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A Viscosity Nonlinear Algorithm for Split Generalized Equilibrium Problem

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Abstract: In this paper, we proposed a viscosity iterative algorithm to approximate a common solution of split generalized equilibrium problem and fixed point problem for a nonexpansive semigroups in real Hilbert spaces. Under certain conditions control on parameters, the iteration sequences generated by the proposed algorithms are proved to be strongly convergent to a solution of split generalized equilibrium problem. Our results can be viewed as a generalization and improvement of various existing results in the current literature. Some numerical examples to guarantee the main result of this paper.

Keywords: Split generalized equilibrium problem, fixed point problem, nonexpansive semigroup.

1. Introduction

The class of nonexpansive mappings have powerful applications to solve various problems arising in the field of applied mathematics, such as variational inequality problem, convex minimization, zeros of a monotone operator, initial value problems of differential equations, game-theoretic model. In 1967, Browder and Petryshyn [3] introduced the concept of strict pseudo contraction as a generalization of nonexpansive mappings. Later on, Alber et al.[1] introduced the notion of total asymptotically nonexpansive mappings which is more general in nature and unifies various definitions of mappings associated with the class of asymptotically nonexpansive mappings. The viscosity iterative algorithms for finding a common element of the set of fixed points for nonlinear operators and the set of solutions of variational inequality problems have been investigated by many authors [22, 32, 35, 36, 37] and references therein. The viscosity technique for nonexpansive mappings in Hilbert space was proposed by Moudafi[18, 21]. This technique allow us to apply this method to convex optimization, linear programming and monoton inclusions [26, 28, 31, 33, 34, 38]. It is well known that the generalized equilibrium problems include variational inequality problems, optimization problems, problems of Nash equilibria, saddle point problems, fixed point problems and complementarity problems as special cases[2, 21, 34, 33].
Moudafi [18] introduced the following split equilibrium problem (SEP):

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Let C be a nonempty subset of a real Hilbert space H_1, Q be a nonempty subset of a real Hilbert space H_2 and let $A : H_1 \to H_2$ be a bounded linear operator. Also $F_1 : C \times C \to R$ and $F_2 : Q \times Q \to R$ be two nonlinear bifunctions. The split equilibrium problem is to find $x^* \in C$ such that

$$F_1(x^*, x) \geq 0, \ \forall x \in C \tag{1}$$

and such that

$$y^* = Ax^* \in Q \ solves \ F_2(y^*, y) \geq 0, \ \forall y \in Q \tag{2}$$

It is remarked that inequality (1) represents the classical equilibrium problem [12] and its solution set is denoted $EP(F_1)$. Moreover, inequalities (1) and (2) constitute a pair of equilibrium problems which aim to find a solution x^* of an equilibrium problem (1) such that its image $y^* = Ax^*$ under a given bounded linear operator A also solves another equilibrium problem (2). The solution set of (SEP) is denoted by $\Omega = \{p \in \mathrm{EP}(F_1) : Ap \in \mathrm{EP}(F_2)\}$ [4, 5, 6, 7, 19, 20].
Recently, Kazmi and Rizvi [13] introduced a split generalized equilibrium problem ($SGEP$):
Find $x^* \in C$ such that

$$F_1(x^*, x) + \psi_1(x^*, x) \geq 0, \ \forall x \in C$$

and

$$y^* = Ax^* \in Q \text{ solves } F_2(y^*, y) + \psi_2(y^*, y) \geq 0, \ \forall y \in Q$$

where $F_1, \psi_1 : C \times C \to \mathbb{R}$ and $F_2, \psi_2 : C \times C \to \mathbb{R}$ be nonlinear bifunctions and $A : H_1 \to H_2$ is bounded linear operator. The solution set of (SGEP) is denote by $\Gamma = \{p \in \mathrm{GEP}(F_1, \psi_1) : Ap \in \mathrm{GEP}(F_2, \psi_2)\}$.
In 2015, Ma and Wang [16] established strong convergence results for the split common fixed point problem of total asymptotically strict pseudo contractions in Hilbert spaces. Quite recently, some methods have been proposed and analyzed in [14, 15] for the split equilibrium problem.
In 2017 Zhang and Gui [39] introduced an iterative algorithm in a Hilbert space as follows:

$$\begin{aligned} u_n &= T_{r_n}^{F_1}(x_n + \delta A^*(T_{s_n}^{F_2} - I)Ax_n) \\ x_{n+1} &= \alpha_n f(x_n) + \tfrac{(1-\alpha_n)}{l} \textstyle\sum_{i=0}^{l} T_i^n u_n, \end{aligned}$$

where $T_i : C \to C$ is a asymptotically nonexpansive mapping for $i = 0, 1, \ldots, n$.

A family $S := \{T(s) : 0 \leq s < \infty\}$ of mapping from C into itself is called a nonexpansive semigroup on C if it satisfies the following conditions:

1. $T(0)x = x$ for all $x \in C$,
2. $T(s+t) = T(s)T(t)$ for all $s, t \geq 0$,
3. $\|T(s)x - T(s)y\| \leq \|x - y\|$ for all $x, y \in C$ and $s \geq 0$,
4. For all $x \in C, s \to T(s)x$ is continuous.

Plubtieng and Punpaeng Theorem introduced the following iterative method for nonexpansive semigroup[24]:

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + (1 - \alpha_n - \beta_n) \frac{1}{s_n} \int_0^{s_n} T(s) x_n ds.$$

In 2010 Kang et.al, introduced and inspired by results in [12], prove a strong convergence of the iterative scheme in a real Hilbert space by

$$x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1 - \beta_n) I - \alpha_n A) \frac{1}{s_n} \int_0^{s_n} T(s) x_n ds,$$

where A is a strong positive bounded linear operator on C.
Cianciaruso et al. [9] considered the following iterative method:

$$\begin{aligned} &F(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0; \\ &x_{n+1} = \alpha_n \gamma f(x_n) + (1 - \alpha_n A) \frac{1}{s_n} \int_0^{s_n} T(s) u_n ds. \end{aligned}$$

Kazmi and Rizvi [13] considered the following iterative method:

$$\begin{aligned} &u_n = T_{r_n}^{(F_1, \psi_1)} (x_n + \delta A^* (T_{r_n}^{(F_2, \psi_2)} - I) A x_n); \\ &x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1 - \beta_n) I - \alpha_n B) \frac{1}{s_n} \int_0^{s_n} T(s) u_n ds. \end{aligned}$$

Recently, Sahebi et al. [25, 26, 27, 28] considered a general viscosity iterative algorithm for finding a common element of the set general equilibrium problem system and the set of fixed points of a nonexpansive semigroup in a Hilbert space. They proved, under the certain appropriate conditions, the iterative algorithm converges strongly to the unique solution of a variational inequality. By intuition from the above mentioned results and the ongoing research in this direction, we aim to employ a viscosity iterative algorithm to approximate a common solution of split generalized equilibrium problem and fixed point problem for a nonexpansive semigroup in real Hilbert spaces.Under certain conditions on parameters, the iteration sequences generated by the proposed algorithms are proved to be strongly convergent to a solution of split generalized equilibrium problem. Some numerical examples and preliminary computational results are also provided. Our results can be viewed as a generalization and improvement of various existing results in the current literature . The rest of paper is organized as follows. The next section presents some preliminary results. Section 3 is devoted to introduce viscosity iterative algorithm for SGEP. section 4 is devoted to prove strong convergence algorithm. The last section presents some numerical examples to demonstrate the proposed algorithms.

2. Preliminaries

Let H be a Hilbert space and C be a nonempty closed and convex subset of H. For each point $x \in H$, there exists a unique nearest point of C, denote by $P_C x$, such that $\|x - P_C x\| \leq \|x - y\|$ for all $y \in C$. P_C is called the metric projection of H onto C. It is well known that P_C is nonexpansive mapping and is characterized by the following property:

$$\langle x - P_C x, y - P_C y \rangle \leq 0 \tag{3}$$

Further, it is well known that every nonexpansive operator $T : H \to H$ satisfies, for all $(x,y) \in H \times H$, inequality

$$\langle (x - T(x)) - (y - T(y)), T(y) - T(x) \rangle \leq (\frac{1}{2}) \|(T(x) - x) - (T(y) - y)\|^2, \tag{4}$$

and therefore, we get, for all $(x,y) \in H \times \mathrm{Fix}(T)$,

$$\langle (x - T(x)), (y - T(y)) \rangle \leq (\frac{1}{2}) \|(T(x) - x)\|^2, \tag{5}$$

see, e.g. [11].

