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IMPLICATIONS OF THE HOMOTHETIC "GENERALIZED PRODUCTION FUNCTION" FOR RETURNS TO SCALE AND LONG-RUN COSTS

Dr. Alpaslan İSAGİLLER

İstanbul Üniversitesi
İktisat Fakültesi
İktisadi Gelişme ve Uluslararası İktisat
Anabilim Dalı

ABSTRACT

In this paper we analyse the implications of the homothetic "generalized production function" of Zellner and Revankar for the behavior of the returns to scale and long-run costs. The spesific implicit production function they derived yields variable returns to scale and U-shaped long-run average cost curve.

ÖZET

Bu çalışmada Zellner ve Revankar'ın homotetik "genelleştirilmiş üretim fonksiyonu"nun ölçeğe göre getiriler ve uzun dönem maliyetler hakkındaki önermelerini analiz etmekteyiz. Geliştirmiş oldukları spesifik örtük üretim fonksiyonu ölçeğe göre değişken getiriler ile U-biçimli uzun dönem ortalama maliyet eğrisi vermektedir.

1. INTRODUCTION

After the "homotheticity" concept has been first introduced by R.W. Shephard into the production theory (Shephard, 1953), the derivation of the specific forms of homothetic production functions became very popular in the late 1960's and thereafter, and several specific forms of such functions have been used increasingly in the empirical studies.

On the other hand, resurgence of interest in "increasing returns" in the 1980's in different areas of applications such as endogenous growth theory and international trade, among others, has put again homothetic functions at the center of analytical investigation, as the family of homothetic functions are the only class of transforms where returns to scale, i.e. proportional changes in output related with equiproportional changes in all production factors, can be accurately analyzed.(Chambers, 1989: 37-40).

In this paper, we will first introduce the homothetic class of "generalized production functions" (GPF's) developed by A. Zellner and N. S. Revankar and then discuss some of their implications, in particular these of the specific form of GPF they used in their empirical study of the U.S. transportation equipment industry with 1957 state data (Zellner and Revankar, 1969: 241-250). We will investigate its implications for the behavior of the returns to scale and long-run costs.

2. GENERALIZED HOMOTHETIC PRODUCTION FUNCTIONS

In their article, Zellner and Revankar essentially propose a method for deriving homothetic class of production functions, which is well-known to us today. They show that a monotone increasing transformation of any given neoclassical production function f (K, L), involving two inputs, physical capital K and labor L, homogenous of degree α_f and with a given elasticity of substitution can yield "generalized production function"

$$V = g(f) \qquad (1)$$

with the same elasticity of substitution and with the returns to scale "variable and satisfying a preassigned relationship to the output level".(V denotes real output rate) In the above equation, the transformation function g satisfies $g(0) = 0$ and $dg/df > 0$ for $0 \leq f < \infty$.

The theorem that Zellner and Revankar developed to derive the GPF's is that the solution to the following differential equation

$$dV/df = (V/f)\,[\alpha(V)/\alpha_f] \qquad (2)$$

is a homothetic generalized production function $V = g(f)$, with the preassigned returns to scale function $\alpha(V)$. From above definition, we should note that the term "generalization" refers both to what is assumed about the elasticity of substitution and the behavior of the returns to scale.

3. VARIABLE RETURNS TO SCALE AND LONG-RUN AVERAGE COST CURVE

From the standart microeconomic theory we know that constant returns to scale leads the firm's long-run average cost (LAC) curve to be a straight horizontal line. Under the assumptions of perfect competition and constant returns to scale, as total output is exactly distributed among production factors (Euler's Theorem), long-run entrepreneurial profits are zero at any factor combinations. If the economy is characterized as perfectly competitive, prices are not affected by a single firm's actions; a single firm's factor demand will not affect factor prices, as well as its product supply, the price of its product: It is a price-taker. Under these conditions, if total expenditures to production factors are increased by a certain rate, total value of output will also increase by the same rate. Therefore, under constant returns to scale assumption, it becomes impossible to determine the optimal scale of production.The optimal output is thus *indeterminate* in the neoclassical standart profit-maximization analysis.

