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Research Article

On some structures of binary soft topological spaces

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Abstract

We initiate binary soft topological spaces which are, in fact, generalization of soft topological spaces in broader sense. The basic structures and concepts of binary soft topological spaces using binary soft points are studied and their fundamental properties are explored. Examples and counter examples are also provided to efficiently describe the presented notions and results. The results, discussed in this paper, are basic and provide an introductory platform but potentially useful research in theoretical as well as applicable directions.

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1. Introduction

Zadeh [24] proposed completely new, elegant approach to vagueness called fuzzy set theory. During the study, it is observed that the fuzzy set operations based on the arithmetic operations with membership functions do not look natural. The reason for these difficulties is, possibly, the inadequacy of the parametrization tool of the theory. While probability theory, theory of fuzzy sets [24], theory of intuitionists fuzzy sets [4], theory of vague sets, theory of interval mathematics [4, 5], theory of rough sets [20], and other mathematical tools are well-known and often useful approaches to describing uncertainty and imperfect knowledge. Each of these theories has its inherent difficulties as pointed out by Molodtsov [17]. All these tools require the pre specification of some parameter to start with. Consequently, Molodtsov [17] initiated a new idea of soft set theory for modelling vagueness and uncertainty, which is free from the difficulties affecting existing methods while modelling the problems with incomplete information. A soft set is a parameterized family of subsets of the universal set. We can say that soft sets are neighborhood systems, and that they are a special case of context-dependent fuzzy sets. In soft set theory, the problem of setting the membership function, among other related problems, simply does not arise. This makes the theory very convenient and easy to apply in practice. Soft set theory has potential applications in many different fields, including the smoothness of functions, game theory, operations research, Riemann integration, Perron integration, probability theory, and measurement theory. Most of these applications have already been

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demonstrated by Molodtsov [18]. At present, work on the soft set theory is progressing rapidly. Maji et al. [15] described the application of soft set theory to a decision-making problem using rough sets approach. In 2005, Pei and Miao [21] and Chen [7] improved the work of Maji et. al [15,16]. Kharal and Ahmad [12] defined and discussed the several properties of soft images and soft inverse images of soft sets. They also applied these notions to the problem of medical diagnosis in medical systems. The algebraic structure of soft set theory and soft topological spaces dealing with uncertainties has also been studied by many researchers [2,3,6–10,12–14,19,21–23].
Recently, Acikgoz and Tas [1] introduced a binary soft set over two initial universal sets and a parameter set and explored its basic structures. In this paper, we initiate binary soft topological spaces which are, in fact, a generalization of soft topological spaces in broader sense and are defined over two initial universes U_1 and U_2 with fixed set of parameters. The concepts of binary soft open set, binary soft closed set, binary soft neighborhood, binary soft closure and binary soft interior are studied and their basic properties are explored. The results, discussed in this paper, are basic and provide an introductory platform but potentially useful research in theoretical as well as applicable directions.

2. Preliminaries

First we recall some definitions and results defined and discussed in [1,2,10,11,15–18,22], which will use in the sequel.

Definition 2.1. Let X be an initial universe and E be a set of parameters. Also let $P(X)$ denotes the power set of X and A be a non-empty subset of E. A pair (F, A) is called a soft set over X, where F is a mapping given by $F : A \to P(X)$. In other words, a soft set over X is a parameterized family of subsets of the universe X . For $e \in A$, $F(e)$ may be considered as the set of e-approximate elements of the soft set (F, A).

Definition 2.2. Let τ be the collection of soft sets over X, then τ is said to be a soft topology on X, if

(1) Φ, $\tilde{X}$ belong to τ.
(2) the union of any number of soft sets in τ belongs to τ.
(3) the intersection of any two soft sets in τ belongs to τ.

The triplet (X, τ, E) is called a soft topological space over X.

Consider U_1 and U_2 are two initial universal sets, E be a set of parameters and $P(U_1)$ and $P(U_2)$ denote the power sets of U_1 and U_2 respectively. Also, let $A, B, C \subseteq E$.

Definition 2.3. A pair (F, A) is said to be a binary soft set over U_1, U_2, where $F : A \to P(U_1) \times P(U_2)$ is defined by, $F(e) = (X, Y)$, for each $e \in A$ such that $X \subseteq U_1, Y \subseteq U_2$.

Example 2.4. Consider the following sets:

$U_1 = \{t_1, t_2, t_3, t_4, t_5\}$ is the set of trousers.
$U_2 = \{b_1, b_2, b_3, b_4, b_5\}$ is the set of blouses.
$E = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8\}$.