It is also known that H satisfies Opial's condition [23], i.e., for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\| \tag{6}$$

holds for every $y \in H$ with $y \neq x$.

Definition 1. A mapping $T : H \to H$ is said to be firmly nonexpansive if

$$\langle Tx - Ty, x - y \rangle \geq \|Tx - Ty\|^2, \ \forall x, y \in H.$$

Lemma 1. [8] The following inequality holds in real space H:

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle, \ \forall x, y \in H.$$

Definition 2. A mapping $T : C \to H$ is said to be monotone, if

$$\langle Tx - Ty, x - y \rangle \geq 0, \quad \forall x, y \in C.$$

T is called α-inverse-strongly-monotone if there exist a positive real number α such that

$$\langle Tx - Ty, x - y \rangle \geq \alpha \|Tx - Ty\|^2, \quad \forall x, y \in C.$$

Lemma 2. [17] Assume that B is a strong positive linear bounded self adjoint operator on a Hilbert space H with coefficient $\bar{\gamma} > 0$ and $0 < \rho \leq \|B\|^{-1}$. Then $\|I - \rho B\| \leq 1 - \rho \bar{\gamma}$.

Lemma 3. [29] Let C be a nonempty bounded closed convex subset of a Hilbert space H and let $S := \{T(s) : 0 \leq s < \infty\}$ be a nonexpansive semigroup on C, for each $x \in C$ and $t > 0$. Then, for any $0 \leq h < \infty$,

$$\lim_{t\to\infty} \sup_{x\in C} \|\frac{1}{t}\int_0^t T(s)x ds - T(h)(\frac{1}{t}\int_0^t T(s)x ds)\| = 0.$$

Lemma 4. [30] Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and $\{\beta_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$. Suppose $x_{n+1} = (1-\beta_n)y_n + \beta_n x_n$, for all integers $n \ge 0$ and $\limsup_{n\to\infty}(\|y_{n+1}-y_n\| - \|x_{n+1}-x_n\|) \le 0$. Then $\lim_{n\to\infty}\|y_n - x_n\| = 0$.

Lemma 5. [38] Let $\{a_n\}$ be a sequence of nonnegative real numbers such that $a_{n+1} \le (1-\alpha_n)a_n + \delta_n,\ n \ge 0$ where α_n is a sequence in $(0,1)$ and δ_n is a sequence in $\mathbb{R}$ such that (i) $\Sigma_{n=1}^{\infty}\alpha_n = \infty$; (ii) $\limsup_{n\to\infty}\frac{\delta_n}{\alpha_n}$ or (iii) $\Sigma_{n=1}^{\infty}\delta_n < \infty$. Then $\lim_{n\to\infty} a_n = 0$.

Assumption 1. Let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying the following assumptions:

1. $F(x,x) \ge 0,\ \forall x \in C$,
2. F is monotone, i.e., $F(x,y) + F(y,x) \le 0,\ \forall x \in C$,
3. F is upper hemicontinuouse, i.e., for each $x,y,z \in C$,
   $\limsup_{t\to 0} F(tz + (1-t)x, y) \le F(x,y)$,
4. For each $x \in C$ fixed, the function $x \to F(x,y)$ is convex and lower semicontinuous;

let $\psi : C \times C \to \mathbb{R}$ be such that

1. $\psi(x,x) \ge 0,\ \forall x \in C$,
2. for each $y \in C$ fixed, the function $x \to \psi(x,y)$ is upper semicontonuous,
3. for each $x \in C$ fixed, the function $y \to \psi(x,y)$ is convex and lower semicontinuous;

Lemma 6. [13] Assume that $F_1, \psi_1 : C \times C \to \mathbb{R}$ satisfying Assumption 1. Let $r > 0$ and $x \in H_1$. Then, there exists $z \in C$ such that $F_1(z,y) + \psi_1(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \ge 0,\ \forall y \in C$.

Lemma 7. [6] Assum that the bifunctions $F_1, \psi_1 : C \times C \to \mathbb{R}$ satisfying Assumption 1 and ψ_1 is monotone. For $r > 0$ and for all $x \in H_1$, define a mapping $T_r^{(F_1,\psi_1)} : H_1 \to C$ as follows:

$$T_r^{(F_1,\psi_1)}x = \{z \in C : F_1(z,y) + \psi_1(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \ge 0\},\ \forall y \in C.$$

Then, the following hold:

(i) $T_r^{(F_1,\psi_1)}$ *is single − valued.*

(ii) $T_r^{(F_1,\psi_1)}$ *is firmly nonexpansive, i.e.,*
$\|T_r^{(F_1,\psi_1)}(x) - T_r^{(F_1,\psi_1)}(y)\|^2 \le \langle T_r^{(F_1,\psi_1)}(x) - T_r^{(F_1,\psi_1)}(y), x - y\rangle,\ x, y \in H_1.$

(iii) $\mathrm{Fix}(T_r^{(F_1,\psi_1)}) = \mathrm{GEP}(F_1,\psi_1)$.

(iv) $\mathrm{GEP}(F_1,\psi_1)$ *is compact and convex.*

Further, assume that $F_2, \psi_2 : Q \times Q \to \mathbb{R}$ satisfying Assumption 1. For $a > 0$ and for all $w \in H_2$, define a mapping $T_a^{(F_2,\psi_2)} : H_2 \to Q$ as follows:

$$T_a^{(F_2,\psi_2)}w = \{d \in Q : F_2(d,e) + \psi_2(d,e) + \frac{1}{a}\langle e - d, d - w\rangle \ge 0\},\ \forall e \in Q.$$

Then, we easily observe that $T_a^{(F_2,\psi_2)}$ satisfy in Lemma 7 and $\mathrm{GEP}(F_2,\psi_2)$ is compact and convex.

Lemma 8. [10] Let $F_1 : C \times C \to \mathbb{R}$ be a bifunction satisfying Assumption 1 hold and let $T_r^{F_1}$ be defined as in Lemma 7, for $r > 0$. Let $x, y \in H_1$ and $r_1, r_2 > 0$. Then,

$$\|T_{r_2}^{F_1} y - T_{r_1}^{F_1} x\| \le \|x - y\| + |\frac{r_2 - r_1}{r_2}| \|T_{r_2}^{F_1} y - y\|.$$

Lemma 9. [34] Let $F_1 : C \times C \to \mathbb{R}$ be a bifunction satisfying Assumption 1 hold and let $T_r^{F_1}$ be defined as in Lemma 7, for $r > 0$. Let $x \in H_1$ and $r_1, r_2 > 0$. Then,

$$\|T_{r_2}^{F_1} x - T_{r_1}^{F_1} x\|^2 \le \frac{r_2 - r_1}{r_2} \langle T_{r_2}^{F_1}(x) - T_{r_1}^{F_1}(x), T_{r_2}^{F_1}(x) - x \rangle.$$

3. Viscosity Iterative Algorithm for SGEP

Let H_1 and H_2 be two real Hilbert spaces and $C \subseteq H_1$, $Q \subseteq H_2$ be nonempty, closed and convex subsets. Let $F_1, \psi_1 : C \times C \to \mathbb{R}$ and $F_2, \psi_2 : Q \times Q \to \mathbb{R}$ are nonlinear mappings satisfying Assumption 1 and F_2 is upper semicontinuous in first argument. Let $S = \{T(s) : s \in [0, +\infty)\}$ be a nonexpansive semigroup on C such that $\Theta = \mathrm{Fix}(S) \cap \Gamma \neq \emptyset$. Also $f : H_1 \to H_1$ be a contraction mapping with constant $\alpha \in (0,1)$, $A : H_1 \to H_2$ be a bounded linear operator and B and D be strongly positive bounded linear self adjoint operators on H_1 with constants $\bar{\gamma}_1, \bar{\gamma}_2 > 0$, such that $0 < \gamma < \frac{\bar{\gamma}_1}{\alpha} < \gamma + \frac{1}{\alpha}$, $\bar{\gamma}_1 \le \|B\| \le 1$ and $\|D\| = \bar{\gamma}_2$.