Wicksell (1959), by leaving constant-returns-to-scale assumption and instead assuming variable returns to scale and thus U-shaped long-run average cost curve, theoretically eliminates the above inconsistency in order to validate the marginal productivity-distribution theory and find a solution to the inderterminacy of optimal scale.

Whereas the production function should possess variable returns to scale in order to be compatible with the Wicksell's approach, economists have instead assumed constant returns to scale in most of the theoretical and applied studies. In order to eliminate this internal contradiction, it has been necessary for the returns to scale to change with the output level so that the production function leads to an U-shaped long-run average cost curve. More technically, the production function coefficient which is the measure of the returns to scale is required to be functionally related to output level, indicating increasing returns at low levels of output and decreasing returns at high levels of output, as referred to by Frisch (1965: 120-121) as the *ultra-passum law* of production.

The homothetic GPF's of Zellner and Revankar can indeed provide such specifications. An earlier approach for variable returns to scale came from S. Clemhout and D. Soskice in 1968 in different studies (Clemhout, 1968: 91-104; Soskice, 1968: 446-448), but the more general treatment of the issue belongs to Zellner and Revenkar.

In all three studies, the authors have however different reasons or arguments for the derivation of the variable-returns-to-scale production functions and not the one mentioned above. Soskice for instance argues that a common procedure for estimating the elasticity of substitution will be inconsistent if the production function is constrained to constant returns to scale and shows this inconsistency with reference to a modified CES function. Zellner and Revankar, on the other hand, derives GPF's mainly to analyse returns to scale at different scales of operations. Although they don't mention the above optimal scale problem in their article, they make the point that "GPF's can show decreasing costs at low levels of output and increased costs at high levels of output". Below we show that the specific form of GPF they estimated in their applied study has the desired properties of a general variable-returns to-scale production function.

This specific form of GPF is derived by taking the original function that is subject to monotone transformation in the Cobb-Douglas form and choosing returns-to-scale function in the following form:

$$\alpha(V) = \alpha / (1 + \theta V) \tag{3}$$

The derived homothetic specific function is an implicit function in the following form:

$$V e^{\theta V} = \gamma K^{\alpha(1-\delta)} L^{\alpha\delta} \qquad \alpha > 0\,,\ 0 < \delta < 1\,,\ \gamma > 0 \tag{4}$$

By deriving the cost functions associated with this specific GPF, we show below that the long-run average cost curve will be U-shaped if $\theta > 0$ and $\alpha > 1$. First we should note that since the conditions of the implicit-function theorem are satisfied for the equation (4), an implicit function F for the real output rate V in the form V = F (K, L) is assured to exist and we can implicitly differentiate (4) to get the partial derivatives of V with respect to capital and labor.

The derivation of the long-run average cost curve is straightforward from the dual cost minimization problem:

$$\min_{K,\,L} C = rK + wL \quad \text{s.t. } V = F(K, L)$$

where w is the wage rate and r is the rate of return to physical capital.

The Lagrangian equation assaciated with the above minimization problem is:

$$\mathcal{L} = rK + wL + \lambda\,[V - F(K, L)] \tag{5}$$

where λ is the Lagrangian multiplier. The first-order (necasary) conditions for a minumum (assuming an interior solution) are:

$$\partial \mathcal{L} / \partial K = r - \partial V / \partial K = 0 \quad (6)$$

$$\partial \mathcal{L} / \partial L = w - \partial V / \partial L = 0 \quad (7)$$

$$\partial \mathcal{L} / \partial \lambda = V - F(K, L) = 0 \quad (8)$$

Note that it can be easily shown that the first-order conditions are satisfied and that the second-order (sufficient) condition holds as the bordered Hessian associated with the Lagrangian function is negative definite.