E is the set of parameters, where e_1 : expensive, e_2 : cheap, e_3 : sport, e_4 : classic, e_5 : colorful, e_6 : plain, e_7 : small, e_8 : large.
The binary soft set (F, A) describes "the special features of both the trousers and the blouses" which Mrs. X is going to buy. Consider $A = \{e_1, e_2, e_3, e_4\} \subseteq E$. Then (F, A) is a binary soft set over U_1, U_2 defined as follows:

$F(e_1) = (\{t_1, t_2\}, \{b_1, b_3\})$,
$F(e_2) = (\{t_3, t_4\}, \{b_2, b_4, b_5\})$,
$F(e_3) = (\{t_2, t_3, t_5\}, \{b_1, b_5\})$,
$F(e_4) = (\{t_1, t_4\}, \{b_2, b_3\})$.

So, we represent the binary soft set (F, A) as:
(F, A) ={expensive trousers and blouses : resp. $\{t_1, t_2\}, \{b_1, b_3\}$; cheap trousers and blouses: resp. $\{t_3, t_4\}, \{b_2, b_4, b_5\}$; sport trousers and blouses: resp.$\{t_2, t_3, t_5\}, \{b_1, b_5\}$; classic trousers and blouses: resp. $\{t_1, t_4\}, \{b_2, b_3\}$}.
We also denote the binary soft set (F, A) as below:
$(F, A) = \{(e_1, (\{t_1, t_2\}, \{b_1, b_3\})), (e_2, (\{t_3, t_4\}, \{b_2, b_4, b_5\})), (e_3, (\{t_2, t_3, t_5\}, \{b_1, b_5\})), (e_4, (\{t_1, t_4\}, \{b_2, b_3\}))\}$.
In this example, we can see the views of Mrs. X, who wants to buy both trousers and blouses under the same parameters.

Definition 2.5. Let (F, A) and (G, B) are two binary soft sets over the universes U_1, U_2. (F, A) is called a binary soft subset of (G, B), if

(1) $A \subseteq B$,

(2) $X_1 \subseteq X_2$ and $Y_1 \subseteq Y_2$ such that $F(e) = (X_1, Y_1)$, $G(e) = (X_2, Y_2)$, for each $e \in A$ such that $X_1, X_2 \subseteq U_1, Y_1, Y_2 \subseteq U_2$.

We denote it by $(F, A)\tilde{\tilde{\subseteq}}(G, B)$. (F, A) is called a binary soft super set of (G, B), if (G, B) is a binary soft subset of (F, A). We write $(F, A)\tilde{\tilde{\supseteq}}(G, B)$.

Definition 2.6. Let (F, A) and (G, B) are two binary soft sets over U_1, U_2. (F, A) is called binary soft equal to (G, B), if (F, A) is binary soft subset of (G, B) and (G, B) is binary soft subset of (F, A). We denote it by $(F, A)\tilde{\tilde{=}}(G, B)$.

Definition 2.7. The complement of binary soft set (F, A) is denoted by $(F, A)^c$ and is defined $(F, A)^c = (F^c, \rceil A)$, where $F^c : \rceil A \to P(U_1) \times P(U_2)$ is a mapping given by $F^c(e) = (U_1 - X, U_2 - Y)$ such that $F(e) = (X, Y)$, for each $e \in A$ such that $X \subseteq U_1, Y \subseteq U_2$. Clearly, $((F, A)^c)^c = (F, A)$.

Definition 2.8. A binary soft set (F, A) over U_1, U_2 is called binary null soft set denoted by $\tilde{\tilde{\Phi}}$, if $F(e) = (\Phi, \Phi)$, for each $e \in A$.

Definition 2.9. A binary soft set (F, A) over U_1, U_2 is called binary absolute soft set denoted by $\tilde{\tilde{A}}$, if $F(e) = (U_1, U_2)$, for each $e \in A$.

Example 2.10. Let U_1, U_2 and A are sets as in Example 2.4. Let (F, A) be a binary soft set defined as follows:

$(F, A) = \{(e_1, (U_1, U_2)), (e_2, (U_1, U_2)), (e_3, (U_1, U_2))\}$.

Therefore,(F, A) is a binary absolute soft set. Clearly, $(\tilde{\tilde{A}})^c \tilde{\tilde{=}} \tilde{\tilde{\Phi}}$ and $(\tilde{\tilde{\Phi}})^c \tilde{\tilde{=}} \tilde{\tilde{A}}$.