Algorithm 1. For given $x_0 \in C$ arbitrary, let the sequence $\{x_n\}$ be generated by:

$$\begin{cases} u_n = T_{r_n}^{(F_1,\psi_1)}(x_n + \delta A^*(T_{a_n}^{(F_2,\psi_2)} - I)Ax_n) \\ x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n D x_n + ((1-\varepsilon_n)I - \beta_n D - \alpha_n B)\frac{1}{s_n}\int_0^{s_n} T(s)u_n ds, \end{cases} \tag{7}$$

where $\delta \in (0, \frac{1}{L^2})$, L is the spectral radius of the operator A^*A and A^* is the adjoint of A, $\{s_n\}$ is positive real sequence, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\varepsilon_n\}$ are the sequence in $(0,1)$ such that $\varepsilon_n \le \alpha_n$, and $\{r_n\} \subset [r, \infty)$ with $r > 0$, $\{a_n\} \subset [a, \infty)$ with $a > 0$ satisfying conditions:

$(C1)\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = \lim_{n\to\infty} \varepsilon_n = 0$, $\Sigma_{n=1}^{\infty} \alpha_n = \infty$;
$(C2)\lim_{n\to\infty} \frac{|s_{n+1} - s_n|}{s_n} = 0$, $\lim_{n\to\infty} s_n = \infty$;
$(C3)\lim_{n\to\infty} |r_{n+1} - r_n| = 0$, $\liminf_{n\to\infty} r_n > 0$, $\lim_{n\to\infty} |a_{n+1} - a_n| = 0$.

Lemma 10. For any $0 < \gamma < \frac{\bar{\gamma}_1}{\alpha} < \gamma + \frac{1}{\alpha}$, there exist a unique fixed point for sequence $\{x_n\}$.

Proof. Using similar argument used in the proof of Lemma 3.1 [28], we can find that the iteration (7) is well defined. ■

Remark 3.1. By 7 (ii) for $\delta \in (0, \frac{1}{2L^2})$,the mapping $I + \delta A^*(T_{a_n}^{(F_2,\psi_2)} - I)A$ is a nonexpansive mapping and $A^*(T_{a_n}^{(F_2,\psi_2)} - I)A$ is a $\frac{1}{2L^2}$-inverse strongly monotone mapping.

Lemma 11. Let $p \in \Theta = \mathrm{Fix}(S) \cap \Gamma$. Then the sequence $\{x_n\}$ generated by Algorithm 1 is bounded.

Proof. Theorem 3.1 [39] implies that

$$\|u_n - p\|^2 \leq \|x_n - p\|^2 + \delta(\delta - \tfrac{1}{L^2})\|A^*(T_{a_n}^{(F_2,\psi_2)} - I)Ax_n\|^2.$$

Since $\delta \in (0, \frac{1}{2L^2})$, we have

$$\|u_n - p\| \leq \|x_n - p\|. \tag{8}$$

Therefore

$$\begin{aligned}
&\|x_{n+1} - p\| \\
&= \|\alpha_n \gamma f(x_n) + \beta_n D x_n + ((1-\varepsilon_n)I - \beta_n D - \alpha_n B)\tfrac{1}{s_n}\textstyle\int_0^{s_n} T(s)u_n ds - p\| \\
&\leq \alpha_n \|\gamma f(x_n) - Bp\| + \beta_n \|Dx_n - Dp\| + \varepsilon_n \|p\| \\
&\quad + \|((1-\varepsilon_n)I - \beta_n D - \alpha_n B)\| \tfrac{1}{s_n}\textstyle\int_0^{s_n} \|T(s)u_n - T(s)p\| ds \\
&\leq \alpha_n(\|\gamma f(x_n) - \gamma f(p)\| + \|\gamma f(p) - Bp\|) + \beta_n \|Dx_n - Dp\| + \varepsilon_n \|p\| \\
&\quad + (1 - \beta_n \bar{\gamma}_2 - \alpha_n \bar{\gamma}_1)\|u_n - p\| \\
&\leq \alpha_n \gamma \alpha \|x_n - p\| + \alpha_n \|\gamma f(p) - Bp\| + \beta_n \bar{\gamma}_2 \|x_n - p\| + \alpha_n \|p\| \\
&\quad + (1 - \beta_n \bar{\gamma}_2 - \alpha_n \bar{\gamma}_1)\|u_n - p\| \\
&= (1 - (\bar{\gamma}_1 - \gamma\alpha)\alpha_n)\|x_n - p\| + \alpha_n(\|p\| + \|\gamma f(p) - Bp\|) \\
&\leq \max\{\|x_n - p\|, \tfrac{\|\gamma f(p) - Bp\| + \|p\|}{\bar{\gamma}_1 - \gamma\alpha}\} \\
&\vdots \\
&\leq \max\{\|x_0 - p\|, \tfrac{\|\gamma f(p) - Bp\| + \|p\|}{\bar{\gamma}_1 - \gamma\alpha}\}.
\end{aligned} \tag{9}$$

Hence $\{x_n\}$ is bounded. ■

Now, set $t_n := \frac{1}{s_n}\int_0^{s_n} T(s)u_n ds$. Then the sequences $\{u_n\}$, $\{t_n\}$ and $\{f(x_n)\}$ are bounded.

Lemma 12. The following properties are satisfying for the algorithm 1

- $P1$. $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$.
- $P2$. $\lim_{n\to\infty} \|x_n - t_n\| = 0$.
- $P3$. $\lim_{n\to\infty} \|(T_{a_n}^{(F_2,\psi_2)} - I)Ax_n\|^2 = 0$.
- $P4$. $\lim_{n\to\infty} \|t_n - u_n\| = 0$.
- $P5$. $\lim_{n\to\infty} \|T(s)t_n - t_n\| = 0$.

Proof. P1: By Theorem 3.1 [39], we have

$$\|u_{n+1} - u_n\| \leq \|x_{n+1} - x_n\| + \delta\|A\|(\tfrac{|a_{n+1} - a_n|}{a_{n+1}}\eta_n)^{\frac{1}{2}} + \tfrac{|r_{n+1} - r_n|}{r_{n+1}}\sigma_{n+1} \tag{10}$$

where

$$\sigma_{n+1} = \sup_{n\in\mathbb{N}} \|T_{r_{n+1}}^{(F_1,\psi_1)}(x_{n+1} + \delta A^*(T_{a_{n+1}}^{(F_2,\psi_2)} - I)Ax_{n+1}) - (x_{n+1} + \delta A^*(T_{a_{n+1}}^{(F_2,\psi_2)} - I)Ax_{n+1})\|$$

$$\eta_n = \sup_{n\in\mathbb{N}} \langle T_{a_{n+1}}^{(F_2,\psi_2)} Ax_n - T_{a_n}^{(F_2,\psi_2)} Ax_n, T_{a_{n+1}}^{(F_2,\psi_2)} Ax_n - Ax_n \rangle,$$

$$\begin{aligned}
\|t_{n+1}-t_n\| &= \|\tfrac{1}{s_{n+1}}\int_0^{s_{n+1}} T(s)u_{n+1}ds - \tfrac{1}{s_n}\int_0^{s_n} T(s)u_n ds\| \\
&= \|\tfrac{1}{s_{n+1}}\int_0^{s_{n+1}} (T(s)u_{n+1}-T(s)u_n)ds + (\tfrac{1}{s_{n+1}}-\tfrac{1}{s_n})\int_0^{s_n}(T(s)u_n - T(s)p)ds \\
&\quad + \tfrac{1}{s_{n+1}}\int_{s_n}^{s_{n+1}}(T(s)u_n - T(s)p)ds\| \\
&\le \|u_{n+1}-u_n\| + \tfrac{|s_{n+1}-s_n|s_n}{s_{n+1}s_n}\|u_n-p\| + \tfrac{|s_{n+1}-s_n|}{s_{n+1}}\|u_n-p\| \\
&= \|u_{n+1}-u_n\| + \tfrac{2|s_{n+1}-s_n|}{s_{n+1}}\|u_n-p\|.
\end{aligned}$$

