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Authors: İlim Kişi, Günay Öztürk, Kadri Arslan
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A new type of canal surface in Euclidean 4-space IE^4

İlim Kişi*[1], Günay Öztürk[2], Kadri Arslan[3]

ABSTRACT

We give, with its sample, a new type of canal surface constructed by means of the parallel transport frame of its spine curve in Euclidean 4-space IE^4. We investigate the curvature features of this surface with respect to the principal curvature functions according to parallel transport frame. Further, we give certain results about Weingarten type canal and tube surfaces. Finally, we give the visualizations of projections of this new type of canal surface in IE^3 for various radius functions.

Keywords: Gaussian curvature, mean curvature, parallel transport frame, Weingarten surface

1. INTRODUCTION

Given a space curve γ called spine curve, a canal surface associated to this curve is defined as a surface swept by a family of spheres of varying radius $r(u)$. If $r(u)$ is constant, the canal surface is a tubular (tube, pipe) surface.

Actually, the concept of canal surface is a generalization of an offset of a planar curve. In [11], do Carmo gives some geometrical properties of tube surfaces and by means of these surfaces proves the theorems named as Fenchel's theorem and the Fary-Milnor theorem.

Apart from being used in pure mathematics, canal surfaces are widely used in many areas especially in CAGD, e.g. construction of blending surfaces, i.e. canal surface with a rational radius, shape reconstruction or robotic path planning (see, [21, 23]). Canal surfaces are also useful in visualising long thin objects such as poles, 3D fonts, brass instruments, or visceral organs of the body.

Tori, Dupin cyclids in [22] and tube surfaces in [18] are the special types of the canal surfaces.

Given a surface M in an Euclidean 3-space IE^3 and its two principal curvatures κ_1 and κ_2, M is a Weingarten surface under the condition that there is a smooth relation $U(\kappa_1, \kappa_2) = 0$. If K and H denote respectively the Gaussian and the mean curvatures of M, $U(\kappa_1, \kappa_2) = 0$ refers to $\Phi(K, H) = 0$, which is equivalent to $\frac{\partial(K,H)}{\partial(u,v)} = K_u H_v - K_v H_u = 0$. Also, if the surface satisfies the equation $aK + bH = c$ for the non-zero real numbers a,b,c, then it is called as a linear Weingarten surface [20].

Frenet-Serret frame gives way to the study of curves in classical differential geometry in

* Corresponding Author

[1] Kocaeli University, Faculty of Arts and Sciences, Mathematics, (ilim.ayvaz@kocaeli.edu.tr) ORCID: 0000-0002-4785-8165

[2] İzmir Demokrasi University, Faculty of Arts and Sciences, Mathematics, ORCID: 0000-0002-1608-0354

[3] Uludağ University, Faculty of Arts and Sciences, Mathematics, ORCID: 0000-0002-1440-7050

Euclidean space. However, the Frenet frame can not be constructed at the points in which curvature vanishes. Hence, an alternative frame is needed. In [6], Bishop defines a new frame for a curve and calls it Bishop frame, which is well defined even if the curve's second derivative in 3-dimensional Euclidean space vanishes. In [6, 16], the advantages of the Bishop frame and the comparison of Bishop frame with the Frenet frame in Euclidean 3-space are given. Euclidean 4-space IE^4 has the same problem as Euclidean 3-space. That is, one of the i-th (1<i<4) derivatives of the curve may be zero.

In [14], using the similar idea, authors consider such curves and construct an alternative frame. They give parallel transport frame of a curve in IE^4. They generalize the notion which is well known in Euclidean 3-space for 4-dimensional Euclidean space IE^4.

In [1-5, 8, 10, 12, 15, 19, 25], authors give some characteristic properties of surfaces in IE^4. Furthermore, in [9, 17] authors consider canal surfaces in IE^4.

In the present study, we consider a canal surface constructed with parallel transport frame of its spine curve in Euclidean 4-space IE^4.

This paper is organized as in the following: Section 2 gives certain preliminaries of a curve and a surface in IE^4. Section 3 introduces a new type of canal surface and give some curvature conditions of this surface in IE^4. Section 4 gives some visualizations of projections of canal surfaces in IE^3 for various radius functions. The figures presented in this paper are generated via the Maple programme.