Definition 2.11. Union of two binary soft sets (F, A) and (G, B) over the universes U_1, U_2 is the binary soft set (H, C), where $C = A \cup B$, and for each $e \in C$ such that $X_1, X_2 \subseteq U_1, Y_1, Y_2 \subseteq U_2$,

$$H(e) = \begin{cases} (X_1, Y_1), & \text{if } e \in A - B \\ (X_2, Y_2), & \text{if } e \in B - A \\ (X_1 \cup X_2, Y_1 \cup Y_2), & \text{if } e \in A \cap B \end{cases},$$

such that $F(e) = (X_1, Y_1)$, for each $e \in A$ and $G(e) = (X_2, Y_2)$, for each $e \in B$. We denote it by $(F, A)\tilde{\tilde{\cup}}(G, B)\tilde{\tilde{=}}(H, C)$.

Definition 2.12. Intersection of two binary soft sets (F, A) and (G, B) over the universes U_1, U_2 is the binary soft set (H, C), where $C = A \cap B$, and $H(e) = (X_1 \cap X_2, Y_1 \cap Y_2)$, for each $e \in C$ such that $F(e) = (X_1, Y_1)$, for each $e \in A$ and $G(e) = (X_2, Y_2)$, for each $e \in B$ such that $X_1, X_2 \subseteq U_1, Y_1, Y_2 \subseteq U_2$. We denote it $(F, A)\tilde{\tilde{\cap}}(G, B)\tilde{\tilde{=}}(H, C)$.

Proposition 2.13. *Let $(F, A), (G, B)$ and (H, C) be three binary soft sets. Then we have:*

(i) $(F, A)\tilde{\tilde{\cup}}(F, A)\tilde{\tilde{=}}(F, A)$.

(ii) $(F, A)\tilde{\tilde{\cup}}(G, B)\tilde{\tilde{=}}(G, B)\tilde{\tilde{\cup}}(F, A)$.

(iii) $(F, A)\tilde{\tilde{\cup}}((G, B)\tilde{\tilde{\cup}}(H, C))\tilde{\tilde{=}}((F, A)\tilde{\tilde{\cup}}(G, B))\tilde{\tilde{\cup}}(H, C)$.

(iv) $(F, A)\tilde{\tilde{\cup}}\ \tilde{\tilde{\Phi}}\tilde{\tilde{=}}(F, A)$.

(v) $(F, A)\tilde{\tilde{\cup}}\tilde{\tilde{A}}\tilde{\tilde{=}}\tilde{\tilde{A}}$.

(vi) $(F, A)\tilde{\tilde{\subseteq}}(F, A)\tilde{\tilde{\cup}}(G, B)$ *and* $(G, B)\tilde{\tilde{\subseteq}}(F, A)\tilde{\tilde{\cup}}(G, B)$.

(vii) $(F, A)\tilde{\tilde{\cup}}(G, B)\tilde{\tilde{=}}\tilde{\tilde{\Phi}}$ *if and only if* $(F, A)\tilde{\tilde{=}}\tilde{\tilde{\Phi}}$ *and* $(G, B)\tilde{\tilde{=}}\tilde{\tilde{\Phi}}$.

(viii) $(F, A)\tilde{\tilde{\subseteq}}(G, B)$ *if and only if* $(F, A)\tilde{\tilde{\cup}}(G, B)\tilde{\tilde{=}}(G, B)$.

Proposition 2.14. *Let $(F, A), (G, B)$ and (H, C) be three binary soft sets. Then we have:*

(i) $(F, A)\ \tilde{\tilde{\cap}}\ (F, A)\tilde{\tilde{=}}(F, A)$.

(ii) $(F, A)\ \tilde{\tilde{\cap}}\ (G, B)\tilde{\tilde{=}}(G, B)\ \tilde{\tilde{\cap}}\ (F, A)$.

(iii) $(F, A)\ \tilde{\tilde{\cap}}\ ((G, B)\ \tilde{\tilde{\cap}}\ (H, C))\tilde{\tilde{=}}((F, A)\ \tilde{\tilde{\cap}}\ (G, B))\ \tilde{\tilde{\cap}}\ (H, C)$.

(iv) $(F, A)\ \tilde{\tilde{\cap}}\ \tilde{\tilde{\Phi}}\tilde{\tilde{=}}\tilde{\tilde{\Phi}}$.

(v) $(F, A)\ \tilde{\tilde{\cap}}\ \tilde{\tilde{A}}\tilde{\tilde{=}}(F, A)$.