By (10) we estimate that

$$\|t_{n+1}-t_n\| \le \|x_{n+1}-x_n\| + \delta\|A\|(\tfrac{|a_{n+1}-a_n|}{a_{n+1}}\eta_n)^{\frac{1}{2}} + \tfrac{|r_{n+1}-r_n|}{r_{n+1}}\sigma_{n+1} + \tfrac{2|s_{n+1}-s_n|}{s_{n+1}}\|u_n-p\|. \tag{11}$$

setting $x_{n+1} = \varepsilon_n x_n + (1-\varepsilon_n)e_n$, then we have

$$\begin{aligned}
e_{n+1}-e_n &= \tfrac{\alpha_{n+1}\gamma f(x_{n+1})+\beta_{n+1}Dx_{n+1}+((1-\varepsilon_{n+1})I-\beta_{n+1}D-\alpha_{n+1}B)t_{n+1}-\varepsilon_{n+1}x_{n+1}}{1-\varepsilon_{n+1}} \\
&\quad - \tfrac{\alpha_n\gamma f(x_n)+\beta_n Dx_n+((1-\varepsilon_n)I-\beta_n D-\alpha_n B)t_n-\varepsilon_n x_n}{1-\varepsilon_n} \\
&= \tfrac{\alpha_{n+1}}{1-\varepsilon_{n+1}}(\gamma f(x_{n+1})-Bt_{n+1}) + \tfrac{\alpha_n}{1-\varepsilon_n}(Bt_n-\gamma f(x_n)) + \tfrac{\beta_{n+1}}{1-\varepsilon_{n+1}}D(x_{n+1}-t_{n+1}) \\
&\quad + \tfrac{\beta_n}{1-\varepsilon_n}D(t_n-x_n) + (t_{n+1}-t_n) + \tfrac{\varepsilon_n}{1-\varepsilon_n}x_n - \tfrac{\varepsilon_{n+1}}{1-\varepsilon_{n+1}}x_{n+1}.
\end{aligned}$$

Using (11), we have

$$\begin{aligned}
\|e_{n+1}-e_n\| &\le \tfrac{\alpha_{n+1}}{1-\varepsilon_{n+1}}\|\gamma f(x_{n+1})-Bt_{n+1}\| + \tfrac{\alpha_n}{1-\varepsilon_n}\|\gamma f(x_n)-Bt_n\| \\
&\quad + \tfrac{\beta_{n+1}}{1-\varepsilon_{n+1}}\|D\|\|x_{n+1}-t_{n+1}\| + \tfrac{\beta_n}{1-\varepsilon_n}\|D\|\|t_n-x_n\| + \|t_{n+1}-t_n\| \\
&\quad + \tfrac{\varepsilon_n}{1-\varepsilon_n}\|x_n\| + \tfrac{\varepsilon_{n+1}}{1-\varepsilon_{n+1}}\|x_{n+1}\| \\
&\le \tfrac{\alpha_{n+1}}{1-\varepsilon_{n+1}}\|\gamma f(x_{n+1})-Bt_{n+1}\| + \tfrac{\alpha_n}{1-\varepsilon_n}\|\gamma f(x_n)-Bt_n\| \\
&\quad + \tfrac{\beta_{n+1}}{1-\varepsilon_{n+1}}\|D\|(\|x_{n+1}\|+\|t_{n+1}\|) + \tfrac{\beta_n}{1-\varepsilon_n}\|D\|(\|t_n\|+\|x_n\|) + \|x_{n+1}-x_n\| \\
&\quad + \delta\|A\|(\tfrac{|a_{n+1}-a_n|}{a_{n+1}}\eta_n)^{\frac{1}{2}} + \tfrac{|r_{n+1}-r_n|}{r_{n+1}}\sigma_{n+1} + \tfrac{2|s_{n+1}-s_n|}{s_{n+1}}\|u_n-p\| \\
&\quad + \tfrac{\varepsilon_n}{1-\varepsilon_n}\|x_n\| + \tfrac{\varepsilon_{n+1}}{1-\varepsilon_{n+1}}\|x_{n+1}\|
\end{aligned}$$

which implies that

$$\begin{aligned}
&\|e_{n+1}-e_n\| - \|x_{n+1}-x_n\| \\
&\le \tfrac{\alpha_{n+1}}{1-\varepsilon_{n+1}}\|\gamma f(x_{n+1})-Bt_{n+1}\| + \tfrac{\alpha_n}{1-\varepsilon_n}\|\gamma f(x_n)-Bt_n\| \\
&\quad + \tfrac{\beta_{n+1}}{1-\varepsilon_{n+1}}\|D\|(\|x_{n+1}\|+\|t_{n+1}\|) + \tfrac{\beta_n}{1-\varepsilon_n}\|D\|(\|t_n\|+\|x_n\|) + \delta\|A\|(\tfrac{|a_{n+1}-a_n|}{a_{n+1}}\eta_n)^{\frac{1}{2}} \\
&\quad + \tfrac{|r_{n+1}-r_n|}{r_{n+1}}\sigma_{n+1} + \tfrac{2|s_{n+1}-s_n|}{s_{n+1}}\|u_n-p\| + \tfrac{\varepsilon_n}{1-\varepsilon_n}\|x_n\| + \tfrac{\varepsilon_{n+1}}{1-\varepsilon_{n+1}}\|x_{n+1}\|.
\end{aligned}$$

Hence, it follows by conditions $(C1)-(C3)$ that

$$\limsup_{n\to\infty}(\|e_{n+1}-e_n\| - \|x_{n+1}-x_n\|) \le 0. \tag{12}$$

From (12) and Lemma 4 we get $\lim_{n\to\infty}\|e_{n+1}-x_n\|=0$, and then

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=\lim_{n\to\infty}(1-\varepsilon_n)\|e_{n+1}-x_n\|=0. \tag{13}$$

Also by (11) we have $\lim_{n\to\infty}\|t_{n+1}-t_n\|=0$.

P2: We can write

$$\begin{aligned}\|x_n-t_n\| &\leq \|x_{n+1}-x_n\|+\|\alpha_n\gamma f(x_n)+\beta_n Dx_n+((1-\varepsilon_n)I-\beta_n D-\alpha_n B)t_n-t_n\|\\ &\leq \|x_{n+1}-x_n\|+\alpha_n\|\gamma f(x_n)-Bt_n\|+\beta_n\|Dx_n-Dt_n\|+\varepsilon_n\|t_n\|\\ &\leq \|x_{n+1}-x_n\|+\alpha_n\|\gamma f(x_n)-Bt_n\|+\beta_n\bar{\gamma}_2\|x_n-t_n\|+\varepsilon_n\|t_n\|.\end{aligned}$$

Then

$$(1-\beta_n\bar{\gamma}_2)\|x_n-t_n\|\leq\|x_{n+1}-x_n\|+\alpha_n\|\gamma f(x_n)-Bt_n\|+\varepsilon_n\|t_n\|.$$

Therefore, we thus

$$\begin{aligned}\|x_n-t_n\| &\leq \tfrac{1}{1-\beta_n\bar{\gamma}_2}\|x_{n+1}-x_n\|+\tfrac{\alpha_n}{1-\beta_n\bar{\gamma}_2}\|\gamma f(x_n)-Bt_n\|+\tfrac{\varepsilon_n}{1-\beta_n\bar{\gamma}_2}\|t_n\|\\ &\leq \tfrac{1}{1-\beta_n\bar{\gamma}_2}\|x_{n+1}-x_n\|+\tfrac{\alpha_n}{1-\beta_n\bar{\gamma}_2}(\|\gamma f(x_n)-Bt_n\|+\|t_n\|).\end{aligned}$$

The condition $(C1)$ together (P_1) implies that

$$\lim_{n\to\infty}\|x_n-t_n\|=0. \tag{14}$$

P3: From (8), we have

$$\begin{aligned}&\|x_{n+1}-p\|^2\\ &=\|\alpha_n\gamma f(x_n)+\beta_n Dx_n+((1-\varepsilon_n)I-\beta_n D-\alpha_n B)t_n-p\|^2\\ &=\|\alpha_n(\gamma f(x_n)-Bp)+\beta_n(Dx_n-Dp)+((1-\varepsilon_n)I-\beta_n D-\alpha_n B)(t_n-p)-\varepsilon_n p\|^2\\ &\leq\|((1-\varepsilon_n)I-\beta_n D-\alpha_n B)(t_n-p)+\beta_n(Dx_n-Dp)-\varepsilon_n p\|^2+2\langle\alpha_n(\gamma f(x_n)-Bp),x_{n+1}-p\rangle\\ &\leq((1-\beta_n\bar{\gamma}_2-\alpha_n\bar{\gamma}_1)\|u_n-p\|+\beta_n\|D\|\|x_n-t_n\|+\varepsilon_n\|p\|)^2+2\alpha_n\langle\gamma f(x_n)-Bp,x_{n+1}-p\rangle\\ &=(1-\beta_n\bar{\gamma}_2-\alpha_n\bar{\gamma}_1)^2\|u_n-p\|^2+(\beta_n)^2\|D\|^2\|x_n-t_n\|^2+(\varepsilon_n)^2\|p\|^2\\ &\quad+2(1-\beta_n\bar{\gamma}_2-\alpha_n\bar{\gamma}_1)\beta_n\|D\|\|u_n-p\|\|x_n-t_n\|+2(1-\beta_n\bar{\gamma}_2-\alpha_n\bar{\gamma}_1)\varepsilon_n\|p\|\|u_n-p\|\\ &\quad+2\beta_n\varepsilon_n\|D\|\|p\|\|x_n-t_n\|+2\alpha_n\langle\gamma f(x_n)-Bp,x_{n+1}-p\rangle\end{aligned} \tag{15}$$