2. BASIC CONCEPTS

Given a unit speed curve $\gamma : I \subseteq IR \to IE^4$ for an interval I in IR, the derivative formulas of Frenet frame are

$$\begin{bmatrix} T' \\ N' \\ B_1' \\ B_2' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 & 0 \\ -\kappa & 0 & \tau & 0 \\ 0 & -\tau & 0 & \sigma \\ 0 & 0 & -\sigma & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix},$$

where $\{T, N, B_1, B_2\}$ is the Frenet frame of γ, and κ, τ and σ are principal curvature functions related to this frame of the curve γ, respectively.

In [14], authors use the same tangent vector $T(s)$ as in Frenet frame for the first vector, and for the other vectors of the new frame they use relatively parallel vector fields $M_1(s)$, $M_2(s)$, and $M_3(s)$ to construct an alternative frame. They call this frame a parallel transport frame along the curve γ. Then they give the following theorem for a parallel transport frame.

Theorem 2.1. [14] Let $\{T, N, B_1, B_2\}$ be the Frenet frame and $\{T, M_1, M_2, M_3\}$ the parallel transport frame along a unit speed curve $\gamma : I \subseteq IR \to IE^4$. The relation between these frames may be expressed as

$$\begin{aligned}
&T = T, \\
&N = \cos\theta(s)\cos\psi(s)M_1 \\
&+ (-\cos\varphi(s)\sin\psi(s) + \sin\varphi(s)\sin\theta(s)\cos\psi(s))M_2 \\
&+ (\sin\varphi(s)\sin\psi(s) + \cos\varphi(s)\sin\theta(s)\cos\psi(s))M_3, \\
&B_1 = \cos\theta(s)\sin\psi(s)M_1 \\
&+ (\cos\varphi(s)\cos\psi(s) + \sin\varphi(s)\sin\theta(s)\sin\psi(s))M_2 \\
&+ (-\sin\varphi(s)\cos\psi(s) + \cos\varphi(s)\sin\theta(s)\sin\psi(s))M_3, \\
&B_2 = -\sin\theta(s)M_1 + \sin\varphi(s)\cos\theta(s)M_2 \\
&+ \cos\varphi(s)\cos\theta(s)M_3,
\end{aligned}$$

(1)

where θ, ψ and φ are the Euler angles. Then the alternative parallel frame equations are

$$\begin{bmatrix} T' \\ M_1' \\ M_2' \\ M_3' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & k_2 & k_3 \\ -k_1 & 0 & 0 & 0 \\ -k_2 & 0 & 0 & 0 \\ -k_3 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} T \\ M_1 \\ M_2 \\ M_3 \end{bmatrix}, \quad (2)$$

where k_1, k_2, k_3 are principal curvature functions according to parallel transport frame of the curve γ and their expressions are as follows:

$$k_1 = \kappa_1 \cos\theta \cos\psi,$$
$$k_2 = \kappa_1(-\cos\varphi \sin\psi + \sin\varphi \sin\theta \cos\psi),$$
$$k_3 = \kappa_1(\sin\varphi \sin\psi + \cos\varphi \sin\theta \cos\psi),$$

where

$$\theta' = \frac{\sigma}{\sqrt{\kappa^2+\tau^2}},$$
$$\psi' = -\tau - \sigma\frac{\sqrt{\sigma^2-(\theta')^2}}{\sqrt{\kappa^2+\tau^2}},$$
$$\varphi' = -\frac{\sqrt{\sigma^2-(\theta')^2}}{\cos\theta},$$

and the following equalities

$$\kappa = \sqrt{{k_1}^2 + {k_2}^2 + {k_3}^2},$$
$$\tau = -\psi' + \varphi' \sin\theta,$$
$$\sigma = \frac{\theta'}{\sin\psi},$$
$$\varphi' \cos\theta + \theta' \cot\psi = 0$$

are hold.

Given a regular surface M in IE^4 with the parametrization $X(u,v):(u,v) \in D \subset IE^2$, at any point p=X(u,v), the vectors X_u and X_v span the tangent space of M. Then the first fundamental form's coefficients are computed as

$$E = \langle X_u, X_u \rangle,\ F = \langle X_u, X_v \rangle,\ G = \langle X_v, X_v \rangle. \quad (3)$$

Here, $\langle\ ,\ \rangle$ is the Euclidean dot product. For the regularity of the surface patch X(u,v), $W^2 = EG - F^2 \neq 0$.