(vi) $(F, A)\ \tilde{\tilde{\cap}}\ (G, B)\ \tilde{\tilde{\subseteq}}\ (F, A)$ *and* $(F, A)\ \tilde{\tilde{\cap}}\ (G, B)\ \tilde{\tilde{\subseteq}}\ (G, B)$.

(vii) $(F, A)\ \tilde{\tilde{\cap}}\ (G, B)\tilde{\tilde{=}}\tilde{\tilde{\Phi}}$ *if and only if* $(F, A)\tilde{\tilde{=}}\tilde{\tilde{\Phi}}$ *or* $(G, B)\tilde{\tilde{=}}\tilde{\tilde{\Phi}}$.

(viii) $(F, A)\ \tilde{\tilde{\subseteq}}\ (G, B)$ *if and only if* $(F, A)\ \tilde{\tilde{\cap}}\ (G, B)\tilde{\tilde{=}}(F, A)$.

Proposition 2.15. *Let (F, A) and (G, B) be two binary soft sets. Then we have:*

(i) $(F, A)\tilde{\tilde{\cup}}(F, A)^c\tilde{\tilde{=}}\tilde{\tilde{A}}$.

(ii) $(F, A)\tilde{\tilde{\cap}}(F, A)^c\tilde{\tilde{=}}\tilde{\tilde{\Phi}}$.

(iii) $(F, A)\ \tilde{\tilde{\subseteq}}(G, B)$ *if and only if* $(G, B)^c\tilde{\tilde{\subseteq}}(F, A)^c$.

(iv) $((F, A)\tilde{\tilde{\cup}}(G, B))^c\tilde{\tilde{=}}(F, A)^c\tilde{\tilde{\cup}}(G, B)^c$.

(v) $((F, A)\tilde{\tilde{\cap}}(G, B))^c\tilde{\tilde{=}}(F, A)^c\tilde{\tilde{\cap}}(G, B)^c$.

Proposition 2.16. *Let $(F, A), (G, B)$ and (H, C) be three binary soft sets. Then we have:*

(i) $(F, A)\tilde{\tilde{\cup}}((G, B)\tilde{\tilde{\cap}}(H, C))\tilde{\tilde{=}}((F, A)\tilde{\tilde{\cup}}(G, B))\tilde{\tilde{\cap}}((F, A)\ \tilde{\tilde{\cup}}(H, C))$.

(ii) $(F, A)\tilde{\tilde{\cap}}((G, B)\tilde{\tilde{\cup}}(H, C))\tilde{\tilde{=}}((F, A)\tilde{\tilde{\cap}}(G, B))\tilde{\tilde{\cup}}((F, A)\tilde{\tilde{\cap}}(H, C))$.

3. Binary soft topology

Consider U_1 and U_2 are two initial universes with a fixed set of parameters E, and we define:

Definition 3.1. The binary soft set (F, E) is called a binary soft point over U_1, U_2, denoted by e_F, if for the element $e \in A$, $F(e) \neq (\Phi, \Phi)$ and $F(e') = (\Phi, \Phi)$, for all $e' \in E - \{e\}$.

Definition 3.2. The binary soft point e_F is said to be in the binary soft set (G, E), denoted by $e_F\ \tilde{\tilde{\in}}\ (G, E)$, if for the element $e \in A$ and $F(e) \subseteq G(e)$.

Proposition 3.3. *Let e_F be a binary soft point over U_1, U_2 and (G, E) be a binary soft set over U_1, U_2. If $e_F\ \tilde{\tilde{\in}}(G, E)$, then $e_F\ \tilde{\tilde{\notin}}(G, E)^c$.*

Proof. If $e_F\ \tilde{\tilde{\in}}(G, E)$, then for $e \in A$ and $F(e) \subseteq G(e)$. This implies $F(e) \nsubseteq G^c(e) = (U_1 - X,\ U_2 - Y)$ such that $G(e) = (X,\ Y)$, where $X \subseteq U_1,\ Y \subseteq U_2$. Therefore, we have $e_F\ \tilde{\tilde{\notin}}(G^c, E)\tilde{\tilde{=}}(G, E)^c$. This completes the proof. $\square$

Remark 3.4. The converse of the above proposition is not true in general and the following example justify our claim.

