weakly null sequence (x_n), which is a Dunford-Pettis set in E, we have $f_n(x_n) \to 0$ as $n \to \infty$.*

The following proposition shows that every order interval in a topological dual Banach lattice is an almost L-Dunford-Pettis set.

Proposition 2.4. *Let E be a Banach lattice. Then, for every $f \in (E')^+$, $[-f, f]$ is an almost L-Dunford-Pettis set in E'.*

Proof. Let (x_n) be a disjoint weakly null sequence, which is a Dunford-Pettis set in E, and put $W = \{x_n : n \in N\}$. Then, W is a relatively weakly compact set of E and $(|x_n|)$ is a disjoint sequence in the solid hull of W. Now, by Theorem 4.34 of [1], we see that $(|x_n|)$ is a weakly null sequence of E. Since

$$f(|x_n|) = \sup\{|g(x_n)| : g \in [-f, f]\} \to 0$$

as $n \to \infty$ for all $f \in (E')^+$, it follows that $[-f, f]$ is an almost L-Dunford-Pettis set in E' for all $f \in (E')^+$, and this ends the proof. □

From Proposition 2.4 and Theorem 1.73 of [1], we get

Corollary 2.5. *Let T be an order bounded operator from a Banach lattice E into another Banach lattice F. Then, $T'([-f, f])$ is an almost L-Dunford-Pettis set in E' for every $f \in (F')^+$.*

Proof. Since T be an order bounded operator from a Banach lattice E into another Banach lattice F, by Theorem 1.73 of [1], we obtain that $T' : F' \to E'$ is also order bounded. Thus, $T'([-f, f])$ is an order bounded subset of E' for all $f \in (F')^+$, and so there exists $g \in (E')^+$ such that $T'([-f, f]) \subset [-g, g]$. Now, from Proposition 2.4, we conclude that $T'([-f, f])$ is an almost L-Dunford-Pettis set in E' for every $f \in (F')^+$, as desired.

□

In order to prove the next theorem, we need the following lemma.

Lemma 2.6. *Let E be a Banach lattice, and let (g_n) be a norm bounded sequence in E^+. Then the sequence defined for $n \geq 2$ by*

$$f_n = \left(g_n - 4^n \sum_{i=1}^{n-1} g_i - 2^{-n} \sum_{i=1}^{\infty} 2^{-i} g_i \right)^+ ,$$

is a disjoint sequence of E^+.

Proof. Let $n > m \geq 2$, then

$$0 \leq f_n \leq (g_n - 4^n g_m)^+,$$

and

$$\begin{aligned} 0 \leq 4^n f_m &\leq 4^n (g_m - 4^{-n} g_n)^+ \\ &= (4^n g_m - g_n)^+ \\ &= (g_n - 4^n g_m)^-. \end{aligned}$$

Since $(g_n - 4^n g_m)^+ \perp (g_n - 4^n g_m)^-$, we deduce that $f_n \perp f_m$, as desired. □

Theorem 2.7. *Let T be an order bounded operator from a Banach lattice E into another Banach lattice F, and let A be a norm bounded solid subset of F'. The following statements are equivalent:*

(1) *$T'(A)$ is an almost L-Dunford-Pettis set in E'.*
(2) *$\{T'(f_n), n \in N\}$ is an almost L-Dunford-Pettis set in E', for each disjoint sequence $(f_n) \subset A^+ = A \cap (F')^+$.*

Proof. (1) ⇒ (2) Obvious.

(2) ⇒ (1) Let (x_n) be a disjoint weakly null sequence, which is a Dunford-Pettis set in E. To finish the proof, we have to prove that $\sup_{g \in A} |T'(g)(x_n)| \to 0$ as $n \to \infty$. Assume by way of contradiction that $\sup_{g \in A} |T'(g)(x_n)|$ does not converge to 0 as $n \to \infty$. So there exists some $\epsilon > 0$ such that $\sup_{g \in A} |T'(g)(x_n)| > \epsilon$ for each n. Hence, there exists $g_n \in A^+$ such that $g_n(|T(x_n)|) > \epsilon$ for all natural number n. Let $g \in A^+$. Then from Corollary 2.5, we see that $T'([g, g])$ is an almost L-Dunford-Pettis sets in E', and we have $g(|T(x_n)|) \to 0$ as $n \to \infty$. Let $n_1 = 1$. Since $g_{n_1}(T(x_n)) \to 0$ as $n \to \infty$, there exists some natural number n_2 such that $n_2 > n_1 = 1$ and $g_{n_1}(|T(x_{n_2})|) < \frac{\epsilon}{2^{2\times 2+2}}$. Also, because $\sum_{k=1}^{2} g_{n_k}(|T(x_n)|) \to 0$ as $n \to \infty$, there exists some natural number n_3 such that $n_3 > n_2 > n_1 = 1$ and $\sum_{k=1}^{2} g_{n_k}(|T(x_{n_3})|) < \frac{\epsilon}{2^{2\times 3+2}}$. By induction, we get a strictly increasing subsequence (n_k) of N such that

