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On The Dynamics of a Nonlinear Difference Equation

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Abstract

In this study we investigate the stability of solutions of difference equation $x_{n+1} = x_{n-3}x_{n-4} - 1$. Moreover, we study periodic and eventually periodic solutions of related difference equation.

Keywords: Difference equations, periodicity, stability, eventually periodicity

Lineer Olmayan Bir Fark Denkleminin Dinamikleri Üzerine

Özet

Bu çalışmada, fark denkleminin çözümlerinin kararlılığını araştırıldı. Ayrıca, ilgili fark denkleminin periyodik ve eninde sonunda periyodik çözümlerini de çalışıldı.

Anahtar Kelimeler: Fark denklemleri, periyodiklik, kararlılık, eninde sonunda periyodiklik

1. Introduction

Due to the fact that most mathematical models need discrete variables, difference equations have great interest in among fields of science. Because, all of difference equations

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consist of these discrete variables. In additions to this, mathematical models apply to different fields in science such as physics, economy, engineering, etc. Many authors have studied the dynamical behaviors of some difference equations, for examples:

In [16], Kent et al studied dynamics of solutions of difference equation

$$x_{n+1} = x_n x_{n-1} - 1.$$

Furthermore, in [29], Wang et al handled convergence of solutions of related difference equation. Moreover, in [19], Liu et al studied some properties of solutions of related difference equation.

In [17], Kent et al investigated long-term behaviors of solutions of difference equation

$$x_{n+1} = x_{n-1} x_{n-2} - 1.$$

In [18], Kent et al dealt with properties of solutions of difference equation

$$x_{n+1} = x_n x_{n-2} - 1.$$

In [15], Kent et al researched nature of solutions difference equation

$$x_{n+1} = x_n x_{n-3} - 1.$$

In [26], Tasdemir et al studied long-term behavior of solutions of the difference equation

$$x_{n+1} = x_{n-1} x_{n-3} - 1.$$

Moreover, in [27], the authors investigated convergence of negative equilibrium of related difference equation.

In [28], Tasdemir et al handled dynamical analysis of difference equation

$$x_{n+1} = x_{n-2} x_{n-3} - 1.$$

Furthermore, there are many books and papers related to difference equations see [1] - [29].

In this paper, we study the dynamics of solutions of the following difference equation

$$x_{n+1} = x_{n-3} x_{n-4} - 1, \quad n = 0, 1, \cdots, \tag{1}$$

with real initial conditions x_{-4}, x_{-3}, x_{-2}, x_{-1}, x_0. Moreover, we deal with the periodic and eventually periodic solutions of related difference equation.

Eq.(1) be a member of the class of equations of the form

$$x_{n+1} = x_{n-k}x_{n-l} - 1, n = 0, 1, \cdots, \tag{2}$$

with special choices of k and l, where $k, l \in \mathbb{N}_0$.

2. Periodic Solutions of Eq.(1)

In this chapter, we investigate the periodic solutions of Eq.(1).

Theorem 1 *Eq.(1) has no two periodic solutions.*

Proof. Let α, β be real numbers, $\alpha \neq \beta$ and for the sake of contradiction that there exist α and β, such that

$$\cdots, \alpha, \beta, \alpha, \beta, \alpha, \cdots$$

is a prime period two solution of Eq.(1). Thus, we have from Eq.(1) that

$$\beta = \beta \cdot \alpha - 1, \tag{3}$$

$$\alpha = \alpha \cdot \beta - 1. \tag{4}$$

From (3) and (4), we get $\alpha = \beta = \overline{x}_1$ or $\alpha = \beta = \overline{x}_2$. This contradicts $\alpha \neq \beta$ and the proof is complete.

The next theorem shows that Eq.(1) has periodic solutions with prime period three.

