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Some Monotonicity Properties on k-Gamma Function

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Abstract

The aim of this work is to obtain some monotonicity properties for the functions involving the logarithms of the k-gamma function for $k > 0$.

Keywords: k-Gamma function, Monotonicity, k-Polygamma function.

k-Gama Fonksiyonu Üzerine Bazı Monotonluk Özellikleri

Özet

Bu çalışmanın amacı, $k > 0$ olmak üzere k-gama fonksiyonunun logaritmasını içeren bazı fonksiyonların monotonluk özelliklerini elde etmektir.

Anahtar Kelimeler: k-Gama fonksiyonu, Monotonluk, k-Poligama fonksiyonu.

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1. Introduction and Preliminaries

The gamma function, which is one of the most important special functions and has many applications in many areas such as physics, engineering etc., is defined by

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$$

for positive real values of x [1]. The psi or digamma function ψ is defined by logarithm derivative of the gamma function as $\psi(x) = \frac{d}{dx}\ln\Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)}$ for $x > 0$. Its series representation is given by

$$\psi(x) = -\gamma + \sum_{n=0}^{\infty}\frac{x-1}{(n+1)(x+n)}$$

for $x > 0$ [8]. The asymptotic representations of the first and second derivative of the function are given by

$$\psi'(z) \sim \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{6z^3} - \frac{1}{30z^5} + \cdots, \quad (z \to \infty, |argz| < \pi) \tag{1}$$

and

$$\psi''(z) \sim -\frac{1}{z^2} - \frac{1}{z^3} - \frac{1}{z^4} - \frac{1}{6z^6} - \cdots, \quad (z \to \infty, |argz| < \pi) \tag{2}$$

respectively [1].

In [11], author shows that for $x \to \infty$

$$\ln\Gamma(x) = \left(x - \frac{1}{2}\right)\ln x - x + \frac{1}{2}\ln(2\pi) + O\left(\frac{1}{x}\right), \tag{3}$$

$$\psi(x) = \ln x - \frac{1}{2x} + O\left(\frac{1}{x^2}\right). \tag{4}$$

These functions are interested by many researchers. Many authors have established some monotonicity results of the gamma function and obtained related inequalities such as in [2-4,7,10] and references therein. For example, in [4], authors used the monotonicity property of the function

$$f(x) = \frac{\ln \Gamma(x+1)}{x \ln x}, \qquad x > 1$$

in order to establish the double-sided inequalities

$$x^{(1-\gamma)x-1} < \Gamma(x) < x^{x-1}, \qquad x > 1$$

where γ denotes the Euler-Mascheroni constant and in [6], they proved that the function f is concave on the interval $[1, \infty)$.

Pochhammer symbol is widely used in combinatorics. Diaz and Pariguan in [5] defined Pochhammer k-symbol and k-generalized gamma function as the following:

Definition 1.2 *Let $x \in \mathbb{C}$, $k \in \mathbb{R}$, and $n \in \mathbb{Z}^+$, the Pochhammer k-symbol is given by*

$$(x)_{n,k} = x(x+k)(x+2k)\ldots(x+(n-1)k)$$

and k-analogue of gamma function is defined by

$$\Gamma_k(x) = \lim_{n\to\infty} \frac{n!\, k^n (nk)^{\frac{x}{k}-1}}{(x)_{n,k}}$$

for $x \in \mathbb{C} \setminus k\mathbb{Z}^-$ and $k > 0$. Its integral representation is given by

$$\Gamma_k(x) = \int_0^\infty t^{x-1}\, e^{-\frac{t^k}{k}} dt$$

for $x \in \mathbb{C}$, $Re(x) > 0$.

They also proved Bohr-Moller theorem and Stirling formula for k-gamma function and obtained several results that are generalizations of the classical gamma function:

Proposition 1.3 *The k-gamma function $\Gamma_k(x)$ satisfies the following properties:*

$$\Gamma_k(x+k) = x\Gamma_k(x), \qquad (5)$$

$$\Gamma_k(k) = 1, \qquad (6)$$

$$\Gamma_k(x) \textit{ is logarithmically convex for } x \in \mathbb{R}, \tag{7}$$

$$\frac{1}{\Gamma_k(x)} = xk^{-\frac{x}{k}} e^{\frac{x}{k}\gamma} \prod_{n=1}^{\infty} \left(\left(1 + \frac{x}{nk}\right) e^{-\frac{x}{nk}} \right) \textit{ where } \gamma = \lim_{n\to\infty} \left(1 + \cdots + \frac{1}{n} - \log n\right), \tag{8}$$