$$\left(\sum_{k=1}^{m-1} g_{n_k}\right)(|T(x_{n_m})|) < \frac{\epsilon}{2^{2m+2}} \text{ for all } m \geq 2.$$

Now, let

$$h = \sum_{k=1}^{\infty} 2^{-k} g_{n_k}$$

and

$$f_m = (g_{n_m} - 4^m \sum_{k=1}^{m-1} g_{n_k} - 2^{-m} h)^+ \text{ for all } m \geq 2.$$

So by Lemma 2.6, we see that (f_m) is a disjoint sequence in $(F')^+$, as $0 \leq f_m \leq g_{n_m}$, $g_{n_m} \in A$ and A is a solid subset of F' then, $f_m \in A^+$. Hence, we have

$$\begin{aligned} f_m(|T(x_{n_m})|) &= (g_{n_m} - 4^m \sum_{k=1}^{m-1} g_{n_k} - 2^{-m} h)^+ (|T(x_{n_m})|) \\ &\geq (g_{n_m} - 4^m \sum_{k=1}^{m-1} g_{n_k} - 2^{-m} h)(|T(x_{n_m})|) \\ &> \epsilon - \frac{\epsilon}{4} - 2^{-m} h(|T(x_{n_m})|). \end{aligned}$$

This prove that $f_m(|T(x_{n_m})|) > \frac{\epsilon}{2}$ for m sufficiently large (because $2^{-m}h(|T(x_{n_m})|) \to 0$). Since $f_m(|T(x_{n_m})|) = \sup\{|T'(y)(x_{n_m})|, |y| \leq f_m\}$, for m sufficiently large there exists some $y_m \in F'$ such that $|y_m| \leq f_m$ and $|T'(y_m)(x_{n_m})| > \frac{\epsilon}{2}$. It is clear that (y_m^+) and (y_m^-) are norm bounded disjoint sequences in A^+ and so, by our hypothesis we obtain

$$\begin{aligned}\frac{\epsilon}{2} &< |T'(y_m)(x_{n_m})| \\ &\leq \left|T'(y_m^+)(x_{n_m})\right| + |T'(y_m^-)(x_{n_m})| \\ &\leq \sup_{k\in N}\left|T'(y_k^+)(x_{n_m})\right| + \sup_{k\in N}\left|T'(y_k^-)(x_{n_m})\right| \to 0,\end{aligned}$$

as $m \to \infty$. This leads to a contradiction, and we are done. □

As a consequence of Theorem 2.7, we obtain the following result.

Corollary 2.8. *Let T be an order bounded operator from a Banach lattice E into another Banach lattice F, and let A be a norm bounded solid subset of F'. The following statements are equivalent:*

(1) *$T'(A)$ is an almost L-Dunford-Pettis set in E'.*
(2) *$f_n(T(x_n)) \to 0$ as $n \to \infty$, for every disjoint weakly null sequence (x_n), which is a Dunford-Pettis set in E^+ and for each disjoint sequence (f_n) in A^+.*

Next, we derive another consequence of Theorem 2.7.

Corollary 2.9. *Let E be a Banach lattice and let A be a norm bounded solid subset of E'. The following statements are equivalent:*

(1) *A is an almost L-Dunford-Pettis set in E'.*
(2) *$\{f_n, n \in N\}$ is an almost L-Dunford-Pettis set in E', for each disjoint sequence $(f_n) \subset A^+ = A \cap (F')^+$.*

3. Almost L-Dunford-Pettis set, aDPcc operator and PDPrcP

The following theorem gives a new characterization of order bounded aDPcc operator from a Banach lattice E into another F in term of almost L-Dunford-Pettis sets in E'.