Theorem 2 *There are three periodic solutions of Eq.(1).*

Proof. Let α, β, γ be real numbers such that at least two are different from each other. Assume that

$$\cdots, \alpha, \beta, \gamma, \alpha, \beta, \gamma, \cdots$$

is a periodic solution of Eq.(1) with prime period three. Then we obtain from Eq.(1) that

$$\alpha = \beta \cdot \gamma - 1, \tag{5}$$

$$\beta = \gamma \cdot \alpha - 1, \tag{6}$$

$$\gamma = \alpha \cdot \beta - 1. \tag{7}$$

From (5) - (7), we have five cases that

$$\alpha = 0, \beta = -1, \gamma = -1, \tag{8}$$

$$\alpha = -1, \beta = 0, \gamma = -1, \tag{9}$$

$$\alpha = -1, \beta = -1, \gamma = 0, \tag{10}$$

$$\alpha = \beta = \gamma = \bar{x}_1, \tag{11}$$

$$\alpha = \beta = \gamma = \bar{x}_2. \tag{12}$$

So, (8), (9) and (10) are periodic solutions of Eq.(1) with prime period three. But (11) and (12) are not three periodic solutions, because these are trivial solutions of Eq.(1). The proof is complete.

Remark 3 *Eq.(1) has three periodic cycle as*

$$\cdots, -1, -1, 0, -1, -1, 0, \cdots.$$

Proof. Let the initial conditions $x_{-4} = -1$, $x_{-3} = -1$, $x_{-2} = 0$, $x_{-1} = -1$ and $x_0 = -1$ as Theorem 2. Therefore, we get from Eq.(1),

$$x_1 = x_{-3}x_{-4} - 1 = (-1)\cdot(-1) - 1 = 0,$$

$$x_2 = x_{-2}x_{-3} - 1 = 0\cdot(-1) - 1 = -1,$$

$$x_3 = x_{-1}x_{-2} - 1 = (-1)\cdot 0 - 1 = -1,$$

$$x_4 = x_0 x_{-1} - 1 = (-1)\cdot(-1) - 1 = 0.$$

The proof is complete as desired.

Theorem 4 *Eq.(1) has eventually periodic solutions with period three. They have three forms:*

Form 1: $(\cdots, x_N, x_{N+1}, x_{N+2}, x_{N+3}, x_{N+4}, -1, -1, 0, \cdots)$, where $N \geq -4$, $x_{N+1} = x_{N+4} = 0$ and $x_{N+2}x_{N+3} = 1$;

Form 2: $(\cdots, x_N, x_{N+1}, x_{N+2}, x_{N+3}, x_{N+4}, -1, 0, -1, \cdots)$, where $N \geq -4$, $x_N = x_{N+3} = 0$, $x_{N+4} = -1$ and $x_{N+1}x_{N+2} = 1$.

Proof. Form 1: Let $\{x_n\}_{n=-4}^{\infty}$ be eventually three periodic solutions of Eq.(1). Thus, for $N \geq -4$, $x_{N+5} = -1$, $x_{N+6} = -1$, $x_{N+7} = 0$ and $x_{N+8} = -1$. Therefore,

$$x_{N+5} = x_{N+1}x_N - 1 = -1,$$

$$x_{N+6} = x_{N+2}x_{N+1} - 1 = -1,$$

$$x_{N+7} = x_{N+3}x_{N+2} - 1 = 0,$$

$$x_{N+8} = x_{N+4}x_{N+3} - 1 = -1,$$

hence,

$$x_{N+1}x_N = 0, \tag{13}$$

$$x_{N+2}x_{N+1} = 0, \tag{14}$$

$$x_{N+3}x_{N+2} = 1, \tag{15}$$

$$x_{N+4}x_{N+3} = 0. \tag{16}$$

Thus we have from (13)-(16),

$$x_{N+1} = x_{N+4} = 0 \text{and} x_{N+2}x_{N+3} = 1.$$

Form 2: Let $\{x_n\}_{n=-4}^{\infty}$ be eventually three periodic solutions of Eq.(1). Hence, for $N \geq -4$, $x_{N+5} = -1$, $x_{N+6} = 0$, $x_{N+7} = -1$ and $x_{N+8} = -1$. Thus,

