On Multiplier of Hyper BCI Algebras

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Abstract
In this paper, we introduce the notion of multiplier of a hyper BCI-algebra, and discuss some properties of hyper BCI-algebras. Also we introduced notion of hyper isotone multiplier.

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Hyper BCI-algebra
Multiplier
Isotone
Fix(H)
Regular

1. INTRODUCTION

The study of BCK-algebras was started by Y. Imai and K. Iseki [1] in 1966 as a generalization of the concept of set-theoretic difference and propositional calculus.

The hyperstructure theory (called also multialgebras) was introduced by F. Marty [2] in 1934.

Moreover the hyper structure was applied to BCI-algebras and was introduced the concepts of hyper BCI-algebras which is a generalization of BCI-algebras by X.X. Long [3] in 2006.

In this paper, we introduce the notion of multiplier of a hyper BCI-algebra, and discuss some properties of hyper BCI-algebras. Also we introduce the notion of hyper isotone multiplier on hyper BCI-algebras.

2. PRELIMINARIES

Definition 2.1. [3] Let \( H \) be a nonempty set and "\( \circ \)" be a hyper operation on \( H \). Then \( H \) is said to be a hyper BCI-algebra, if it contains a constant \( 0 \) and the following conditions hold:

(b1) \((x \circ z) \circ (y \circ z) \ll x \circ y\),
(b2) \((x \circ y) \circ z = (x \circ z) \circ y\),
(b3) \(x \ll x\),
(b4) \(x \ll y\), \(y \ll x \Rightarrow x = y\),
(b5) \(0 \circ (0 \circ x) \ll x\)

for all \( x, y, z \in H \) where \( x \ll y \) is defined by \( 0 \in x \circ y \) and for every \( A, B \subseteq H \) \( A \ll B \) is defined by for all \( a \in A \), there exists \( b \in B \) such that \( a \ll b \). In such case "\( \ll \)" is called the hyper order in \( H \).
Let \((H,\varnothing,0)\) be a hyper BCI-algebra. By \(H^+\) we mean \(H^+ = \{x \in H \mid 0 \in 0 \circ x\}\)

We have \(0 \in H^+\), thus \(H^+ \neq \emptyset\).

**Proposition 2.2.** [4] Let \((H,\varnothing,0)\) be a hyper BCI-algebra, the following hold:

(i) \(x \ll x \circ 0\)

(ii) \(A \ll A\)

(iii) \(y \ll z\) implies \(x \circ z \ll x \circ y\),

for all \(x, y, z \in H\) and for all nonempty subsets \(A\) and \(B\) of \(H\).

**Definition 2.3.** [4] Let \((H,\varnothing,0)\) be a hyper BCI-algebra. Then the set \(S_k = \{x \in H : x \circ H \ll \{x\}\}\) is called as hyper BCK-part of \(H\). If \(H \neq S_k\), then \(H\) is said to be a proper hyper BCI-algebra.

A hyper BCI-algebra \(H\) is called a

(i) weak proper hyper BCI-algebra if \(H\) is proper and \(H^+ = H\). In the other word if \(0\) is the smallest element of \(H\),

(ii) strong proper hyper BCI-algebra if \(H^+ \neq H\). We note that if \(x \notin H^+\), then \(0 \notin 0 \circ x\). Thus \(0 \circ x \ll \{0\}\)

Therefore, \(0 \circ H \ll \{0\}\) and \((H,\varnothing,0)\) is proper.

**Definition 2.4.** [4] Let \(I\) be a nonempty subset of hyper BCI-algebra \(H\) and \(0 \in I\). Then \(I\) is said to be a

(i) weak hyper BCI-ideal of \(H\) if \(x \circ y \subseteq I\) and \(y \in I\) imply that \(x \in I\), for all \(x, y \in H\),

(ii) hyper BCI-ideal of \(H\) if \(x \circ y \ll I\) and \(y \in I\) imply that \(x \in I\), for all \(x, y \in H\),

(iii) strong hyper BCI-ideal of \(H\) if \(x \circ y \equiv I\) and \(y \in I\) imply that \(x \in I\), for all \(x, y \in H\), where \(x \circ y \equiv I\) means \(x \circ y \cap I \neq \emptyset\).

**Definition 2.5.** [5] Let \(I\) be a nonempty subset of a hyper BCI-algebra \(H\) and \(0 \in I\). Then \(I\) is called to be hyper subalgebra of \(H\) if \(x \circ y \subseteq I\) for all \(x, y \in I\).