Theorem 3.1. *For an order bounded operator T from a Banach lattice E into another F. The following statements are equivalent:*

(1) *T is an aDPcc operator.*
(2) *$T'(B_{F'})$ is an almost L-Dunford-Pettis set in E'.*
(3) *$\{T'(f_n), n \in N\}$ is an almost L-Dunford-Pettis set in E', for each disjoint sequence $(f_n) \subset B_{F'}^+$.*
(4) *$f_n(T(x_n)) \to 0$ as $n \to \infty$, for every disjoint weakly null sequence (x_n), which is a Dunford-Pettis set in E^+ and for each disjoint sequence $(f_n) \subset B_{F'}^+$.*

Proof. (1) ⇔ (2) Let (x_n) be a disjoint weakly null sequence, which is a Dunford-Pettis subset of E'. Since

$$\|T(x_n)\| = \sup_{f\in T'(B_{F'})} |f(x_n)|,$$

then, it is clear that T is an aDPcc operator if and only if $T'(B_{F'})$ is an almost L-Dunford-Pettis in E'.

(2) ⇔ (3) Follows from Theorem 2.7.
(3) ⇔ (4) Follows from Proposition 2.3. □

As a simple consequence of Theorem 3.1, we get a characterization of PDPrcP in Banach lattices.

Corollary 3.2. *Let E be a Banach lattice. The following statements are equivalent:*

(1) *E has the PDPrcP.*
(2) *$B_{E'}$ is an almost L-Dunford-Pettis set.*
(3) *$\{f_n, n \in N\}$ is an almost L-Dunford-Pettis set in E', for each disjoint sequence $(f_n) \subset B^+_{E'}$.*
(4) *$f_n(x_n) \to 0$ as $n \to \infty$, for every disjoint weakly null sequence (x_n), which is a Dunford-Pettis set in E^+ and for each disjoint sequence $(f_n) \subset B^+_{E'}$.*

In the next result, we obtain a new characterization of PDPrcP in Banach lattices in term of almost L-Dunford-Pettis sets.

Theorem 3.3. *A Banach lattice E has the PDPrcP if and only if every bounded subset of E' is an almost L-Dunford-Pettis set.*

Proof. For the "if" part, since $B_{E'}$ is an almost L-Dunford-Pettis set, by Corollary 3.2 we conclude that E has the PDPrcP.

For the "only if" part, assume by way of contradiction that there exists a bounded subset A, which is not an almost L-Dunford-Pettis set of E'. Then, there exists a disjoint weakly null sequence (x_n), which is a Dunford-Pettis set of E such that $\sup_{f\in A} |f(x_n)| > \epsilon > 0$ for some $\epsilon > 0$ and each n. Hence, for every n there exists some f_n in A such that $|f_n(x_n)| > \epsilon$.

On the other hand, since $(f_n) \subset A$, there exists some $K > 0$ such that $\|f_n\|_{E'} \leq K$ for all n. Thus,

$$|f_n(x_n)| \leq K \, \|x_n\|,$$

for each n, so by our hypothesis, $|f_n(x_n)| \to 0$ as $n \to \infty$, which is impossible. This completes the proof. □

Let us define the following.

Definition 3.4. Let E be a Banach lattice, E has the property (a) if for every weakly null sequence (x_n), which is a Dunford-Pettis set in E we have $|x_n| \to 0$ for $\sigma(E, E')$ as $n \to \infty$.

Remark 3.5. Let E be a Banach lattice. Note that E is discrete with order continuous norm $\Rightarrow$ the lattice operations of E are weakly sequentially continuous (see Proposition 2.5.23 of [5]) $\Rightarrow$ E has the property (a).

We need to recall of the following characterization of aDPcc operators, which is established in Theorem 3.9 of [4].

Theorem 3.6. *An operator T from a Banach lattice E into a Banach space Y is aDPcc if and only if $\|T(x_n)\| \to 0$ as $n \to \infty$ for every weakly null sequence (x_n), which is a Dunford-Pettis set in E^+.*

In the following result, we establish a sufficient condition such that the class of aDPcc operators and the class of DPcc operators coincide.

Theorem 3.7. *Let E be a Banach lattice and Y be a Banach space such that E has the property (a), then each aDPcc operator from E into Y is DPcc.*

Proof. Let T be an aDPcc operator from E into Y. We prove that T is DPcc, let (x_n) be a weakly null sequence, which is a Dunford-Pettis set in E. Since E has the property (a) then (x_n^+) and (x_n^-) be weakly null sequences in E^+, and it is clear that are Dunford-Pettis sets. Now, it follows from Theorem 3.6 that $\|T(x_n^+)\| \to 0$ and $\|T(x_n^-)\| \to 0$ as $n \to \infty$. Thus,

$$\|T(x_n)\| = \|T(x_n^+) - T(x_n^-)\| \leq \|T(x_n^+)\| + \|T(x_n^-)\| \to 0 \text{ as } n \to \infty,$$

and we are done. □

Now, from Theorem 3.7 and Corollary 3.20 of [4], we derive

Corollary 3.8. *Let E and F be two Banach lattices such that E has the property (a) or F is discrete with order continuous norm, then each positive aDPcc operator from E into F is DPcc.*

The following result give a necessary and sufficient condition such that each order interval in a topological dual Banach lattice is an L-Dunford-Pettis set.