$$x_{N+5} = x_{N+1}x_N - 1 = -1,$$

$$x_{N+6} = x_{N+2}x_{N+1} - 1 = 0,$$

$$x_{N+7} = x_{N+3}x_{N+2} - 1 = -1,$$

$$x_{N+8} = x_{N+4}x_{N+3} - 1 = -1,$$

therefore,

$$x_{N+1}x_N = 0, \tag{17}$$

$$x_{N+2}x_{N+1} = 1, \tag{18}$$

$$x_{N+3}x_{N+2} = 0, \tag{19}$$

$$x_{N+4}x_{N+3} = 0. \tag{20}$$

Thus we have from (17)-(20),

$$x_N = x_{N+3} = 0, x_{N+4} = -1 \text{and} x_{N+1}x_{N+2} = 1.$$

So, the proof is complete as desired.

3. On The Stability of Eq.(1)

Going throughout this chapter, we study the stability analysis of Eq.(1).

Lemma 5 *Eq.(1) has two equilibriums as a golden number and its conjugate:*

$$\overline{x}_{1,2} = \frac{1 \pm \sqrt{5}}{2}. \tag{21}$$

Proof. Let $x_n = \overline{x}$ for all $n \geq -4$. Thus, we have from Eq.(1)

$$\overline{x} = \overline{x} \cdot \overline{x} - 1 \tag{22}$$

$$\overline{x}^2 - \overline{x} - 1 = 0. \tag{23}$$

Therefore, from (23),

$$\overline{x}_1 = \frac{1 + \sqrt{5}}{2},$$

$$\overline{x}_2 = \frac{1 - \sqrt{5}}{2}.$$

Consequently, we get the two equilibrium points of Eq.(1) which are positive and negative respectively.

Lemma 6 *Let $\overline{x}$ is a equilibrium point of Eq.(1). Then the linearized equation of Eq.(1) is*

$$z_{n+1} - \overline{x} \cdot z_{n-3} - \overline{x} \cdot z_{n-4} = 0. \tag{24}$$

Proof. Let I be some interval of real numbers and let

$$f : I^5 \to I$$

be a continuously differentiable function such that f is defined by

$$f\left(x_n, x_{n-1}, x_{n-2}, x_{n-3}, x_{n-4}\right) = x_{n-3}x_{n-4} - 1.$$

Therefore we get,

$$q_0 = \frac{\partial f}{\partial x_n}\left(\overline{x}, \overline{x}, \overline{x}, \overline{x}, \overline{x}\right) = [0]\left(\overline{x}, \overline{x}, \overline{x}, \overline{x}, \overline{x}\right) = 0,$$

$$q_1 = \frac{\partial f}{\partial x_{n-1}}\left(\overline{x}, \overline{x}, \overline{x}, \overline{x}, \overline{x}\right) = [0]\left(\overline{x}, \overline{x}, \overline{x}, \overline{x}, \overline{x}\right) = 0,$$

$$q_2 = \frac{\partial f}{\partial x_{n-2}}\left(\overline{x}, \overline{x}, \overline{x}, \overline{x}, \overline{x}\right) = [0]\left(\overline{x}, \overline{x}, \overline{x}, \overline{x}, \overline{x}\right) = 0,$$

$$q_3 = \frac{\partial f}{\partial x_{n-3}}\left(\bar{x}, \bar{x}, \bar{x}, \bar{x}, \bar{x}\right) = \left[x_{n-4}\right]\left(\bar{x}, \bar{x}, \bar{x}, \bar{x}, \bar{x}\right) = \bar{x},$$

$$q_4 = \frac{\partial f}{\partial x_{n-4}}\left(\bar{x}, \bar{x}, \bar{x}, \bar{x}, \bar{x}\right) = \left[x_{n-3}\right]\left(\bar{x}, \bar{x}, \bar{x}, \bar{x}, \bar{x}\right) = \bar{x}.$$

Hence, the linearized equation associated with Eq.(1) about the equilibrium point $\bar{x}$ is

$$z_{n+1} = q_0 \cdot z_n + q_1 \cdot z_{n-1} + q_2 \cdot z_{n-2} + q_3 \cdot z_{n-3} + q_4 \cdot z_{n-4},$$

then

$$z_{n+1} - \bar{x} \cdot z_{n-3} - \bar{x} \cdot z_{n-4} = 0.$$

The proof is complete as a result.