At any point p in M, there is a decomposition $T_p IE^4 = T_p M + T_p^{\perp} M$, where $T_p^{\perp} M$ is the orthogonal component of $T_p M$ in IE^4. Let $\widetilde{\nabla}$ be the Riemannian connection of IE^4. Then the induced Riemannian connection on M for any given local vector fields X_1, X_2 tangent to M is defined as

$$\nabla_{X_1} X_2 = (\widetilde{\nabla}_{X_1} X_2)^T, \quad (4)$$

where T represents the tangential component.

Let $\chi(M)$ and $\chi^{\perp}(M)$ be the spaces of the smooth vector fields tangent and normal to M, respectively. The second fundamental map is defined as follows:

$$\begin{aligned} &h : \chi(M) \times \chi(M) \to \chi^{\perp}(M) \\ &h(X_i, X_j) = \widetilde{\nabla}_{X_i} X_j - \nabla_{X_i} X_j,\ 1 \le i,j \le 2. \end{aligned} \quad (5)$$

This map is well-defined, symmetric, and bilinear.

Proposition 2.2. [7] Let M be a surface in IE^4 given with the parametrization X(u,v). If the coefficient of the first fundamental form F=0, the second fundamental form of M becomes

$$\begin{aligned} h(X_u,X_u) &= X_{uu} - \frac{1}{E}\langle X_{uu}, X_u \rangle X_u + \frac{1}{G}\langle X_{uv}, X_u \rangle X_v, \\ h(X_u,X_u) &= X_{uv} - \frac{1}{E}\langle X_{uv}, X_u \rangle X_u - \frac{1}{G}\langle X_{uv}, X_v \rangle X_v, \\ h(X_u,X_u) &= X_{vv} + \frac{1}{E}\langle X_{uv}, X_v \rangle X_u - \frac{1}{G}\langle X_{vv}, X_v \rangle X_v. \end{aligned} \quad (6)$$

Proposition 2.3. [7] Let M be a surface in IE^4 given with the parametrization X(u,v). Then for the basis X_u, X_v of $T_P(M)$ the Gaussian curvature and the mean curvature vector of M are defined as follows respectively,

$$K = \frac{1}{W^2}\left(\langle h(X_u,X_u), h(X_v,X_v)\rangle - \langle h(X_u,X_v), h(X_u,X_v)\rangle\right) \quad (7)$$

and

$$\vec{H} = \frac{1}{2W^2}(Eh(X_v,X_v) - 2Fh(X_u,X_v) + Gh(X_u,X_u)), \quad (8)$$

where $W^2 = EG - F^2$.

3. CANAL SURFACE ACCORDING TO PARALLEL TRANSPORT FRAME IN IE^4

In [13], authors give the following parametrization for a canal surface:

$$M : X(u,v) = \gamma(u) + r(u)(B_1(u)\cos v + B_2(u)\sin v),$$

where $\gamma = \gamma(u)$ is a space curve parametrized by arclength with the Frenet frame $\{T(u), N(u), B_1(u), B_2(u)\}$.

Using the similar idea, we give the following parametrization:

$$M : X(u,v) = \gamma(u) + r(u)(M_2(u)\cos v + M_3(u)\sin v), \quad (9)$$

where $r(u)$ is a differentiable function and $\{T(u), M_1(u), M_2(u), M_3(u)\}$ is parallel transport frame of the curve γ in IE^4.