$$\Gamma_k(x) = k^{\frac{x}{k}-1} \Gamma\left(\frac{x}{k}\right). \tag{9}$$

This new generalization of the classical gamma function has attracted many researchers. For example, Krasniqi in [9] used the equation (8) in order to obtain the following series representations of k-digamma function and k-polygamma function respectively by

$$\Psi_k(x) = \frac{\ln k - \gamma}{k} - \frac{1}{x} + \sum_{n=1}^{\infty} \frac{x}{nk(x+nk)} \tag{10}$$

and

$$\Psi_k^{(r)}(x) = (-1)^{r+1} r! \sum_{n=0}^{\infty} \frac{1}{(x+nk)^{r+1}} \tag{11}$$

for $r = 1,2, \ldots$ where $\psi_k(x) = \frac{d}{dx} \ln \Gamma_k(x) = \frac{\Gamma_k'(x)}{\Gamma_k(x)}$.

2. Main Results

The objective of this paper is to develop some new monotonicity results involving the logarithms of k-gamma function for some real values of x, which are generalizations of inequalities in [4].

Lemma 2.1 *The inequality*

$$\frac{2k}{u^3} > \frac{1}{u^2} - \frac{1}{(u+k)^2}$$

holds true for $k > 0$ *and* $u > 0$.

Proof. Since $u, k > 0$, we have

$$2u^2 + 4uk + 2k^2 > 2u^2 + uk.$$

Then

$$2(u+k)^2 > (2u+k)u.$$

Hence we get

$$\frac{2k}{u^3} > \frac{(2u+k)k}{u^2(u+k)^2}$$

and the result follows.

Theorem 2.2 *For $x > -k$ and $k > 0$, the function*

$$f(x) = \psi_k'(x+k) + x\psi_k''(x+k \tag{12}$$

is positive.

Proof. By taking logarithms of the equation (9), we get

$$ln\Gamma_k(x) = \left(\frac{x}{k} - 1\right) ln\, k + ln\Gamma\left(\frac{x}{k}\right) \tag{13}$$

and differentiating the equation (13) with respect to x leads us that

$$\psi_k(x) = \frac{ln\, k}{k} + \frac{1}{k}\,\psi\left(\frac{x}{k}\right), \psi_k'(x) = \frac{1}{k^2}\psi'\left(\frac{x}{k}\right) \text{ and } \psi_k''(x) = \frac{1}{k^3}\psi''\left(\frac{x}{k}\right).$$

Then from the equations (1) and (2), we have

$$\lim_{x\to\infty} f(x) = 0.$$

For positivity of the function f, we need to show that the function f is decreasing. So by using the equation (11), we obtain

$$f(x) = \sum_{n=1}^{\infty} \frac{nk - x}{(x+nk)^3}.$$

Then we get

$$f(x) - f(x+k) = \frac{k-x}{(x+k)^3} + \sum_{n=2}^{\infty} \frac{nk-x}{(x+nk)^3} - \sum_{n=1}^{\infty} \frac{nk-x-k}{(x+k+nk)^3}$$

$$= \frac{k-x}{(x+k)^3} + \sum_{n=1}^{\infty} \frac{2k}{(x+(n+1)k)^3} = -\frac{1}{(x+k)^2} + \sum_{n=1}^{\infty} \frac{2k}{(x+nk)^3}.$$

Lemma 2.1 leads us that

$$f(x) - f(x+k) = -\frac{1}{(x+k)^2} + \sum_{n=1}^{\infty} \frac{2k}{(x+nk)^3}$$

$$> -\frac{1}{(x+k)^2} + \sum_{n=1}^{\infty} \left[\frac{1}{(x+nk)^2} - \frac{1}{(x+nk+k)^2}\right] > 0$$

as desired.

Corollary 2.3 *The function*

$$g(x) = x^2\psi_k'(x+k) - x\psi_k(x+k) + \ln\Gamma_k(x+k) \qquad (14)$$

is a decreasing function on $(-k, 0)$ *and an increasing function on* $[0, \infty)$ *for* $x > -k$.

Proof. In order to obtain the result, we just need to show that the first derivative of the function g is positive on $(-k, 0)$ and negative on $(0, \infty)$ respectively.

$$g'(x) = 2x\psi_k'(x+k) + x^2\psi_k''(x+k) - \psi_k(x+k) - x\psi_k'(x+k) + \psi_k(x+k)$$

$$= x\psi_k'(x+k) + x^2\psi_k''(x+k) = xf(x)$$

where $f(x)$ is defined as in theorem 2.2. Since $f(x) > 0$ for $x > -k$ in Theorem 2.2, we obtain desired results.