Lemma 7 *The characteristic equation of Eq.(1) about its equilibrium point* $\bar{x}$ *is*

$$\lambda^5 - \bar{x} \cdot \lambda - \bar{x} = 0. \tag{25}$$

Proof. We have from linearized equation of Eq.(1) that

$$\lambda^5 - \bar{x} \cdot \lambda - \bar{x} = 0.$$

Hereby, we endeavor the stability of equilibrium points of Eq.(1) which are positive and negative respectively.

Theorem 8 *The positive equilibrium* $\bar{x}_1 = \frac{1+\sqrt{5}}{2}$ *of Eq.(1) is unstable.*

Proof. We consider (25) for $\bar{x}_1 = \frac{1+\sqrt{5}}{2}$. Thus, we obtain five roots of (25):

$$\lambda_1 \approx 1.3007,$$

$$\lambda_{2,3} \approx 0.18947 \pm 1.2090i,$$

$$\lambda_{4,5} \approx -0.8398 \pm 0.35418i.$$

Therefore,

$$|\lambda_1| \approx 1.3007 > 1,$$

$$|\lambda_{2,3}| \approx 1.2238 > 1,$$

$$|\lambda_{4,5}| \approx 0.91143 < 1.$$

Hence, we have

$$|\lambda_1| > |\lambda_{2,3}| > 1 > |\lambda_{4,5}|.$$

Consequently, the positive equilibrium $\bar{x}_1 = \frac{1+\sqrt{5}}{2}$ of Eq.(1) is unstable.

Theorem 9 *The negative equilibrium* $\bar{x}_2 = \frac{1-\sqrt{5}}{2}$ *of Eq.(1) is unstable.*

Proof. We consider (25) for $\bar{x}_2 = \frac{1-\sqrt{5}}{2}$. Therefore, we get five roots of (25):

$$\lambda_1 \approx -0.70937,$$

$$\lambda_{2,3} \approx 0.79105 \pm 0.66559i,$$

$$\lambda_{4,5} \approx -0.43637 \pm 0.79042i.$$

Thus,

$$|\lambda_1| \approx 0.70937 < 1,$$

$$|\lambda_{2,3}| \approx 0.90287 < 1,$$

$$|\lambda_{4,5}| \approx 1.0338 > 1.$$

Hence, we have that

$$|\lambda_1| < |\lambda_{2,3}| < 1 < |\lambda_{4,5}|.$$

After all, the negative equilibrium $\bar{x}_2 = \frac{1-\sqrt{5}}{2}$ of Eq.(1) is unstable.

4. Numerical Examples

In this chapter, we present some figures for Eq.(1) so as to confirm the above theoretical results.

Example 10 *If the initial conditions* $x_{-4} = -1, x_{-3} = -1, x_{-2} = 0, x_{-1} = -1$ *and* $x_0 = -1$, *the Eq.(1) has periodic solutions with period three. Furthermore, periodic cycle of solutions of Eq.(1) is*

$$\cdots,-1,-1,0,-1,-1,0,\cdots.$$

Figure 1 verify our theoretical results (see Theorem 2).