**Definition 2.6.** [6] Let \((H_1,\varnothing_1,0_1)\) and \((H_2,\varnothing_2,0_2)\) be two hyper BCI-algebras and \(f : H_1 \rightarrow H_2\) be a function. Then \(f\) is defined a homomorphism if and only if

\[f(x \circ_1 y) = f(x) \circ_2 f(y),\text{ for all } x, y \in H_1,\]

If \(f\) is one to one \((onto)\) then \(f\) is monomorphism \((epimorphism)\) and if \(f\) is both one to one and onto, then \(f\) is a isomorphism and \((H_1,\varnothing_1,0_1)\) and \((H_2,\varnothing_2,0_2)\) are isomorphic.

3. MULTIPLIER OF HYPER BCI-ALGEBRAS

In the following, the notion of multiplier of a hyper BCI-algebra is given.

**Definition 3.1.** Let \((H,\varnothing,0)\) be a hyper BCI-algebra. A map \(d : H \rightarrow H\) is said to be a multiplier if for all \(x, y \in H\) \(d(x \circ y) = d(x) \circ y\).
Example 3.1. Let $H = \{0, \alpha, \beta\}$ and $(H, \circ, 0)$ be a hyper BCI-algebra with Cayley table as follows

**Table 1. Cayley table**

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>$\alpha$</th>
<th>$\beta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>${0}$</td>
<td>${0}$</td>
<td>${\beta}$</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>${\alpha}$</td>
<td>${0, \alpha}$</td>
<td>${\beta}$</td>
</tr>
<tr>
<td>$\beta$</td>
<td>${\beta}$</td>
<td>${\beta}$</td>
<td>${0}$</td>
</tr>
</tbody>
</table>

Define a map $d_1 : H \rightarrow H$ such that $d_1(x) = \begin{cases} 
\beta, & x = 0, \alpha \\
0, & x = \beta 
\end{cases}$

Hence it is easily checked that $d_1$ is a multiplier of hyper BCI-algebra.

Therefore $H$ is strong proper hyper BCI-algebra.

If $I_1 = \{0, \beta\} \subseteq H$ then $I_1$ is ideal of $H$.

Example 3.2. Cayley table given in Example 3.1 and $d_2 = I_H$ then $d_2$ is multiplier of $H$.

Example 3.3. Cayley table given in Example 3.1 and define a map $d_3 : H \rightarrow H$

$\begin{align*}
d_3(x) = \begin{cases} 
0, & x = 0, \alpha \\
b, & x = \beta 
\end{cases}
\end{align*}$

Hence it is easily checked that $d_3$ is a multiplier of hyper BCI-algebra.

If $I_2 = \{0, \alpha\} \subseteq H$ then $I_2$ is ideal of $H$. And also $d_3$ is an invariant map: $d_3(I_2) \subseteq I_2$.

**Proposition 3.2.** Let $(H, \circ, 0)$ be a hyper BCI-algebra and $d$ be a multiplier of $H$. Then it satisfies $d(x \circ d(x)) \ll 0$ for all $x \in H$.

**Proof.** Using (b3):

$\begin{align*}
d(x \circ d(x)) & \ll 0 \\
0 & \in d(x) \circ d(x) \\
0 & \in d(x) \circ d(x) \circ 0
\end{align*}$

**Definition 3.3.** Let $(H, \circ, 0)$ be a hyper BCI-algebra and a map $d : H \rightarrow H$ is called to be a regular if $d(0) = 0$.

**Example 3.4.** $d_3$ given in Example 3.3 is multiplier of hyper BCI-algebra and regular. That is $d_3(0) = 0$.

**Proposition 3.4.** Let $(H, \circ, 0)$ be a hyper BCI-algebra and a map $d : H \rightarrow H$ is a regular multiplier of $H$. Then the following hold for all $x, y \in H$:
(i) \(d(x) \ll x\),

(ii) \(d(x \circ y) \ll d(x) \circ d(y)\).

**Proof.**

(i) \(0 = \rho(0) = d(0 \circ x) = d(x) \circ x\), for all \(x \in X\). We get \(d(x) \ll x\).

(ii) Let \(y \in X\). Using (i) and Prop.2.2.(iii), we have \(d(x \circ y) = d(x) \circ y\).

Therefore we get \(d(x) \circ y \ll d(x) \circ d(y)\).