$$
\begin{aligned}
&\leq (1-\beta_n\bar{\gamma}_2-\alpha_n\bar{\gamma}_1)^2(\|x_n-p\|^2+\delta(\delta-\tfrac{1}{L^2})\|A^*(T_{a_n}^{(F_2,\psi_2)}-I)Ax_n\|^2)+(\beta_n)^2\|D\|^2\|x_n-t_n\|^2\\
&\quad+(\varepsilon_n)^2\|p\|^2+2(1-\beta_n\bar{\gamma}_2-\alpha_n\bar{\gamma}_1)\beta_n\|D\|\|u_n-p\|\|x_n-t_n\|+2(1-\beta_n\bar{\gamma}_2-\alpha_n\bar{\gamma}_1)\varepsilon_n\|p\|\|u_n-p\|\\
&\quad+2\beta_n\varepsilon_n\|D\|\|p\|\|x_n-t_n\|+2\alpha_n\langle\gamma f(x_n)-Bp,x_{n+1}-p\rangle\\
&\leq \|x_n-p\|^2+(\beta_n\bar{\gamma}_2+\alpha_n\bar{\gamma}_1)^2\|x_n-p\|^2+(1-\beta_n\bar{\gamma}_2-\alpha_n\bar{\gamma}_1)^2\delta(\delta-\tfrac{1}{L^2})\|A^*(T_{a_n}^{(F_2,\psi_2)}-I)Ax_n\|^2\\
&\quad+(\beta_n)^2\|D\|^2\|x_n-t_n\|^2+(\alpha_n)^2\|p\|^2+2(1-\beta_n\bar{\gamma}_2-\alpha_n\bar{\gamma}_1)\beta_n\|D\|\|u_n-p\|\|x_n-t_n\|\\
&\quad+2(1-\beta_n\bar{\gamma}_2-\alpha_n\bar{\gamma}_1)\alpha_n\|p\|\|u_n-p\|+2\beta_n\varepsilon_n\|D\|\|p\|\|x_n-t_n\|+2\alpha_n\langle\gamma f(x_n)-Bp,x_{n+1}-p\rangle.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&(1-\beta_n\bar{\gamma}_2-\alpha_n\bar{\gamma}_1)^2\delta(\tfrac{1}{L^2}-\delta)\|A^*(T_{a_n}^{(F_2,\psi_2)}-I)Ax_n\|^2\\
&\leq \|x_n-p\|^2-\|x_{n+1}-p\|^2+(\beta_n\bar{\gamma}_2+\alpha_n\bar{\gamma}_1)^2\|x_n-p\|^2+(\beta_n)^2\|D\|^2\|x_n-t_n\|^2+(\alpha_n)^2\|p\|^2\\
&\quad+2(1-\beta_n\bar{\gamma}_2-\alpha_n\bar{\gamma}_1)\beta_n\|D\|\|u_n-p\|\|x_n-t_n\|+2(1-\beta_n\bar{\gamma}_2-\alpha_n\bar{\gamma}_1)\alpha_n\|p\|\|u_n-p\|\\
&\quad+2\beta_n\varepsilon_n\|D\|\|p\|\|x_n-t_n\|+2\alpha_n\langle\gamma f(x_n)-Bp,x_{n+1}-p\rangle\\
&\leq (\|x_n-p\|+\|x_{n+1}-p\|)\|x_n-x_{n+1}\|+(\beta_n\bar{\gamma}_2+\alpha_n\bar{\gamma}_1)^2\|x_n-p\|^2+(\beta_n)^2\|D\|^2\|x_n-t_n\|^2\\
&\quad+(\alpha_n)^2\|p\|^2+2(1-\beta_n\bar{\gamma}_2-\alpha_n\bar{\gamma}_1)\beta_n\|D\|\|u_n-p\|\|x_n-t_n\|+2(1-\beta_n\bar{\gamma}_2-\alpha_n\bar{\gamma}_1)\alpha_n\|p\|\|u_n-p\|\\
&\quad+2\beta_n\varepsilon_n\|D\|\|p\|\|x_n-t_n\|+2\alpha_n\langle\gamma f(x_n)-Bp,x_{n+1}-p\rangle.
\end{aligned}
$$

This implies by (C_1), (P_1) and (14) that

$$
\lim_{n\to\infty}\|(T_{a_n}^{(F_2,\psi_2)}-I)Ax_n\|^2=0. \tag{16}
$$

P4: we have

$$
\|u_n-p\|^2\leq\|x_n-p\|^2-\|u_n-x_n\|^2+2\delta\|u_n-x_n\|\|A^*(T_{a_n}^{(F_2,\psi_2)}-I)Ax_n\|,
$$

The equations (15) and (16) imply that

$$
\begin{aligned}
\|x_{n+1}-p\|^2 &\leq (1-\beta_n\bar{\gamma}_2-\alpha_n\bar{\gamma}_1)^2\|u_n-p\|^2+(\beta_n)^2\|D\|^2\|x_n-t_n\|^2+(\varepsilon_n)^2\|p\|^2\\
&\quad+2(1-\beta_n\bar{\gamma}_2-\alpha_n\bar{\gamma}_1)\beta_n\|D\|\|u_n-p\|\|x_n-t_n\|+2(1-\beta_n\bar{\gamma}_2-\alpha_n\bar{\gamma}_1)\varepsilon_n\|p\|\|u_n-p\|\\
&\quad+2\beta_n\varepsilon_n\|D\|\|p\|\|x_n-t_n\|+2\alpha_n\langle\gamma f(x_n)-Bp,x_{n+1}-p\rangle\\
&\leq (1-\beta_n\bar{\gamma}_2-\alpha_n\bar{\gamma}_1)^2(\|x_n-p\|^2-\|u_n-x_n\|^2+2\delta\|A(u_n-x_n)\|\|(T_{a_n}^{(F_2,\psi_2)}-I)Ax_n\|)\\
&\quad+(\beta_n)^2\|D\|^2\|x_n-t_n\|^2+(\varepsilon_n)^2\|p\|^2+2(1-\beta_n\bar{\gamma}_2-\alpha_n\bar{\gamma}_1)\beta_n\|D\|\|u_n-p\|\|x_n-t_n\|\\
&\quad+2(1-\beta_n\bar{\gamma}_2-\alpha_n\bar{\gamma}_1)\varepsilon_n\|p\|\|u_n-p\|+2\beta_n\varepsilon_n\|D\|\|p\|\|x_n-t_n\|\\
&\quad+2\alpha_n\langle\gamma f(x_n)-Bp,x_{n+1}-p\rangle\\
&\leq \|x_n-p\|^2+(\beta_n\bar{\gamma}_2+\alpha_n\bar{\gamma}_1)^2\|x_n-p\|^2-(1-\beta_n\bar{\gamma}_2-\alpha_n\bar{\gamma}_1)\|u_n-x_n\|^2\\
&\quad+2(1-\beta_n\bar{\gamma}_2-\alpha_n\bar{\gamma}_1)^2\delta\|A(u_n-x_n)\|\|(T_{a_n}^{(F_2,\psi_2)}-I)Ax_n\|+(\beta_n)^2\|D\|^2\|x_n-t_n\|^2\\
&\quad+(\alpha_n)^2\|p\|^2+2(1-\beta_n\bar{\gamma}_2-\alpha_n\bar{\gamma}_1)\beta_n\|D\|\|u_n-p\|\|x_n-t_n\|\\
&\quad+2(1-\beta_n\bar{\gamma}_2-\alpha_n\bar{\gamma}_1)\alpha_n\|p\|\|u_n-p\|+2\beta_n\varepsilon_n\|D\|\|p\|\|x_n-t_n\|\\
&\quad+2\alpha_n\langle\gamma f(x_n)-Bp,x_{n+1}-p\rangle.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&(1-\beta_n\bar{\gamma}_2-\alpha_n\bar{\gamma}_1)^2\|u_n-x_n\|^2\\
&\le \|x_n-p\|^2-\|x_{n+1}-p\|^2+(\beta_n\bar{\gamma}_2+\alpha_n\bar{\gamma}_1)^2\|x_n-p\|^2\\
&\quad +2(1-\beta_n\bar{\gamma}_2-\alpha_n\bar{\gamma}_1)^2\delta\|A(u_n-x_n)\|\|(T_{a_n}^{(F_2,\psi_2)}-I)Ax_n\|+(\beta_n)^2\|D\|^2\|x_n-t_n\|^2+(\alpha_n)^2\|p\|^2\\
&\quad +2(1-\beta_n\bar{\gamma}_2-\alpha_n\bar{\gamma}_1)\beta_n\|D\|\|u_n-p\|\|x_n-t_n\|+2(1-\beta_n\bar{\gamma}_2-\alpha_n\bar{\gamma}_1)\alpha_n\|p\|\|u_n-p\|\\
&\quad +2\beta_n\varepsilon_n\|D\|\|p\|\|x_n-t_n\|+2\alpha_n\langle\gamma f(x_n)-Bp,x_{n+1}-p\rangle\\
&\le (\|x_n-p\|+\|x_{n+1}-p\|)\|x_n-x_{n+1}\|+(\beta_n\bar{\gamma}_2+\alpha_n\bar{\gamma}_1)^2\|x_n-p\|^2\\
&\quad +2(1-\beta_n\bar{\gamma}_2-\alpha_n\bar{\gamma}_1)^2\delta\|A(u_n-x_n)\|\|(T_{a_n}^{(F_2,\psi_2)}-I)Ax_n\|+(\beta_n)^2\|D\|^2\|x_n-t_n\|^2+(\alpha_n)^2\|p\|^2\\
&\quad +2(1-\beta_n\bar{\gamma}_2-\alpha_n\bar{\gamma}_1)\beta_n\|D\|\|u_n-p\|\|x_n-t_n\|+2(1-\beta_n\bar{\gamma}_2-\alpha_n\bar{\gamma}_1)\alpha_n\|p\|\|u_n-p\|\\
&\quad +2\beta_n\varepsilon_n\|D\|\|p\|\|x_n-t_n\|+2\alpha_n\langle\gamma f(x_n)-Bp,x_{n+1}-p\rangle.
\end{aligned}
$$