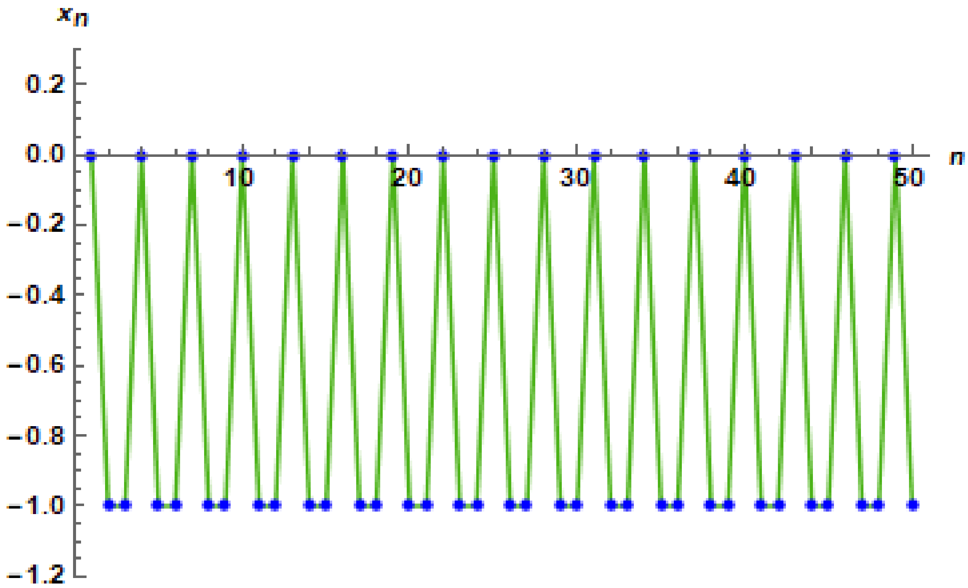

Figure 1. Eq.(1) with the given initial conditions has three periodic solutions.

Example 11 *If the initial conditions* $x_{-4}=\frac{65}{9}, x_{-3}=\frac{9}{5}, x_{-2}=\frac{20}{27}, x_{-1}=\frac{81}{10}$ *and* $x_0=\frac{2}{9}$, *then Eq.(1) has eventually periodic solutions with period three. Figure 2 verify our theoretical results (see Theorem 4).*

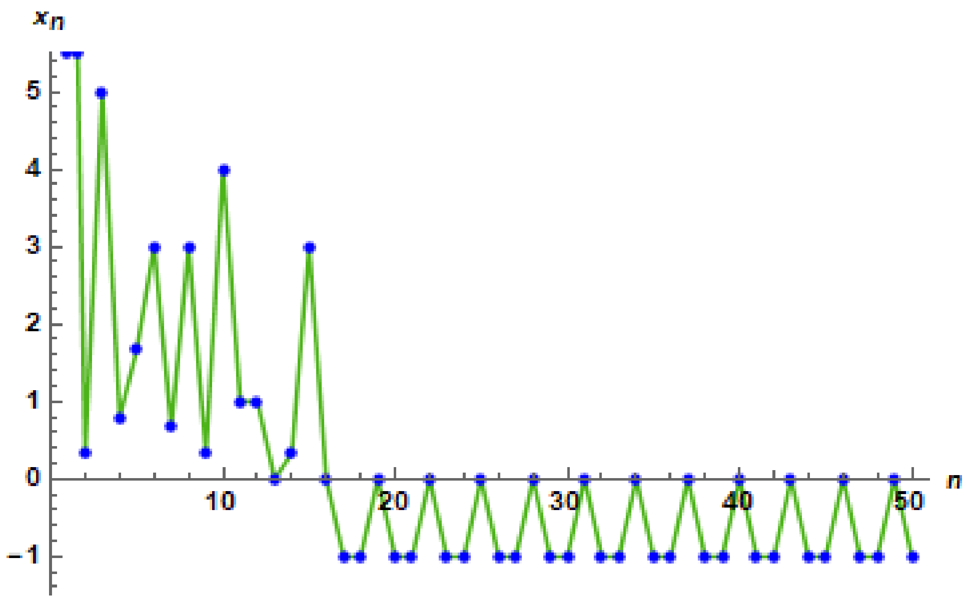

Figure 2. *Eq.(1) with the given initial conditions has eventually three periodic solutions*.

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