From the first-order conditions, we find out the (conditional) factor demand function for capital and labor respectively as follows:

$$K = (Ve^{\theta V})^{1/\alpha} / \{\gamma^{1/\alpha}[r\delta / w(1-\delta)]^{\delta}\} \quad (9)$$

$$L = V^{1/\alpha} e^{\theta V/\alpha} / \{\gamma^{1/\alpha}[w(1-\delta) / r\delta]^{(1-\delta)}\} \;. \quad (10)$$

Substituting the factor demand functions into the cost equation C = rK+wL, the long-run total cost (LTC) function is obtained:

$$LTC = (Ve^{\theta V})^{1/\alpha} \; [r / (1-\delta)] \; / \{\gamma^{1/\alpha}[r\delta / w(1-\delta)]^{\delta}\} \quad (11)$$

Defining $A \equiv [r / (1-\delta)] / \{\gamma^{1/\alpha}[r\delta / w(1-\delta)]^{\delta}\}$ as a constant, we can write the long-run total cost simply as:

$$LTC = AV^{1/\alpha} e^{\theta V/\alpha}. \quad (12)$$

Dividing LTC byV, we obtain long-run average cost (LAC) function:

$$LAC = AV^{(1-\alpha)/\alpha} e^{\theta V/\alpha} \quad (13)$$

Now we can analyse the relations between long-run average cost curve LAC and the returns to scale function $\alpha(V)$ for possible values of the parameters θ and α.

A) First, let us assume $\theta > 0$.

If $\theta > 0$, the returns to scale is α at $V = 0$ and its limit goes to zero as V increases infinitely, i.e. $\lim_{V\to\infty} [\alpha(V) = \alpha / (1 + \theta V)] = 0$.

i) Assume $\alpha > 1$.

Differentiating LAC with respect to V, we have the slope of the LAC curve:

$$dLAC / dV = (A/\alpha)\, V^{(1-\alpha)/\alpha} e^{\theta V/\alpha} [(1-\alpha)V^{-1} + \theta] \quad (14)$$

The sign of the slope depends on the expression inside the parenthesis.

$$dLAC/dV < 0 \Rightarrow [(1-\alpha)V^{-1} + \theta] < 0 \Rightarrow V < (\alpha-1)/\theta .$$

The slope of the LAC curve is 0 at V = (α-1) / θ. At this point LAC curve reaches its unique, single minimum. Note that the second derivative of the LAC function,

$$d^2LAC/dV^2 = (A/\alpha)V^{(1-\alpha)/\alpha} e^{\theta V/\alpha} \{(1/\alpha)[(1-\alpha)V^{-1} + \theta]^2 + (\alpha-1)V^{-2}\} \tag{15}$$