Hence we have \(d(x \circ y) \ll d(x) \circ d(y)\).

**Example 3.5.** \(d_3\) given in Example 3.3. is multiplier of hyper BCI-algebra. And also it is a homomorphism.

**Definition 3.5.** Let \((H, \circ, 0)\) be a hyper BCI-algebra and a map \(d : H \to H\), if \(x \ll y\) then \(d(x) \ll d(y)\) for all \(x, y \in H\), \(d\) is said to be hyper isotone.

**Example 3.6.** \(d_3\) given in Ex. 3.3 is hyper isotone.

**Proposition 3.6.** Let \((H, \circ, 0)\) be a hyper BCI-algebra and \(d\) be a regular multiplier of \(H\). If \(d : H \to H\) is an endomorphism, then \(d\) is hyper isotone.

**Proof.** Let \(x, y \in X\) and \(x \ll y\)

Therefore we get \(0 \in x \circ y\). \(d\) be a regular multiplier of \(H\) and \(d : H \to H\) is an endomorphism so we have \(d(0) \in d(x \circ y) \ll d(x) \circ d(y)\). Hence we find \(d(x) \ll d(y)\).

**Definition 3.7.** Let \((H, \circ, 0)\) be a hyper BCI-algebra and \(d_1, d_2\) be two maps. Then a map \(d_1 \bullet d_2 : H \to H\) is defined by \((d_1 \bullet d_2)(x) = d_1(d_2(x))\) for all \(x \in H\).

**Proposition 3.8.** Let \((H, \circ, 0)\) be a hyper BCI-algebra and be \(d_1, d_2\) two maps. \(d_1, d_2 : H \to H\) are multipliers of \(H\). Then \(d_1 \bullet d_2\) is a multiplier of \(H\).

Proof. Let \(x, y \in H\), we get,

\[
(d_1 \bullet d_2)(x \circ y) = d_1(d_2(x \circ y)) = d_1(d_2(x) \circ y) = (d_1 \bullet d_2)(x) \circ y
\]

And so \(d_1 \bullet d_2\) is a multiplier of \(H\).

**Definition 3.9.** Let \((H, \circ, 0)\) be a hyper BCI-algebra and \(d\) be a multiplier of \(H\). A set \(\text{Fix}_d(H)\) is defined by \(\text{Fix}_d(H) := \{x \in H \mid d(x) = x\}\).

**Proposition 3.10.** Let \((H, \circ, 0)\) be a hyper BCI-algebra and \(d\) be a regular multiplier of \(H\). If \(x \in \text{Fix}_d(H)\) and \(y \in H\) imply \((d \bullet d)(x \circ y) = (x \circ y)\).
Proof. Let $x, y \in H$, we have,

$$(d \cdot d)(x \circ y) = d(d(x \circ y))$$

$$= d(d(x) \circ y)$$

$$= d(d(x)) \circ y$$

$$= d(x) \circ y$$

$$= x \circ y$$

**Proposition 3.11.** Let $(H, \circ, 0)$ be a hyper BCI-algebra and $d$ be a multiplier of $H$. Then $\text{Fix}_d(H)$ is a hyper subalgebra of $H$.

**Proof.** Let $x, y \in \text{Fix}_d(H)$. We have $d(x \circ y) = d(x) \circ y = x \circ y$.

Hence we find $\text{Fix}_d(H)$ is a hyper subalgebra of $H$.

**Proposition 3.12.** Let $(H, \circ, 0)$ be a hyper BCI-algebra and $d$ be a multiplier of $H$. If $x \in H$ and $y \in \text{Fix}_d(H)$ then $x \land y \in \text{Fix}_d(H)$.

**Proof.** Let $y \in \text{Fix}_d(H)$, we get,

$$d(x \land y) = d(y \circ (y \circ x))$$

$$= d(y) \circ (y \circ x)$$

$$= y \circ (y \circ x)$$

$$= x \land y$$

**Proposition 3.13.** Let $(H, \circ, 0)$ be a hyper BCI-algebra and $d$ be a multiplier of $H$. If $x \in H$ and $y \in \text{Fix}_d(H)$ then $d(x \circ y) = d(x) \circ d(y)$.

**Proof.** Let $y \in \text{Fix}_d(H)$ and $x \in H$

$$d(x \circ y) = d(x) \circ y$$

$$= d(x) \circ d(y).$$

**CONFLICTS OF INTEREST**

No conflict of interest was declared by the authors.
REFERENCES