Corollary 3.1. Let $\{T, N, B_1, B_2\}$ be the Frenet frame and $\{T, M_1, M_2, M_3\}$ the parallel transport frame along a unit speed curve $\gamma = \gamma(s) : I \to IE^4$. Then, the parallel transport frame vectors can be given as follows:

$$\begin{aligned}
&T = T,\\
&M_1 = \cos\theta(u)\cos\psi(u)N\\
&+\cos\theta(u)\sin\psi(u)B_1 - \sin\theta(u)B_2,\\
&M_2 = (-\cos\varphi(u)\sin\psi(u) + \sin\varphi(u)\sin\theta(u)\cos\psi(u))N\\
&+(\cos\varphi(u)\cos\psi(u) + \sin\varphi(u)\sin\theta(u)\sin\psi(u))B_1\\
&+\sin\varphi(u)\cos\theta(u)B_2,\\
&M_3 = (\sin\varphi(u)\sin\psi(u) + \cos\varphi(u)\sin\theta(u)\cos\psi(u))N\\
&+(-\sin\varphi(u)\cos\psi(u) + \cos\varphi(u)\sin\theta(u)\sin\psi(u))B_1\\
&+\cos\varphi(u)\cos\theta(u)B_2.
\end{aligned}$$

Proof. If the equations (1) is written in the matrix form, the transition matrix is obtained as follows:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta\cos\psi & -\cos\varphi\sin\psi + \sin\varphi\sin\theta\cos\psi & \sin\varphi\sin\psi + \cos\varphi\sin\theta\cos\psi \\ 0 & \cos\theta\sin\psi & \cos\varphi\cos\psi + \sin\varphi\sin\theta\sin\psi & -\sin\varphi\cos\psi + \cos\varphi\sin\theta\sin\psi \\ 0 & -\sin\theta & \sin\varphi\cos\theta & \cos\varphi\cos\theta \end{bmatrix}.$$

By calculating the inverse of this transition matrix, we write the desired result.

Example 3.2. Consider the unit speed curve $\gamma(u) = (a\cos cu, a\sin cu, b\cos du, b\sin du)$ in IE^4, where $a^2c^2 + b^2d^2 = 1$. Then the canal surface associated to the spine curve γ in IE^4 has the following parametrization

$$X(u,v) = \left(\begin{array}{l} a\cos cu + \dfrac{r(u)}{4\kappa}\left\{ \begin{array}{l} (ac^2\cos cu + \sqrt{3}\kappa bd\sin cu - 2\sqrt{3}bd^2\cos cu)\cos v \\ +(-\sqrt{3}ac^2\cos cu - 3\kappa bd\sin cu - 2bd^2\cos cu)\sin v \end{array} \right\}, \\ a\sin cu + \dfrac{r(u)}{4\kappa}\left\{ \begin{array}{l} (ac^2\sin cu - \sqrt{3}\kappa bd\cos cu - 2\sqrt{3}bd^2\sin cu)\cos v \\ +(-\sqrt{3}ac^2\sin cu + 3\kappa bd\cos cu - 2bd^2\sin cu)\sin v \end{array} \right\}, \\ b\cos du + \dfrac{r(u)}{4\kappa}\left\{ \begin{array}{l} (bd^2\cos du - \sqrt{3}\kappa ac\sin du + 2\sqrt{3}ac^2\cos du)\cos v \\ +(-\sqrt{3}bd^2\cos du + 3\kappa ac\sin du + 2ac^2\cos du)\sin v \end{array} \right\}, \\ b\sin du + \dfrac{r(u)}{4\kappa}\left\{ \begin{array}{l} (bd^2\sin du + \sqrt{3}\kappa ac\cos du + 2\sqrt{3}ac^2\sin du)\cos v \\ +(-\sqrt{3}bd^2\sin du - 3\kappa ac\cos du + 2ac^2\sin du)\sin v \end{array} \right\} \end{array} \right)$$

where $\kappa = \sqrt{k_1^2 + k_2^2 + k_3^2}$, $0 \le u \le 2\pi$, a, b, c, d are real constants and $c, d > 0$.

The tangent space of M is spanned by the vectors

$$\begin{aligned} X_u &= fT + r'\cos vM_2 + r'\sin vM_3, \\ X_v &= -r\sin vM_2 + r\cos vM_3, \end{aligned} \quad (10)$$

where

$$f(u,v) = 1 - k_2(u)r(u)\cos v - k_3(u)r(u)\sin v. \quad (11)$$

Thus, the coefficients of the first fundamental form become

$$\begin{aligned} E &= \langle X_u, X_u \rangle = f^2 + (r')^2, \\ F &= \langle X_u, X_v \rangle = 0, \\ G &= \langle X_v, X_v \rangle = r^2. \end{aligned} \quad (12)$$

Proposition 3.3. Let M be a canal surface according to parallel transport frame given with the parametrization (9) in IE^4. Then for all $p \in M$, the surface patch of M is regular if and only if $r^2(f^2 + (r')^2) \neq 0$.