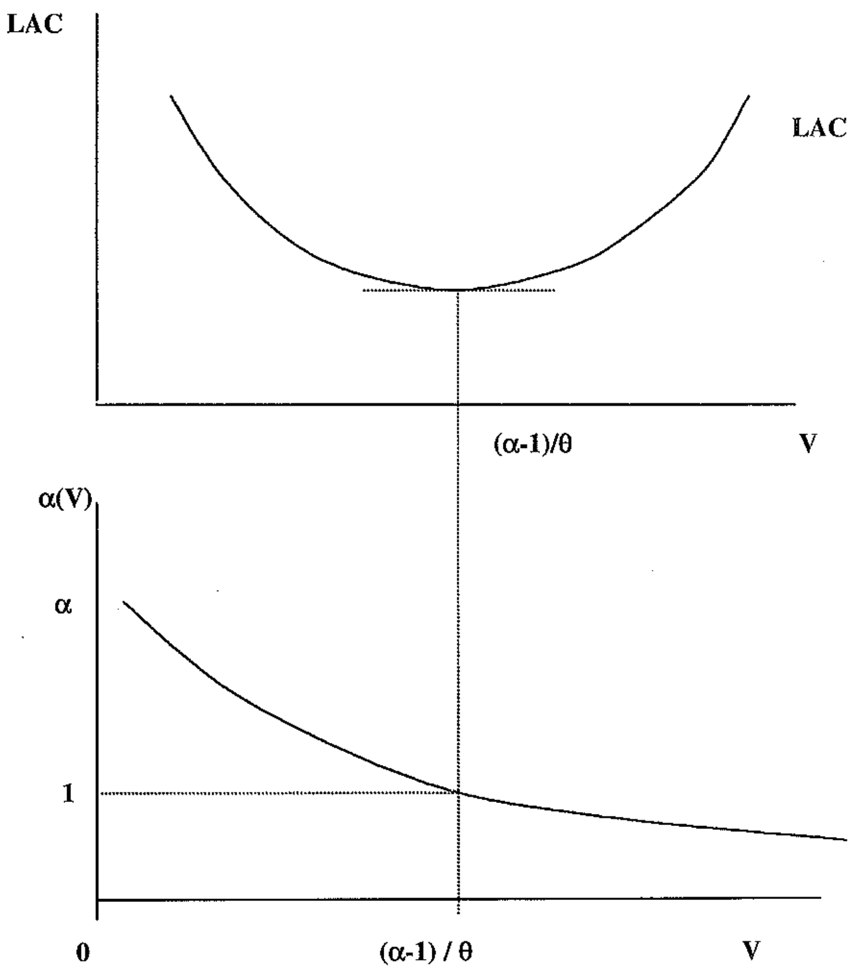

Figure (1). LAC and Returns-To-Scale Curves For θ > 0 and α > 1 is positive at all values of V, satisfying the second order (sufficient) condition for a minumum. For the values of V greater than $(\alpha\text{-}1) / \theta$, the slope is rising. Therefore as depicted in Fig.(1), the LAC function has an U-shaped curve with a unique, single minumum at $V = (\alpha\text{-}1) / \theta$.

Since $\alpha'(V) = -\alpha\theta / (1+\theta V)^2 < 0$ and $\alpha''(V) = 2\alpha\theta^2 / (1 + \theta V)^3 > 0$, the returns-to-scale function $\alpha(V)$ is convex and has negatively sloped curve in the Fig. (1). For all $V < (\alpha\text{-}1) / \theta$, $\alpha(V) > 1$ and the specific GPF exhibits increasing (at a decreasing degree) returns to scale. For all $V > (\alpha\text{-}1) / \theta$, as $\alpha(V) < 1$, the case of decreasing (at an increasing degree, as $\alpha(V)$ approaches to zero) returns to scale prevails. At the minimum of LAC curve where $V = (\alpha\text{-}1) / \theta$, $\alpha(V) = 1$ and the implicit production function has *locally* the *single* case of "constant" returns to scale. (For some other examples of production functions with local constant returns to scale, see Kohn and Vaage, 1998: 65-71)

The optimal scale for a producer is determined by $(\alpha\text{-}1) / \theta$ output level at which the LAC curve reaches its minumum and increased returns to scale are exhausted. The equilibrium of the firm will be set at the minumum of LAC curve where the average cost, the marginal cost and the price of the output are all equal.

ii) Assume $0 < \alpha \leq 1$.

For all positive values of V, $dLAC/dV > 0$ and $\alpha(V) < 1$: Thus, LAC curve is positively sloped and returns to scale are diminishing as illustrated in Fig.(2). Note that if $\alpha=1$, then the specific GPF, the returns to scale and LAC functions take the following forms respectively:

$$Ve^{\theta V} = \gamma K^{(1-\delta)} L^{\delta}, \quad \alpha(V) = 1 / (1 + \theta V), \quad LAC = A\, e^{\theta V} \tag{16}$$

The cases of $0 < \alpha < 1$ and $\alpha = 1$ are depicted separately in Fig.(2), the latter case with dashed curves.

B) Now, assume $\theta = 0$.

The implicit production function reduces to Cobb-Douglas function, homogenous of degree α :

$$V = \gamma K^{\alpha(1-\delta)} L^{\alpha\delta} \qquad \alpha > 0,\ 0 < \delta < 1 \tag{17}$$

Then, the returns to scale is now constant : $\alpha(V) = \alpha$.

If $\alpha > 1$, the returns to scale are increasing, but at an unchanging rate, and the LAC curve is concave and downward sloping. If $\alpha < 1$, the returns to scale are decreasing (at a constant rate again), and the associated LAC curve is in this case upward sloping, its concavity depending on the value of α around ½. Obviously if $\alpha = 1$, the case of constant returns to scale is obtained and LAC curve is a straight line at A.

C) Negative θ is a "less plausible case" in which α(V) increases from α at V = 0 as V increases.