It follows by Condition $(C1)$, $(P2)$ and $(P3)$ that

$$\lim_{n\to\infty}\|x_n-u_n\|=0. \tag{17}$$

This implies by (14) and (17)that

$\|t_n-u_n\|\le\|t_n-x_n\|+\|x_n-u_n\|\to 0$. Therefore, $\lim_{n\to\infty}\|t_n-u_n\|=0$.

P5: Let $E:=\{w\in C:\|w-p\|\le\|x_0-p\|,\frac{1}{\bar{\gamma}_1-\gamma\alpha}\|\gamma f(p)-Bp\|+\|p\|\}$, Then E is a nonempty bounded closed convex subset of C which is $T(s)$-invariant for each $s\in[0,+\infty)$ and contains $\{x_n\}$. So, without loss of generality, we may assume that $S:=\{T(s):s\in[0,+\infty)\}$ is a nonexpansive semigroup on E. From Theorem 1 [13] we have

$$\|T(s)x_n-x_n\| \quad \le 2\|\tfrac{1}{s_n}\textstyle\int_0^{s_n}T(s)u_nds-x_n\|+\|T(s)\tfrac{1}{s_n}\int_0^{s_n}T(s)u_nds-\tfrac{1}{s_n}\int_0^{s_n}T(s)u_nds\|.$$

By Lemma 3 and (14), we obtain $\lim_{n\to\infty}\|T(s)x_n-x_n\|=0$.
Therefore

$$
\begin{aligned}
\|T(s)t_n-x_n\| &\le \|T(s)t_n-T(s)x_n\|+\|T(s)x_n-x_n\|\\
&\le \|t_n-x_n\|+\|T(s)x_n-x_n\|\to 0,
\end{aligned}
$$

$$
\begin{aligned}
\|T(s)t_n-t_n\| &\le \|T(s)t_n-T(s)x_n\|+\|T(s)x_n-x_n\|+\|x_n-t_n\|\\
&\le \|t_n-x_n\|+\|T(s)x_n-x_n\|+\|x_n-t_n\|\to 0.
\end{aligned}
$$

Then we have $\lim_{n\to\infty}\|T(s)t_n-t_n\|=0$. ■

4. Convergence Algorithm

Theorem 1. The Algorithm defined by (7) convergence strongly to $z \in \mathrm{Fix}(S) \cap \Gamma$, which is a unique solution of the variational inequality

$$\langle (\gamma f - A)z, y - z \rangle \leq 0, \quad \forall y \in \Theta.$$

Proof. Let $s = P_\Theta$. We get

$$\begin{aligned} \|s(I-B+\gamma f)(x) - s(I-B+\gamma f)(y)\| &\leq \|(I-B+\gamma f)(x) - (I-B+\gamma f)(y)\| \\ &\leq \|I-B\|\|x-y\| + \gamma\|f(x)-f(y)\| \\ &\leq (1-\bar{\gamma}_1)\|x-y\| + \gamma\alpha\|x-y\| \\ &= (1-(\bar{\gamma}_1 - \gamma\alpha))\|x-y\|. \end{aligned}$$

Then $s(I-B+\gamma f)$ is a contraction mapping from H_1 into itself. Therefore by Banach contraction principle, there exists $z \in H_1$ such that $z = s(I-B+\gamma f)z = P_\Theta(I-B+\gamma f)z$.
We show that $\limsup_{n\to\infty}\langle (\gamma f - B)z, x_n - z\rangle \leq 0$ where $z = P_\Theta(I-B+\gamma f)$. To show this inequality, we choose a subsequence $\{t_{n_i}\}$ of $\{t_n\} \subseteq E$ such that

$$\limsup_{n\to\infty}\langle (\gamma f - B)z, t_n - z\rangle = \limsup_{n\to\infty}\langle (\gamma f - B)z, t_{n_i} - z\rangle. \tag{18}$$

Since $\{t_{n_i}\}$ is bounded, there exists a subsequence $\{t_{n_{i_j}}\}$ of $\{t_{n_i}\}$ which converges weakly to some $w \in C$. Without loss of generality, we can assume that $t_{n_i} \rightharpoonup w$. Now, we prove that $w \in \mathrm{Fix}(S) \cap \Gamma$. Let us first show that $w \in \mathrm{Fix}(S)$. Assume that $w \notin \mathrm{Fix}(S)$. Since $t_{n_i} \rightharpoonup w$ and $T(s)w \neq w$, from Opial's conditions (6) and Lemma 12 (P5), we have

$$\begin{aligned} \liminf_{n\to\infty}\|t_{n_i} - w\| &< \liminf_{n\to\infty}\|t_{n_i} - T(s)w\| \\ &\leq \liminf_{n\to\infty}(\|t_{n_i} - T(s)t_{n_i}\| + \|T(s)t_{n_i} - T(s)w\|) \\ &\leq \liminf_{n\to\infty}\|t_{n_i} - w\|, \end{aligned}$$

which is a contradiction. Thus, we obtain $w \in \mathrm{Fix}(S)$. We show that $w \in \Gamma$. Since $u_n = T_{r_n}^{(F_1,\psi_1)}(x_n + \delta A^*(T_{a_n}^{(F_2,\psi_2)} - I)Ax_n)$, where $d_n = x_n + \delta A^*(T_{a_n}^{(F_2,\psi_2)} - I)Ax_n$.
we have

$$F_1(u_n, y) + \psi_1(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - d_n\rangle \geq 0, \qquad \forall y \in C.$$

It follows from the monotonically of F_1 that

$$\psi_1(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - d_n\rangle \geq F_1(u_n, y), \qquad \forall y \in C$$

which implies that

$$\psi_1(u_n, y) + \langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_n} + \delta A^*(\frac{(T_{a_{n_i}}^{(F_2,\psi_2)} - I)Ax_{n_i}}{r_n})\rangle \geq F_1(y, u_{n_i}), \qquad \forall y \in C.$$