Proof. Assume that the surface patch is regular. Then from the equations (12), $W^2 = r^2(f^2 + (r')^2) \neq 0$. Conversely, if the condition holds, it is easy to see that the surface patch is regular.

The second partial derivatives of X(u,v) are expressed as follows:

$$\begin{aligned} X_{uu} &= gT + fk_1M_1 + (fk_2 + r''\cos v)M_2 + (f(k_3 + r''\sin v)M_3, \\ X_{uv} &= f_vT - r'\sin vM_2 + r'\cos vM_3, \\ X_{vv} &= -r\cos vM_2 - r\sin vM_3 \end{aligned} \quad (13)$$

where

$$g = g(u,v) = f_u(u,v) - k_2(u)r'(u)\cos v - k_3(u)r'(u)\sin v. \quad (14)$$

Hence, from the equations (10) and (13), we get

$$\begin{aligned}\langle X_{uu}, X_u \rangle &= f f_u + r'r'',\\ \langle X_{uv}, X_u \rangle &= f f_v,\\ \langle X_{uv}, X_v \rangle &= rr',\\ \langle X_{uv}, X_v \rangle &= 0.\end{aligned} \tag{15}$$

Further, by the use of the equations (10), (12), and (15), the second fundamental form of M becomes

$$\begin{aligned}h(X_u, X_u) = &\frac{1}{r(f^2+(r')^2)}(f^2 r'(f-1) - rr'(fr''-gr'))T\\ &+ fk_1 M_1\\ &+ \frac{f\cos v}{r(f^2+(r')^2)}(f^2 - f^3 + r(fr''-gr'))M_2\\ &+ \frac{f\sin v}{r(f^2+(r')^2)}(f^2 - f^3 + r(fr''-gr'))M_3,\end{aligned}$$

(16)

$$h(X_u, X_v) = \frac{f_v r'}{f^2+(r')^2}(r'T - f\cos v M_2 - f\sin v M_3) \tag{17}$$

$$h(X_v, X_v) = \frac{fr}{f^2+(r')^2}(r'T - f\cos v M_2 - f\sin v M_3). \tag{18}$$

From the equations (15)-(17), we get the following result:

Proposition 3.7. Let M be a canal surface according to parallel transport frame given with the parametrization (9) in IE^4. Then the Gaussian curvature of M at point p is

$$K = \frac{1}{r^2(f^2+(r')^2)^2}(f^4 - f^3 - fr(fr''-gr') - f_v^2 (r')^2). \tag{19}$$

As a consequence of (19), we obtain the following result:

Corollary 3.8. Let M be a tube surface with constant $r = r(u)$. Then the Gaussian curvature of M becomes

$$K = -\frac{k_2\cos v + k_3 \sin v}{fr} = \frac{f-1}{fr^2}. \tag{20}$$

Proposition 3.9. Let M be a canal surface according to parallel transport frame given with the parametrization (9) in IE^4. If γ is a straight line, then the Gaussian curvature of M at point p is

$$K = -\frac{r''}{r(1+(r')^2)^2}. \tag{21}$$

Proof. Let γ be a straight line. Then the curvatures k_1, k_2, k_3 of γ are identically zero. By (11) and (14), we find f=1, g=0 which shows that the equation (21) holds.

Corollary 3.10. Let M be a canal surface according to parallel transport frame given with the parametrization (9) in IE^4. When γ is a straight line, the surface M is flat if and only if r is a linear function of the form r(u)=au+b for some real constants a, b.

Proposition 3.11. Let M be a canal surface according to parallel transport frame given with the parametrization (9) in IE^4. Then the mean curvature vector of M at point p is

$$\begin{aligned}\vec{H} = \frac{1}{2r(f^2+(r')^2)^2}\{&(fr'(f^2+(r')^2) - rr'(fr''-gr') - f^2r'(1-f))T\\ &+ frk_1(f^2+(r')^2)M_1\\ &+ (-f^2\cos v(f^2+(r')^2) + f^3\cos v(1-f) + fr(fr''-gr')\cos v)M_2\\ &+ (-f^2\sin v(f^2+(r')^2) + f^3\sin v(1-f) + fr(fr''-gr')\sin v)M_3\}.\end{aligned}$$

(22)

Proof. Substituting the equations (16)-(18) into (6), we obtain the vector given in (22).