Example 3.5. Consider the following sets:
$U_1 = \{b_1, b_2, b_3, b_4\}$ is the set of paints.
$U_2 = \{t_1, t_2, t_3, t_4\}$ is the set of t-shirts.
$E = \{e_1, e_2, e_3, e_4, e_5\}$.
Let $e_{2_F} = (e_2, (\{b_1, b_2, b_3\}, \{t_1, t_2, t_3\}))$ and
$(G, A) = \{(e_1, (\{b_1, b_4\}, \{t_1, t_4\})), (e_2, (\{b_1, b_3\}, \{t_1, t_3\}))\}$. Then $e_{2_F} \tilde{\tilde{\notin}} (G, A)$ and also $e_{2_F} \tilde{\tilde{\notin}} (G, A)^c = \{(e_1, (\{b_2, b_3\}, \{t_2, t_3\})), (e_2, (\{b_2, b_4\}, \{t_2, t_4\})), (e_3, (U_1, U_2))\}$.

Definition 3.6. The difference (H, E) of two binary soft sets (F, E) and (G, E) over U_1, U_2, denoted by $(F, E) \tilde{\tilde{\setminus}} (G, E)$ and is defined as $H(e) = F(e) \setminus G(e)$, for all $e \in E$.

Example 3.7. Consider the following sets:
$U_1 = \{b_1, b_2, b_3, b_4, b_5\}$ is the set of paints.
$U_2 = \{t_1, t_2, t_3, t_4, t_5\}$ is the set of t-shirts.
$E = \{e_1, e_2\}$.
Let us take the following two binary soft sets (F, E) and (G, E) as:
$(F, E) = \{(e_1, (\{b_1, b_4\}, \{t_1, t_4\})), (e_2, (\{b_1, b_3\}, \{t_1, t_3\}))\}$.
$(G, E) = \{(e_1, (\{b_1, b_2, b_5\}, \{t_1, t_3, t_5\})), (e_2, (\{b_1, b_2, b_4\}, \{t_1, t_2, t_5\}))\}$
Then $H(e) = F(e) \setminus G(e)$, for all $e \in E$. Thus
$H(e_1) = F(e_1) \setminus G(e_1) = (\{b_4\}, \{t_4\})$.
$H(e_2) = F(e_2) \setminus G(e_2) = (\{b_3\}, \{t_3\})$.
So, $(H, E) = \{(e_1, (\{b_4\}, \{t_4\})), (e_2, (\{b_3\}, \{t_3\}))\}$.

Definition 3.8. The binary soft relative complement of a binary soft set (F, E) is denoted by $(F, E)'$ and is defined by $(F, E)' = (F', E)$ where $F' : E \to P(U_1) \times P(U_2)$ is a mapping given by $F'(e) = (U_1 - X, U_2 - Y)$ where $F(e) = (X, Y)$, for all $e \in E$ such that $X \subseteq U_1, Y \subseteq U_2$.

Proposition 3.9. *Let (F, E) and (G, E) be two binary soft sets over U_1, U_2. Then*
(1)$((F, E) \tilde{\tilde{\cup}} (G, E))' \tilde{\tilde{=}} (F, E)' \tilde{\tilde{\cap}} (G, E)'$.
(2)$((F, E) \tilde{\tilde{\cap}} (G, E))' \tilde{\tilde{=}} (F, E)' \tilde{\tilde{\cup}} (G, E)'$.

Proof. (1) Let $(F, E) \tilde{\tilde{\cup}} (G, E) \tilde{\tilde{=}} (H, E)$, where $H(e) = F(e) \cup G(e)$, for all $e \in E$. Then $H'(e) = (F(e) \cup G(e))^c = (F(e))^c \cap (G(e))^c = (F^c(e)) \cap (G^c(e))$, for all $e \in E$. Thus $(H, E)' \tilde{\tilde{=}} (F, E)' \tilde{\tilde{\cap}} (G, E)'$.
(2) Let $(F, E) \tilde{\tilde{\cap}} (G, E) \tilde{\tilde{=}} (H, E)$ where $H(e) = F(e) \cap G(e)$ for all $e \in E$. Then $H'(e) = (F(e) \cap G(e))^c = (F(e))^c \cup (G(e))^c = (F^c(e)) \cup (G^c(e))$ for all $e \in E$. Thus $(H, E)' \tilde{\tilde{=}} (F, E)' \tilde{\tilde{\cup}} (G, E)'$. Hence the proof is completed. □

Now we define binary soft topology as:

Definition 3.10. Let τ be the collection of binary soft sets over U_1, U_2 and E denotes the set of parameters. Then τ is said to be binary soft topology over U_1, U_2, if
(1) $\tilde{\tilde{\Phi}}, \tilde{\tilde{E}}$ belong to τ.
(2) the union of any number of binary soft sets in τ belongs to τ.
(3) the intersection of any two binary soft sets in τ belongs to τ.
The (U_1, U_2, τ, E) is called a binary soft topological space over U_1, U_2.