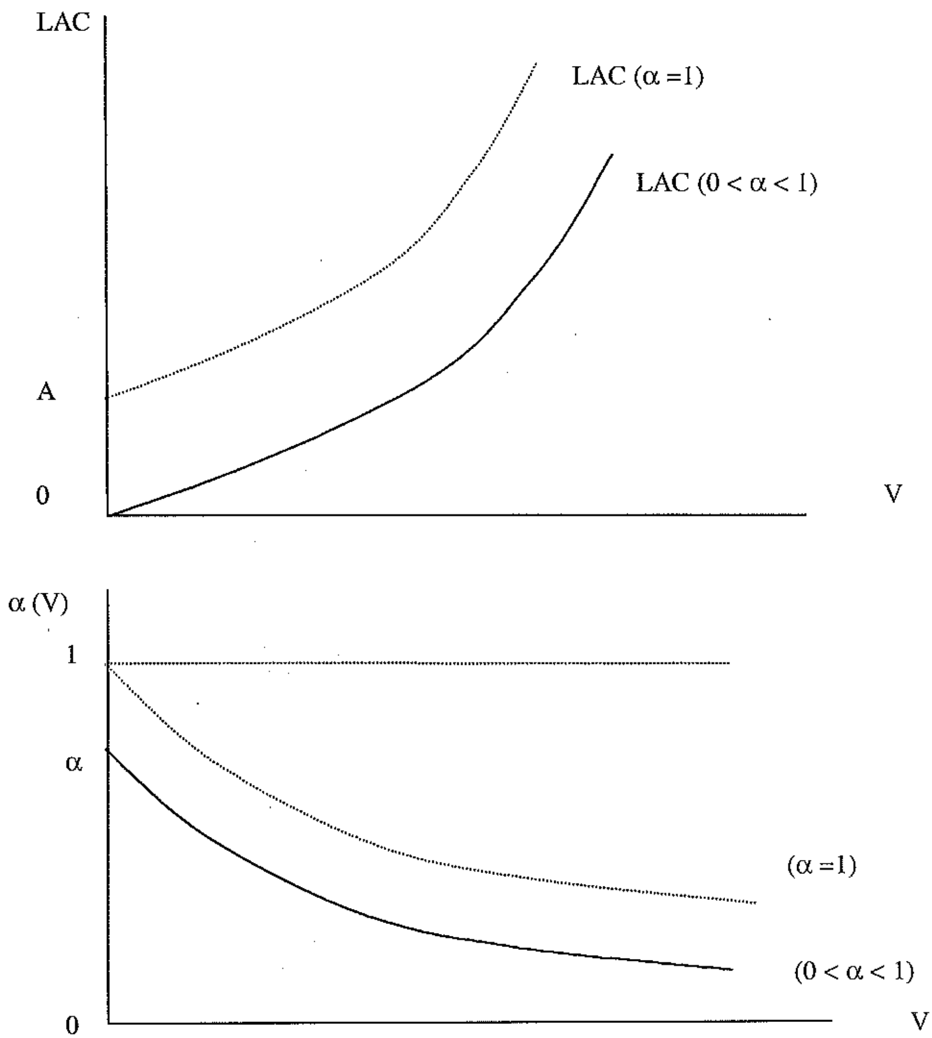

Figure (2). LAC and Returns-To-Scale Curves For $\theta > 0$ and $0 < \alpha \leq 1$

4. CONCLUDING REMARKS

In this note we considered the specific form of the homothetic class of "generalized production functions" derived by Zellner and Revankar that can yield variable returns to scale and U-shaped long-run average cost curve. Although G. S. Suzawa (1978) criticizes Zellner and Revankar for relying on "ad hoc" preassigned returns to scale functions, the GPF's allow us to study whether an economy or a particular industry is characterized by variable returns tő scale.

Zellner and Revankar derives the specific form of GPF by subjecting Cobb-Douglas production function to monotone increasing transformation. The resulting homothetic production function has the same elasticity of substitution.Thus we can get a higher generalization through transformation of a "variable-elasticity-of–substitution" (VES) production function [for example, Revankar's (1971) VES function] so that the resulting function has both the properties of variable elasticity of substitution and variable returns to scale.

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