Corollary 3.12. Let M be a canal surface according to parallel transport frame given with the parametrization (9) in IE^4. Then the mean curvature of M at point p is

$$H = \frac{1}{2r(f^2+(r')^2)^{3/2}}\begin{pmatrix} f^2(f^2+(r')^2)^2 - 2frr'^2(fr''-gr') \\ -2f^3(f^2+(r')^2)(1-f) + r^2(fr''-gr')^2 \\ +2f^2r(fr''-gr') + f^4(1-f)^2 \\ +f^2r^2k_1^2(f^2+(r')^2) - 4f^3r(fr''-gr') \end{pmatrix}^{1/2}.$$

Corollary 3.13. Let M be a tube surface with constant $r = r(u)$. Then the mean curvature vector of M becomes

$$\vec{H} = \frac{1}{2fr}(rk_1M_1 + \cos v(-2f+1)M_2 + \sin v(-2f+1)M_3). \tag{23}$$

Corollary 3.14. Let M be a tube surface with constant $r = r(u)$. Then the mean curvature of M at point p is

$$H = \frac{1}{2fr}(4f^2 - 4f + r^2 k_1^{\,2} + 1)^{1/2}. \tag{24}$$

Proposition 3.15. Let M be a canal surface according to parallel transport frame given with the parametrization (9) in IE^4. If γ is a straight line, then the mean curvature vector of M at point p is

$$\vec{H} = \frac{1 + (r')^2 - rr''}{2r(1 + (r')^2)^2}(r'T - \cos vM_2 - \sin vM_3)\ .(25)$$

Proof. Let γ be a straight line. Then the curvatures k_1, k_2, k_3 of γ are identically zero. By (11) and (14), we find f=1, g=0 which shows that the equation (25) holds.

Corollary 3.16. Let M be a canal surface according to parallel transport frame given with the parametrization (9) in IE^4. If γ is a straight line, then the mean curvature of M at point p is

$$H = \frac{1 + (r')^2 - rr''}{2r(1 + (r')^2)^{3/2}}. \tag{26}$$

Proposition 3.17. Let M be a canal surface according to parallel transport frame given with the parametrization (9) in IE^4. If γ is a straight line, the surface M is minimal if and only if

$$2r + 2\sqrt{r^2 - c_1^{\,2}} = e^{\frac{u}{c_1} + c_2}.$$

Proof. Let M is minimal. Then from the equation (26), $1 + (r')^2 - rr'' = 0$. If we take $r' = p(u)$, the last equation becomes

$$\frac{dr}{r} = \frac{pdp}{p^2 + 1}. \tag{27}$$

The solution of the equation (27) is as follows:

$$r^2 = c_1^2(p^2 + 1).$$

Again taking $p(u) = r'$, we obtain the following ordinary differential equation:

$$\frac{dr}{\sqrt{r^2 - c_1^{\,2}}} = \frac{du}{c_1}.$$

Integrating both sides of the last equation, we get the solution.

As a consequence of (25), we obtain the following result:

Proposition 3.18. Let M be a tube surface with constant $r = r(u)$. If γ is a straight line, the mean curvature vector of M at point p is

$$\vec{H} = \frac{1}{2r}(-\cos vM_2 - \sin vM_3).$$

Corollary 3.19. Let M be a tube surface with constant $r = r(u)$. If γ is a straight line, M has constant mean curvature of the form

$$H = \frac{1}{2r}.$$

Proposition 3.20. Let M be a tube surface with constant $r = r(u)$ in IE^4. If γ is a straight line, then M is a Weingarten surface.

Proof. Considering the equations (21) and (26), we see that K and H are the functions of the variable u. Thus

$$K_v = 0 = H_v,$$

which means $K_u H_v - K_v H_u = 0$.