Definition 3.11. Let (U_1, U_2, τ, E) be a binary soft topological space over U_1, U_2, then the members of τ are said to be binary soft open sets over U_1, U_2. A binary soft set (F, E) over U_1, U_2 is said to be a binary soft closed set over U_1, U_2, if its binary soft relative complement $(F, E)'$ belongs to τ.

Remark 3.12. In binary soft topology, if we take $U_2 = \Phi$, then binary soft topology coincides with soft topology. Thus binary soft topology is indeed the generalization of soft topology.

The proof of the following proposition follows directly from the definition of binary soft topological spaces and De-Morgan's laws for binary soft sets which are given in Proposition 3.9.

Proposition 3.13. *Let (U_1, U_2, τ, E) be a binary soft topological space over U_1, U_2. Then*
(1) $\tilde{\tilde{\Phi}}, \tilde{\tilde{E}}$ *are binary soft closed sets over U_1, U_2.*
(2) *the intersection of any number of binary soft closed sets is a binary soft closed set over U_1, U_2.*
(3) *the union of any two binary soft closed sets is a binary soft closed set over U_1, U_2.*

Definition 3.14. Let U_1 and U_2 are initial universal sets, E be the set of parameters and $\tau = \left\{\tilde{\tilde{\Phi}}, \tilde{\tilde{E}}\right\}$. Then τ is called the binary soft indiscrete topology over U_1, U_2 and (U_1, U_2, τ, E) is said to be a binary soft indiscrete space over U_1, U_2.

Definition 3.15. Let U_1 and U_2 are initial universal sets, E be the set of parameters and let τ be the collection of all binary soft sets which can be defined over U_1, U_2. Then τ is called the binary soft discrete topology over U_1, U_2 and (U_1, U_2, τ, E) is said to be a binary soft discrete space over U_1, U_2.

Proposition 3.16. *Let (U_1, U_2, τ_1, E) and (U_1, U_2, τ_2, E) be two binary soft topological spaces over the same universes U_1, U_2, then $(U_1, U_2, \tau_1 \cap \tau_2, E)$ is a binary soft topological space over U_1, U_2.*

Proof. (1) $\tilde{\tilde{\Phi}}, \tilde{\tilde{E}}$ belong to $\tau_1 \cap \tau_2$.
(2) Let $\{(F_i, E) : i \in I\}$ be a family of binary soft sets in $\tau_1 \cap \tau_2$. Then $(F_i, E) \in \tau_1$ and $(F_i, E) \in \tau_2$, for all $i \in I$. So $\tilde{\tilde{\cup}}_{i\in I}(F_i, E) \in \tau_1$ and $\tilde{\tilde{\cup}}_{i\in I}(F_i, E) \in \tau_2$. Thus $\tilde{\tilde{\cup}}_{i\in I}(F_i, E) \in \tau_1 \cap \tau_2$.
(3) Let $(F, E), (G, E) \in \tau_1 \cap \tau_2$. Then $(F, E), (G, E) \in \tau_1$ and $(F, E), (G, E) \in \tau_2$. Since $(F, E)\tilde{\tilde{\cap}}(G, E) \in \tau_1$ and $(F, E)\tilde{\tilde{\cap}}(G, E) \in \tau_2$, so $(F, E)\tilde{\tilde{\cap}}(G, E) \in \tau_1 \cap \tau_2$.
Thus $\tau_1 \cap \tau_2$ defines a binary soft topology over U_1, U_2 and $(U_1, U_2, \tau_1 \cap \tau_2, E)$ is a binary soft topological space over U_1, U_2. This completes the proof. ☐

Remark 3.17. The union of two binary soft topologies over U_1, U_2 may not be a binary soft topology over U_1, U_2.