From $\|u_n - x_n\| \to 0$, we get $u_{n_i} \rightharpoonup w$ and $\frac{u_{n_i} - x_{n_i}}{r_n} \to 0$.
Since $\lim_{n\to\infty} \|A^*(T_{a_n}^{(F_2,\psi_2)} - I)Ax_n\| = 0$, then $A^*(\frac{(T_{a_{n_i}}^{(F_2,\psi_2)} - I)Ax_{n_i}}{r_n}) \to 0$.
Therefore

$$\psi_1(u_{n_i}, y) \geq F_1(y, u_{n_i}), \qquad \psi_1(w,y) \geq F_1(y,w).$$

Let $y_t = ty + (1-t)w$ for all $t \in (0,1]$. Since $y \in C$ and $w \in C$, we get $y_t \in C$. It follows from Assumption 1 that

$$\begin{aligned} 0 = F_1(y_t,y_t) + \psi_1(y_t,y_t) &\leq tF_1(y_t,y) + (1-t)F_1(y_t,w) \\ &\quad + t\psi_1(y_t,y) + (1-t)\psi_1(y_t,w) \\ &= t(F_1(y_t,y) + \psi_1(y_t,y)) \\ &\quad + (1-t)(F_1(y_t,w) + \psi_1(y_t,w)) \\ &\leq F_1(y_t,y) + \psi_1(y_t,y), \end{aligned}$$

so $0 \leq F_1(y_t,y) + \psi_1(y_t,y)$.
Letting $t \to 0$, we obtain $0 \leq F_1(w,y) + \psi_1(w,y)$. This implies that $w \in \mathrm{GEP}(F_1,\psi_1)$. Now we show that $Aw \in \mathrm{GEP}(F_2,\psi_2)$. Since $\|u_n - x_n\| \to 0$, $u_n \rightharpoonup w$ as $n \to \infty$ and $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \rightharpoonup w$ and since A is bounded linear operator so that $Ax_{n_j} \rightharpoonup Aw$.
From $\|(T_{a_n}^{(F_2,\psi_2)} - I)Ax_n\| \to 0$, we have $T_{a_n}^{(F_2,\psi_2)}Ax_{n_j} \rightharpoonup Aw$. Therefore from Lemma 7, we have

$$F_2(T_{a_{n_j}}^{(F_2,\psi_2)}Ax_{n_j}, v) + h_2(T_{a_{n_j}}^{(F_2,\psi_2)}Ax_{n_j}, v) + \frac{1}{a_{n_j}}\langle v - T_{a_{n_j}}^{(F_2,\psi_2)}Ax_{n_j}, T_{a_{n_j}}^{(F_2,\psi_2)}Ax_{n_j} - Aw\rangle \geq 0, \qquad \forall v \in Q.$$

Since F_2 is upper semicontinuous in first argument, from above inequality, we obtain

$$F_2(Aw,v) + \psi_2(Aw,v) \geq 0, \qquad \forall v \in Q,$$

which means that $Aw \in \mathrm{GEP}(F_2,\psi_2)$ and hence $w \in \Gamma$.
We claim that $\limsup_{n\to\infty}\langle(\gamma f - B)z, x_n - z\rangle \leq 0$, where $z = P_\Theta(I - B + \gamma f)$. Now from (3), we have

$$\begin{aligned} \limsup_{n\to\infty}\langle(\gamma f - B)z, x_n - z\rangle &= \limsup_{n\to\infty}\langle(\gamma f - B)z, t_n - z\rangle \\ &\leq \limsup_{n\to\infty}\langle(\gamma f - B)z, t_{n_i} - z\rangle \\ &= \langle(\gamma f - B)z, w - z\rangle \\ &\leq 0. \end{aligned} \tag{19}$$

Next, we show that $x_n \to z$. It follows from (8) that

$$\begin{aligned}
\|x_{n+1}-z\|^2 &= \alpha_n\langle\gamma f(x_n)-Bz,x_{n+1}-z\rangle+\beta_n\langle Dx_n-Dz,x_{n+1}-z\rangle-\varepsilon_n\langle z,x_{n+1}-z\rangle\\
&\quad+\langle((1-\varepsilon_n)I-\beta_nD-\alpha_nB)(t_n-z),x_{n+1}-z\rangle\\
&\le \alpha_n(\gamma\langle f(x_n)-f(z),x_{n+1}-z\rangle+\langle\gamma f(z)-Bz,x_{n+1}-z\rangle)+\beta_n\|D\|\|x_n-z\|\|x_{n+1}-z\|\\
&\quad-\varepsilon_n\|z\|\|x_{n+1}-z\|+\|(1-\varepsilon_n)I-\beta_nD-\alpha_nB\|\|t_n-z\|\|x_{n+1}-z\|\\
&\le \alpha_n\alpha\gamma\|x_n-z\|\|x_{n+1}-z\|+\alpha_n\langle\gamma f(z)-Bz,x_{n+1}-z\rangle+\beta_n\bar\gamma_2\|x_n-z\|\|x_{n+1}-z\|\\
&\quad-\varepsilon_n\|z\|\|x_{n+1}-z\|+(1-\beta_n\bar\gamma_2-\alpha_n\bar\gamma_1)\|x_n-z\|\|x_{n+1}-z\|\\
&= (1-\alpha_n(\bar\gamma_1-\alpha\gamma))\|x_n-z\|\|x_{n+1}-z\|-\varepsilon_n\|z\|\|x_{n+1}-z\|+\alpha_n\langle\gamma f(z)-Bz,x_{n+1}-z\rangle\\
&\le \tfrac{1-\alpha_n(\bar\gamma_1-\alpha\gamma)}{2}(\|x_n-z\|^2+\|x_{n+1}-z\|^2)-\varepsilon_n\|z\|\|x_{n+1}-z\|+\alpha_n\langle\gamma f(z)-Bz,x_{n+1}-z\rangle\\
&\le \tfrac{1-\alpha_n(\bar\gamma_1-\alpha\gamma)}{2}\|x_n-z\|^2+\tfrac{1}{2}\|x_{n+1}-z\|^2-\varepsilon_n\|z\|\|x_{n+1}-z\|+\alpha_n\langle\gamma f(z)-Bz,x_{n+1}-z\rangle.
\end{aligned}$$

This implies that

$$\begin{aligned}
2\|x_{n+1}-z\|^2 &\le (1-\alpha_n(\bar\gamma_1-\alpha\gamma))\|x_n-z\|^2+\|x_{n+1}-z\|^2\\
&\quad-2\alpha_n\|z\|\|x_{n+1}-z\|+2\alpha_n\langle\gamma f(z)-Bz,x_{n+1}-z\rangle.
\end{aligned}$$

Then

$$\begin{aligned}
\|x_{n+1}-z\|^2 &\le (1-\alpha_n(\bar\gamma_1-\alpha\gamma))\|x_n-z\|^2-2\alpha_n\|z\|\|x_{n+1}-z\|+2\alpha_n\langle\gamma f(z)-Bz,x_{n+1}-z\rangle\\
&= (1-k_n)\|x_n-z\|^2+2\alpha_n l_n,
\end{aligned}\tag{20}$$

where $k_n=\alpha_n(\bar\gamma_1-\alpha\gamma)$ and $l_n=\langle\gamma f(z)-Bz,x_{n+1}-z\rangle-\|z\|\|x_{n+1}-z\|$.
Since $\lim_{n\to\infty}\alpha_n=0$ and $\Sigma_{n=0}^{\infty}\alpha_n=\infty$, it is easy to see that $\lim_{n\to\infty}k_n=0$, $\Sigma_{n=0}^{\infty}k_n=\infty$ and $\limsup_{n\to\infty}l_n\le 0$. Hence, from (19) and (20) and Lemma 5, we deduce that $x_n\to z$, where $z=P_\Theta(I-B+\gamma f)z$. ■

Remark 4.1. Putting $r_n=a_n,\{\varepsilon_n\}=0,D=0$ and $\delta\in(0,\frac{1}{L})$ we obtain method introduced in Theorem 1 [13]. Taking $H_1=H_2=H,\psi_1=\psi_2=0,\{\beta_n\}=\{\varepsilon_n\}=0$ and $A=D=0$, then Theorem 4.1 of [9] is obtained. Taking $H_1=H_2=H,F_1=F_2=\psi_1=\psi_2=0,\{\varepsilon_n\}=0,A=0$ and $B=D=I$, then Theorem 3.3 of [24] is obtained.

5. Numerical Examples

In this section, we give some examples and numerical results for supporting our main theorem. All the numerical results have been produced in Matlab 2017 on a Linux workstation with a 3.8 GHZ Intel annex processor and 8 Gb of memory.