Theorem 2.4

(i) *Let* $h(x) = x\psi_k(x+k) - \ln\Gamma_k(x+k)$. *Then, the function* $h(x)$ *increases for* $x \geq 0$ *and decreases for* $-k < x < 0$. *Also, we have*

$$\lim_{x\to\infty} \frac{h(x)}{x} = \frac{1}{k}.$$

(ii) *Let* $h(x) = x\psi_k(x+k) - \ln\Gamma_k(x+k)$. *Then, the function* $h(x)$ *increases for* $x \geq 0$ *and decreases for* $-k < x < 0$. *Also, we have*

$$\lim_{x\to\infty} \frac{h(x)}{x} = \frac{1}{k}.$$

(iii) *The function* $H(x) = \ln x - \frac{1}{h(x)} \ln \Gamma_k(x+k)$ *approximately increases for* $x, k > 0$ *and* $x \gtrsim \frac{7-4k}{3}$.

Proof. Differentiating the function $h(x)$ with respect to x and using the equation (11) lead us that

$$h'(x) = \psi_k(x+k) + x\psi_k'(x+k) - \psi_k(x+k)$$

$$= x\psi_k'(x+k) = x\sum_{n=1}^{\infty} \frac{1}{(x+nk)^2}.$$

Hence, we obtain monotonicity of the function h. By replacing $\frac{x}{k}$ instead of x in the equation (13), adding the term $\ln x$ in both sides of the equation and using the equations (3), (4) and (9), we get

$$\ln \Gamma_k(x+k) = -\frac{\ln k}{2} + \left(\frac{x}{k} - \frac{1}{2}\right)\ln x - \frac{x}{k} + \frac{1}{2}\ln 2\pi + O\left(\frac{1}{x}\right). \quad (15)$$

By differentiating the equation (15), we obtain

$$\psi_k(x+k) = \frac{1}{k}\ln x + \left(\frac{x}{k} - \frac{1}{2}\right)\frac{1}{x} - \frac{1}{k} + O\left(\frac{1}{x^2}\right). \quad (16)$$

Hence the limit follows from the equations (15) and (16). Now let us prove ii. By differentiating the function H and using the Theorem 2.4 (i), we get

$$H'(x) = \frac{1}{x} + \frac{h'(x)}{h^2(x)}\ln \Gamma_k(x+k) - \frac{\psi_k(x+k)}{h(x)}$$

$$= \frac{1}{x} - \frac{1}{h^2(x)}[h(x)\psi_k(x+k) - h'(x)\ln \Gamma_k(x+k)]$$

$$= \frac{1}{xh^2(x)}\big[h(x)\big(x\psi_k(x+k) - \ln \Gamma_k(x+k) - x\psi_k(x+k)\big) - xh'(x)\ln \Gamma_k(x)\big]$$

$$= \frac{\ln \Gamma_k(x+k)}{xh^2(x)}\,[xh'(x) - h(x)] = \frac{g(x)\ln \Gamma_k(x+k)}{xh^2(x)}$$

where the function $g(x)$ is defined as in Corollary 2.3.

By using the equation (9), we get

$$\ln \Gamma_k(x+k) = \frac{x}{k}\ln k \; + \; \ln \Gamma\left(\frac{x}{k}+1\right)$$

for $x > 0$ and $k > 0$. The points which make the right hand side of the above equation positive are shown in the following Figure 1:

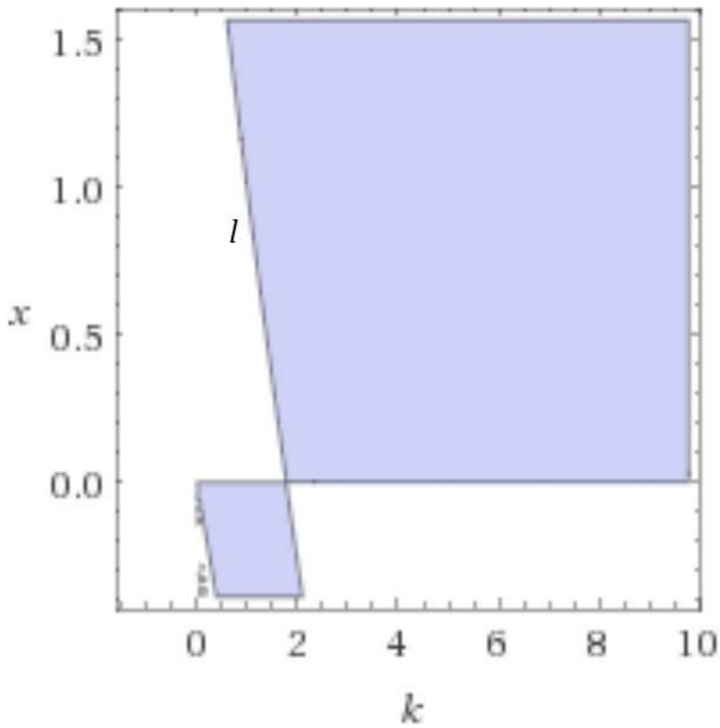

Figure 1.