Example 3.18. Let $U_1 = \{b_1, b_2, b_3\}, U_2 = \{t_1, t_2, t_3\}$, $E = \{e_1, e_2\}$ and
$\tau_1 = \{\tilde{\tilde{\Phi}}, \tilde{\tilde{E}}, (F_1, E), (F_2, E), (F_3, E), (F_4, E), (F_5, E)\}$,
$\tau_2 = \{\tilde{\tilde{\Phi}}, \tilde{\tilde{E}}, (G_1, E), (G_2, E), (G_3, E), (G_4, E), (G_5, E)\}$
be two binary soft topologies defined over U_1, U_2, where $(F_1, E), (F_2, E), (F_3, E), (F_4, E), (F_5, E), (G_1, E), (G_2, E), (G_3, E), (G_4, E)$ and (G_5, E) are binary soft sets over U_1, U_2, defined as follows:

$$\begin{array}{ll}
F_1(e_1) = (\{b_2\}, \{t_2\}), & F_1(e_2) = (\{b_1\}, \{t_1\}), \\
F_2(e_1) = (\{b_2, b_3\}, \{t_2, t_3\}), & F_2(e_2) = (\{b_1, b_2\}, \{t_1, t_2\}), \\
F_3(e_1) = (\{b_1, b_2\}, \{t_1, t_2\}), & F_3(e_2) = (U_1, U_2), \\
F_4(e_1) = (\{b_1, b_2\}, \{t_1, t_2\}), & F_4(e_2) = (\{b_1, b_3\}, \{t_1, t_3\}), \\
F_5(e_1) = (\{b_2\}, \{t_2\}), & F_5(e_2) = (\{b_1, b_2\}, \{t_1, t_2\}).
\end{array}$$

And

$$\begin{array}{ll} G_1(e_1) = (\{b_2\}, \{t_2\}), & G_1(e_2) = (\{b_1\}, \{t_1\}), \\ G_2(e_1) = (\{b_2, b_3\}, \{t_2, t_3\}), & G_2(e_2) = (\{b_1, b_2\}, \{t_1, t_2\}), \\ G_3(e_1) = (\{b_1, b_2\}, \{t_1, t_2\}), & G_3(e_2) = (\{b_1, b_2\}, \{t_1, t_2\}), \\ G_4(e_1) = (\{b_2\}, \{t_2\}), & G_4(e_2) = (\{b_1, b_3\}, \{t_1, t_3\}), \\ G_5(e_1) = (\{b_1, b_2\}, \{t_1, t_2\}), & G_5(e_2) = (U_1, U_2). \end{array}$$

Consider $\tau = \tau_1 \cup \tau_2 =$

$\{\tilde{\tilde{\Phi}}, \tilde{\tilde{E}}, (F_1, E), (F_2, E), (F_3, E), (F_4, E), (F_5, E), (G_1, E), (G_2, E), (G_3, E), (G_4, E), (G_5, E)\}$.

If we take $(F_2, E) \mathbin{\tilde{\tilde{\cap}}} (G_3, E) = (H, E)$ then
$H(e_1) = F_2(e_1) \cap G_3(e_1) = (\{b_2, b_3\}, \{t_2, t_3\}) \cup (\{b_1, b_2\}, \{t_1, t_2\}) = (U_1, U_2)$, and
$H(e_2) = F_2(e_2) \cap G_3(e_2) = (\{b_1, b_2\}, \{t_1, t_2\}) \cup (\{b_1, b_2\}, \{t_1, t_2\}) = (\{b_1, b_2\}, \{t_1, t_2\})$.
This follows $(H, E) \notin \tau$. Thus τ is not a binary soft topology over U_1, U_2.

4. Binary soft neighborhood, binary soft closure and binary soft interior

We define binary soft neighborhood as follows.

Definition 4.1. A binary soft set (G, E) in a binary soft topological space (U_1, U_2, τ, E) is called a binary soft neighborhood (briefly: nbd) of the binary soft point e_F over U_1, U_2, if there exists a binary soft open set (H, E) such that $e_F \mathrel{\tilde{\tilde{\in}}} (H, E) \mathrel{\tilde{\tilde{\subseteq}}} (G, E)$.
The binary soft neighborhood system of a binary soft point e_F, denoted by $\tilde{\tilde{N}}_\tau(e_F)$, is the family of all its binary soft neighborhoods.

Definition 4.2. A binary soft set (G, E) over U_1, U_2, in a binary soft topological space (U_1, U_2, τ, E) is called a binary soft neighborhood (briefly: nbd) of the binary soft set (F, E), if there exists a binary soft open set (H, E) such that $(F, E) \mathrel{\tilde{\tilde{\subseteq}}} (H, E) \mathrel{\tilde{\tilde{\subseteq}}} (G, E)$.