Example 5.1. Let $H_1=H_2=\mathbb{R}^2$, the set of all real numbers, with the inner product defined by $\langle(x,y),(z,t)\rangle=xz+yt,\ \forall(x,y),(z,t)\in\mathbb{R}^2$, and induced usual norm $\|(x,y)\|=(x^2+y^2)^{\frac{1}{2}}$. Let $C=[-1,2]\times[0,3]$, $Q=[-4,0]\times[0,3]$. Define bifunctions $F_1,\psi_1:C\times C\to\mathbb{R}$ by $F_1((x,y),(z,t))=(x,y-3)(z-x,t-y)$ and $\psi_1((x,y),(z,t))=(3x,2y+3)(z-x,t-y)$. Define $F_2,\psi_2:Q\times Q\to\mathbb{R}$

by $F_2((u,v),(k,l)) = (u-2,v)(k-u,l-v)$ and $\psi_2((u,v),(k,l)) = (-3u+2,\frac{v}{2})(k-u,l-v)$. Furthermore, define $f(x,y) = (\frac{1}{8}x,\frac{1}{7}y)$, $A(x,y) = (2x,2y)$, $B(x,y) = (-2x,-2y)$, $D(x,y) = (\frac{1}{2}x,\frac{1}{2}y)$. We define a nonexpansive semigroup mappings on C as follows:

$$T(s)(x,y) = \frac{1}{1+2s}(x,y)$$

Choose $\alpha_n = \frac{1}{2n}$, $\beta_n = \frac{1}{n^2+n}$, $\varepsilon_n = \frac{1}{3n^2}$, $s_n = 2n$, $r_n = 1+\frac{1}{n}$ and $a_n = \frac{2n}{3n-1}$. The sequences $\{(x_n,y_n)\} \subset \mathbb{R}^2$, $\{u_n\} \subset C$, and $\{z_n\} \subset Q$ generated by the iterative schemes

$$z_n = T_{a_n}^{(F_2,\psi_2)}(A(x_n,y_n));\ u_n = T_{r_n}^{(F_1,\psi_1)}((x_n,y_n)+\frac{1}{16}A^*(z_n - A(x_n,y_n))); \tag{21}$$

$$(x_{n+1},y_{n+1}) = \frac{1}{n}(\frac{1}{8}x_n,\frac{1}{7}y_n)+\frac{1}{2(n^2+n)}(x_n,y_n)+((1-\frac{1}{3n^2})I-\frac{1}{(n^2+n)}D-\frac{1}{2n}B)\frac{1}{s_n}\int_0^{s_n}\frac{1}{1+2s}u_n ds \tag{22}$$

It is easy to check that all the conditions in Theorem 1 satisfy with $w = \{(0,0)\} \in \mathrm{Fix}(S)\cap\Gamma$. After simplification, schemes (21) and (22) reduce to

$$z_n = (\tfrac{(2-6n)}{n+1}x_n, \tfrac{6n-2}{6n-1}y_n);$$

$$u_n = (\tfrac{n}{(n+1)(5n+1)}x_n, \tfrac{42n^2-8n}{(48n-8)(2n+3)}y_n);$$

$$(x_{n+1},y_{n+1}) = \frac{1}{n}(\frac{1}{8}x_n,\frac{1}{7}y_n)+\frac{1}{2(n^2+n)}(x_n,y_n)+\frac{1}{4n}\ln(1+4n)(1-\frac{1}{3n^2}-\frac{1}{2(n^2+n)}+\frac{1}{n})u_n$$

Choose $(x_1,y_1) = (2,1)$. Figure 1 indicates the behavior of x_n for algorithm (1), which converges to the same solution, that is, $w = \{(0,0)\} \in \mathrm{Fix}(S)\cap\Gamma$ as a solution of this example.

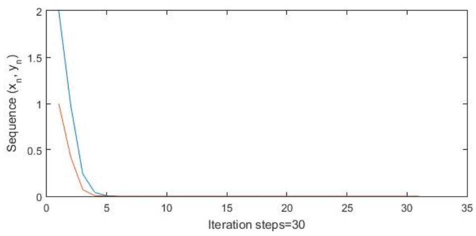

FIGURE 1. The graph of $\{(x_n,y_n)\}$ with initial value $(2,1)$.

Example 5.2. Let $H_1 = H_2 = \mathbb{R}$ with the inner product $\langle x,y\rangle = xy$, $\forall x,y \in \mathbb{R}$, and induced usual norm $|\ .\ |$. Let $C = [0,2]$, $Q = [-4,-2]$. Define $F_1,\psi_1 : C\times C \to \mathbb{R}$ by $F_1(x,y) = (x-6)(y-x)$ and $\psi_1(x,y) = 2x(y-x)$, $\forall x,y \in C$. Define $F_2,\psi_2 : Q\times Q \to \mathbb{R}$ by $F_2(u,v) = (u+16)(v-u)$ and $\psi_2(u,v) = 3u(v-u)$. Let $f(x) = \frac{1}{6}x$, $A(x) = -2x$, $B(x) = \frac{1}{2}x$, $D(x) = x$, and for each $x \in C$, $T(s)x = x$. Furthermore, we take $\alpha_n = \frac{1}{n}$, $\beta_n = \frac{1}{2(n+1)^2}$, $\varepsilon_n = \frac{1}{n^2}$, $s_n = n$, $r_n = 1+\frac{2}{n}$ and

$a_n = \frac{n}{2n+1}$, in algorithm (1), we obtain

$$x_{n+1} = (\frac{1}{3n} + \frac{1}{2(n+1)^2})x_n + ((1 - \frac{1}{n^2})I - \frac{1}{2(n+1)^2}D - \frac{1}{n}B)\frac{1}{s_n}\int_0^{s_n} u_n ds$$

After simplification, schemes we have

$$x_{n+1} = (\frac{1}{3n} + \frac{1}{2(n+1)^2})x_n + u_n(1 - \frac{1}{n^2} - \frac{1}{2(n+1)^2} - \frac{1}{2n})$$

Figure 2 indicates the behavior of x_n with initial point $x_1 = 0.1$.

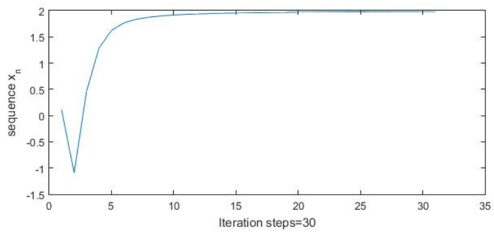

FIGURE 2. The graph of $\{x_n\}$ with initial value $x_1 = 0.1$.

Example 5.3. Let $H_1 = H_2 = \mathbb{R}$ with the inner product defined by $\langle x,y\rangle = xy$, and induced usual norm $|\ .\ |$. Let $C = [0,4]$,$Q = [0,2]$.Define $F_1 : C \times C \to \mathbb{R}$ by $F_1(x,y) = x(y-x)$. Define $F_2 : Q \times Q \to \mathbb{R}$ by $F_2(u,v) = -2u(u-v)$. Let $f(x) = \frac{1}{8}x$, $A(x) = -x$, $B(x) = x$, $D(x) = 3x$, and for each $x \in C$, $T(s)x = e^{-2s}x$. Furthermore, we take $\alpha_n = \frac{1}{\sqrt{n}}$, $\beta_n = \frac{1}{n+1}$, $\varepsilon_n = \frac{1}{2n^2}$, $s_n = 2n$, $r_n = 1 + \frac{1}{n}$ and $a_n = \frac{2n}{3n-1}$ in algorithm (1), we obtain

$$x_{n+1} = (\frac{1}{4\sqrt{n}} + \frac{3}{n+1})x_n + ((1 - \frac{1}{2n^2})I - \frac{1}{n+1}D - \frac{1}{\sqrt{n}}B)\frac{1}{s_n}\int_0^{s_n} e^{-2s}u_n ds$$

After simplification,we have

$$x_{n+1} = (\frac{1}{4\sqrt{n}} + \frac{3}{n+1})x_n + (1 - \frac{1}{2n^2} - \frac{3}{n+1} - \frac{1}{\sqrt{n}})(\frac{1-e^{-4n}}{2n})u_n$$

Figure 3 indicates the behavior of x_n with initial point $x_1 = 1$.

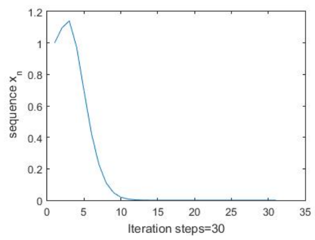

FIGURE 3. The graph of $\{x_n\}$ with initial value $x_1 = 1$.

6. Conclusions

We have proposed an iterative algorithm for finding a common solution of a system of generalized equilibrium problems, a split equilibrium and a hierarchial fixed point prob- lems over the common fixed points set of nonexpansive semigroups in Hilbert spaces.We proved that the proposed iterative has strong convergence under some mild conditions imposed on algorithm parameters.

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