Proposition 3.9. *Let E be a Banach lattice. The following statements are equivalent:*

(1) *For every $f \in (E')^+$, $[-f, f]$ is an L-Dunford-Pettis set in E'.*
(2) *E has the property (a).*

Proof. Let (x_n) be a weakly null sequence, which is a Dunford-Pettis set of E, then the result follows from the equality:

$$f(|x_n|) = \sup\{|g(x_n)| : g \in [-f, f]\},$$

for every $f \in (E')^+$ and every n. □

We need the following proposition.

Proposition 3.10. *A Banach space X has the DPrcP if and only if the closed unit ball $B_{X'}$ of X' is L-Dunford-Pettis.*

Proof. Let (x_n) be a weakly null sequence, which is a Dunford-Pettis set of X, then the result follows from the equality:

$$\|x_n\| = \sup_{f\in B_{X'}} |f(x_n)|,$$

for every n. □

Remark 3.11. It is clear that every L-Dunford-Pettis set in a dual Banach lattice is almost L-Dunford-Pettis, but the converse is not true in general. In fact, if we put $E = L^1[0, 1] \oplus L^2[0, 1]$ then, E has the PDPrcP but does not have the DPrcP (see Example 3.4 of [4]), hence from Corollary 3.2 and Proposition 3.10, we see that the closed unit ball $B_{E'}$ is an almost L-Dunford-Pettis set but it is not L-Dunford-Pettis.

Now, we are in a position to give our major result, and we characterize Banach lattice E such that each almost L-Dunford-Pettis set of E' is L-Dunford-Pettis.

Theorem 3.12. *Let E be a Banach lattice. The following statements are equivalent:*

(1) *Each almost L-Dunford-Pettis set of E' is L-Dunford-Pettis.*
(2) *E has the property (a).*
(3) *Each aDPcc operator from E to any Banach lattice F is DPcc.*
(4) *Each aDPcc operator from E to ℓ^∞ is DPcc.*

Proof. (1) ⇒ (2) Let $f \in (E')^+$ then, $[-f, f]$ is an almost L-Dunford-Pettis set in E' (see Proposition 2.4), and by our hypothesis, we have that $[-f, f]$ is an L-Dunford-Pettis set in E'. Now, from Proposition 3.9, we see that E has the propery (a).

(2) ⇒ (3) Let T be an aDPcc operator from E to any Banach lattice F, since E has the property (a) then, by Theorem 3.7, T is DPcc operator.

(3) ⇒ (4) Obvious.

(4) ⇒ (1) Suppose by way of contradiction that there exist an almost L-Dunford-Pettis set A in E' which is not L-Dunford-Pettis. As A is not L-Dunford-Pettis subst of E', so there exists a weakly null sequence (x_n), which is a Dunford-Pettis subset of E such that $\sup_{f\in A} |f(x_n)| > \epsilon > 0$ for some $\epsilon > 0$ and each n. Hence, for every n there exists some f_n in A such that $|f_n(x_n)| > \epsilon$.

On the other hand, consider the operator $T : E \to \ell^\infty$ defined by

$$T(x) = (f_n(x))_{n=0}^{\infty} \text{ for all } x \in E.$$

We show that T is aDPcc operator. Since A is almost L-Dunford-Pettis subst of E', then for every disjoint weakly null sequence (y_m), which is a Dunford-Pettis of E, we obtain

$$\begin{aligned} \|T(y_m)\|_\infty &= \|(f_n(y_m))_{n=0}^{\infty}\|_\infty \\ &= \sup_{n\in N} |f_n(y_m)| \\ &\leq \sup_{f\in A} |f(y_m)| \to 0, \end{aligned}$$

as $m \to \infty$, this prove that T is aDPcc, and by our hypothesis we see that T is DPcc.

Now, we have

$$\epsilon < |f_n(x_n)| \leq \|T(x_n)\|_\infty \to 0, \text{ as } n \to \infty,$$

which is impossible, and this ends the proof. □

Consequently, we obtain some sufficient conditions such that the PDPrcP and DPrcP in Banach lattice conicide.