Proposition 3.21. Let M be a tube surface with constant $r = r(u)$ in IE^4. M is a Weingarten surface if and only if one of the three conditions holds:

i) The first curvature function of the spine curve γ vanishes, i.e., $k_1 = 0$.

ii) The first curvature function of the spine curve γ is constant, i.e., $(k_1)_u = 0$.

iii) For the second and the third curvatures of the spine curve γ, the equation

$$\frac{k_3(u)}{k_2(u)} = \tan v = c, \quad c \in IR$$

holds.

Proof. By using the equations (20) and (24), we obtain

$$K_u = \frac{f_u}{f^2 r^2}, \qquad K_v = \frac{f_v}{f^2 r^2}, \tag{28}$$

and

$$H_u = \frac{1}{4f^2r^2}\begin{Bmatrix} fr(4f^2-4f+r^2k_1{}^2+1)^{-1/2}(8ff_u - 4f_u + 2r^2k_1(k_1)_u) \\ -2f_u r(4f^2-4f+r^2k_1{}^2+1)^{1/2} \end{Bmatrix}$$

$$H_u = \frac{1}{4f^2r^2}\begin{Bmatrix} fr(4f^2-4f+r^2k_1{}^2+1)^{-1/2}(8ff_v - 4f_v) \\ -2f_v r(4f^2-4f+r^2k_1{}^2+1)^{1/2} \end{Bmatrix}. \tag{29}$$

Thus,

$$K_u H_v - K_v H_u = 0 \Leftrightarrow k_1 (k_1)_u f_v = 0,$$

which yields the expected result.

Proposition 3.22. Let M be a tube surface with constant $r = r(u)$. If γ is a straight line, M is a linear Weingarten surface.

Proof. Assume that M is a tube surface with constant $r = r(u)$ in IE^4 and γ is a straight line. Then we know that K=0 and $H = \frac{1}{2r}$. For the non-zero real numbers a,b,c, we get

$$a.0 + b.\frac{1}{2r} = c,$$

which has the solution $(a, 2rc, c)$, $a, c \in IR - \{0\}$.

4. VISUALIZATION

Canal surfaces are very popular in geometric modeling. In this section, we visualize the surfaces given with the patch

$$X(u,v) = (x(u,v), y(u,v), z(u,v), w(u,v))$$

in IE^4 by the use of Maple program. We plot the graph of the projection of the given surface by using maple plotting command

$$\text{plot3d}([x, y, z + w], u = a..b, v = c..d). \tag{30}$$

After than, we construct some 3D geometric shape models by using the canal surfaces defined in Example 3.2. for the following values;

a) $r(u) = 2u + 6$,

b) $r(u) = u^2$,

c) $r(u) = \cos(u^2)$.

We plot the graph of the projection of these surfaces in IE^3 by the use of plotting command (30). (see, Figure 1, Figure 2, Figure 3).

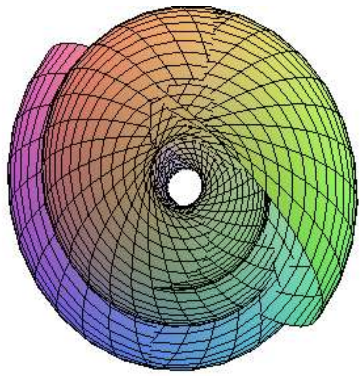

Figure 1: Canal surface with $r(u) = 2u + 6$

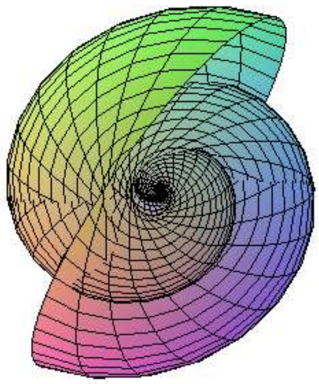

Figure 2: Canal surface with $r(u) = u^2$

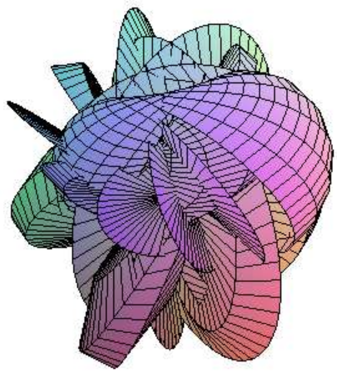

Figure 3: Canal surface with $r(u) = \cos(u^2)$

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