$x = 0$ is a solution of the last equation for all $k > 0$ and lower line segment is $x = -k$. The tangent of the line l which passes from the points $(k, x) = (0{,}649, 1.5)$ and $(k, x) = (1.379, 0.5)$ approximately equals to $\frac{4}{3}$. So, we calculate equation of the line with the point (1,1), which is also on the line, we get $x = \frac{7\text{-}4k}{3}$. The upper blue area of Figure 1 shows that for $x > 0$ and $x > \frac{7-4k}{3}$, $\ln \Gamma_{\mathrm{k}}(x+k) > 0$ and also the lower blue area of Figure 1 shows that for $x < 0$, $-k < x$ and $x < \frac{7-4k}{3}$, $\ln \Gamma_{\mathrm{k}}(x+k) > 0$. So the proof follows.

Now we can give the following:

Corollary 2.5 *The function* $F(x) = \frac{\ln \Gamma_k(x+k)}{x \ln x}$ *is an increasing function for* $x \gtrsim \frac{7-4k}{3}$ *and* $k > 0$. *Furthermore* $\lim_{x\to\infty} F(x) = \frac{1}{k}$.

Proof. We have

$$\begin{aligned}(x \ln x)^2\, F'(x) &= x\psi_k(x+k) - \ln \Gamma_k(x+k) \ln x \; - \ln \Gamma_k(x+k) \\ &= h(x)H(x)\end{aligned}$$

where h and H are the functions in Theorem 2.4 (i) and (ii) respectively. Hence wet get the monotonicity result for $F(x)$.

By using the equation (15), we have

$$\lim_{x\to\infty} F(x) = \lim_{x\to\infty} \frac{-\frac{\ln k}{2} + \left(\frac{x}{k} - \frac{1}{2}\right) \ln x \; - \frac{x}{k} + \frac{1}{2} \ln 2\pi}{x \ln x}$$

$$= \lim_{x\to\infty} \frac{\left(\frac{x}{k} - \frac{1}{2}\right) \ln x}{x \ln x} + \lim_{x\to\infty} \frac{-\frac{\ln k}{2} - \frac{x}{k} + \frac{1}{2} \ln 2\pi}{x \ln x} = \frac{1}{k}$$

as desired.

Before we give other result we need following property.

Lemma 2.6 *The inequality*

$$\frac{2k}{u^3} < \frac{1}{2(u-k)^2} - \frac{1}{2(u+k)^2}$$

holds for $u > k$ *and* $k > 0$.

Proof. Since $k < u$, we have

$$u^4 - 2u^2k^2 + k^4 < u^4.$$

Then we can write

$$\frac{2k}{u^3} < \frac{2uk}{(u-k)^2(u+k)^2} = \frac{(u+k)^2 - (u-k)^2}{2(u-k)^2(u+k)^2} = \frac{1}{2(u-k)^2} - \frac{1}{2(u+k)^2}$$

as desired.

Theorem 2.8 *Let* $g(x) = x^2\psi_k'(x+k) + x^3\psi_k''(x+k)$ *for* $x > 0$. *Then*

$$0 < g(x) < \frac{1}{2}.$$

Proof. Since $g(x) = x^2 f(x)$, where $f(x)$ as in Theorem 2.2, the lower bound follows by Theorem 2.2. For the upper bound, let us define the function G by

$$G(x) = \frac{1}{2x^2} - f(x)$$

for $x > 0$. Since the function G tends to zero as $x \to \infty$, we need to show that $G(x) > G(x+k)$. By Lemma 2.6, we get

$$\begin{aligned} G(x) - G(x+k) &= \frac{1}{2x^2} - \frac{1}{2(x+k)^2} - [f(x) - f(x+k)] \\ &= \frac{1}{2x^2} - \frac{1}{2(x+k)^2} + \frac{1}{(x+k)^2} - \sum_{n=1}^{\infty} \frac{2k}{(x+nk)^3} \\ &> \frac{1}{2x^2} + \frac{1}{2(x+k)^2} - \sum_{n=1}^{\infty} \left[\frac{1}{2(x+nk-k)^2} - \frac{1}{2(x+nk+k)^2}\right] \\ &= \frac{1}{2x^2} + \frac{1}{2(x+k)^2} - \left[\frac{1}{2x^2} + \frac{1}{2(x+k)^2}\right] = 0 \end{aligned}$$

and the proof is completed.

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