Corollary 3.13. *Let E be a Banach lattice. Suppose that one of the following assertions is valid:*

(1) *Each almost L-Dunford-Pettis set of E' is L-Dunford-Pettis.*
(2) *E has the property (a).*
(3) *The lattice operations of E are weakly sequentially continuous.*
(4) *E is discrete.*
(5) *Each aDPcc operator from E to ℓ^∞ is DPcc.*

Then, E has the PDPrcP if and only if E has the DPrcP.

Proof. (1), (2) and (5) Follows from Theorem 3.12, in particular, we put in assertion (3) of this Theorem $F = E$ and $T = Id_E : E \to E$ the identity operator.

(3) Follows from Remark 3.5 and (2).

(4) If E has the PDPrcP, then, its norm is order continuous, and as E is discrete so by Remark 3.5 and assertion (3), we deduce that E has the DPrcP, and this completes the proof. □

4. aL-Dunford-Pettis property in Banach lattices

Let E be a Banach lattice, note that each relatively weakly compact subset A of a dual topological Banach lattice E' is L-Dunford-Pettis (see Proposition 2.3 of [6]), and hence A is almost L-Dunford-Pettis. The converse of this property is not true in general, in fact, the closed unit ball B_{ℓ^∞} of ℓ^∞ is almost L-Dunford-Pettis set (see Corollary 3.2), but it is not relatively weakly compact.

Now, we give the following definition.

Definition 4.1. A Banach lattice E has the aL-Dunford-Pettis property, if every almost L-Dunford-Pettis set in E' is relatively weakly compact.

Note that an aDPcc operator is not weakly compact in general. In fact, Id_{ℓ^1} is aDPcc, but it is not weakly compact.

Used the idea of aL-Dunford-Pettis property in Banach lattice, we establish the weak compactness of aDPcc operators.

Theorem 4.2. *Let E be a Banach lattice, then, the following assertions are equivalent:*

(1) *E has the aL-Dunford-Pettis property,*
(2) *for each Banach space Y, every aDPcc operator from E into Y is weakly compact,*
(3) *every aDPcc operator from E into ℓ^∞ is weakly compact.*

Proof. (1) $\Rightarrow$ (2) Suppose that E has the aL-Dunford-Pettis property and $T : E \to Y$ is aDPcc operator. Thus $T'(B_{Y'})$ is an almost L-Dunford-Pettis set in E'. So by hypothesis, it is relatively weakly compact and T is a weakly compact operator.

(2) $\Rightarrow$ (3) Obvious.

(3) $\Rightarrow$ (1) If E does not have the aL-Dunford-Pettis property, there exists an almost L-Dunford-Pettis subset A of E' which is not relatively weakly compact. So there is a sequence $(f_n) \subseteq A$ with no weakly convergent subsequence. Now, we show that the operator $T : E \to \ell^\infty$ defined by $T(x) = (f_n(x))$ for all $x \in E$ is aDPcc but it is not weakly compact. As $(f_n) \subseteq A$ is almost L-Dunford-Pettis set, then for every disjoint weakly null sequence (x_m), which is a Dunford-Pettis set in E we have

$$\|T(x_m)\| = \sup_n |f_n(x_m)| \to 0, \text{ as } m \to \infty,$$

so T is aDPcc operator. Hence $T'((\lambda_n)_{n=1}^\infty) = \sum_{n=1}^\infty \lambda_n f_n$ for every $(\lambda_n)_{n=1}^\infty \in \ell^1 \subset (\ell^\infty)'$. If e'_n is the usual basis element in ℓ^1 then $T'(e'_n) = f_n$, for all $n \in N$. Thus, T' is not a weakly compact operator and neither is T. This finishes the proof. □

As a consequence of Theorem 4.2, we derive the following result.

Corollary 4.3. *A PDPrc space has the aL-Dunford-Pettis property if and only if it is reflexive.*

Proof. ($\Rightarrow$) If a Banach lattice E has the PDPrcP, then the identity operator Id_E on E is aDPcc. As E has the aL-Dunford-Pettis property, it follows from Theorem 4.2 that Id_E is weakly compact, and hence E is reflexive.

($\Leftarrow$) Obvious. □

Remark 4.4. Note that the Banach lattice ℓ^1 is not reflexive and has the PDPrcP, then from Corollary 4.3, we conclude that ℓ^1 does not have the aL-Dunford-Pettis property.

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