Theorem 4.3. *The binary soft neighborhood system $\tilde{\tilde{N}}_\tau(e_F)$ at binary soft point e_F in a binary soft topological space (U_1, U_2, τ, E) has the following properties:*
(a) If $(G, E) \mathrel{\tilde{\tilde{\in}}} \tilde{\tilde{N}}_\tau(e_F)$, then $e_F \mathrel{\tilde{\tilde{\in}}} (G, E)$.
(b) If $(G, E) \mathrel{\tilde{\tilde{\in}}} \tilde{\tilde{N}}_\tau(e_F)$ and $(G, E) \mathrel{\tilde{\tilde{\subseteq}}} (H, E)$, then $(H, E) \mathrel{\tilde{\tilde{\in}}} \tilde{\tilde{N}}_\tau(e_F)$.
(c) If (G, E), $(H, E) \mathrel{\tilde{\tilde{\in}}} \tilde{\tilde{N}}_\tau(e_F)$, then $(G, E) \mathbin{\tilde{\tilde{\cap}}} (H, E) \mathrel{\tilde{\tilde{\in}}} \tilde{\tilde{N}}_\tau(e_F)$.
(d) If $(G, E) \mathrel{\tilde{\tilde{\in}}} \tilde{\tilde{N}}_\tau(e_F)$, then there is a $(M, E) \mathrel{\tilde{\tilde{\in}}} \tilde{\tilde{N}}_\tau(e_F)$ such that $(G, E) \mathrel{\tilde{\tilde{\in}}} N_\tau(e_H)$, for each $e_H \mathrel{\tilde{\tilde{\in}}} (M, E)$.

Proof. (a) If $(G, E) \mathrel{\tilde{\tilde{\in}}} \tilde{\tilde{N}}_\tau(e_F)$, then there is a $(H, E) \mathrel{\tilde{\tilde{\in}}} \tau$ such that $e_F \mathrel{\tilde{\tilde{\in}}} (H, E) \mathrel{\tilde{\tilde{\subseteq}}} (G, E)$. Therefore, we have $e_F \mathrel{\tilde{\tilde{\in}}} (G, E)$.

(b) Let $(G, E) \mathrel{\tilde{\tilde{\in}}} \tilde{\tilde{N}}_\tau(e_F)$ and $(G, E) \mathrel{\tilde{\tilde{\subseteq}}} (H, E)$. Since $(G, E) \mathrel{\tilde{\tilde{\in}}} \tilde{\tilde{N}}_\tau(e_F)$, then there is a $(M, E) \mathrel{\tilde{\tilde{\in}}} \tau$ such that $e_F \mathrel{\tilde{\tilde{\in}}} (M, E) \mathrel{\tilde{\tilde{\subseteq}}} (G, E)$. Therefore, we have $e_F \mathrel{\tilde{\tilde{\in}}} (M, E) \mathrel{\tilde{\tilde{\subseteq}}} (G, E) \mathrel{\tilde{\tilde{\subseteq}}} (H, E)$ and so $(H, E) \mathrel{\tilde{\tilde{\in}}} \tilde{\tilde{N}}_\tau(e_F)$.

(c) If $(G, E), (H, E) \mathrel{\tilde{\tilde{\in}}} \tilde{\tilde{N}}_\tau(e_F)$, then there exist $(M, E), (S, E) \mathrel{\tilde{\tilde{\in}}} \tau$ such that $e_F \mathrel{\tilde{\tilde{\in}}} (M, E) \mathrel{\tilde{\tilde{\subseteq}}} (G, E)$ and $e_F \mathrel{\tilde{\tilde{\in}}} (S, E) \mathrel{\tilde{\tilde{\subseteq}}} (H, E)$. Hence $e_F \mathrel{\tilde{\tilde{\in}}} (M, E) \mathbin{\tilde{\tilde{\cap}}} (S, E) \mathrel{\tilde{\tilde{\subseteq}}} (G, E) \mathbin{\tilde{\tilde{\cap}}} (H, E)$. Since $(M, E) \mathbin{\tilde{\tilde{\cap}}} (S, E) \mathrel{\tilde{\tilde{\in}}} \tau$, we have $(G, E) \mathbin{\tilde{\tilde{\cap}}} (H, E) \mathrel{\tilde{\tilde{\in}}} \tilde{\tilde{N}}_\tau(e_F)$.

(d) If $(G, E) \mathrel{\tilde{\tilde{\in}}} \tilde{\tilde{N}}_\tau(e_F)$, then there is a $(S, E) \mathrel{\tilde{\tilde{\in}}} \tau$ such that $e_F \mathrel{\tilde{\tilde{\in}}} (S, E) \mathrel{\tilde{\tilde{\subseteq}}} (G, E)$. Put $(M, E) = (S, E)$. Then for every $e_H \mathrel{\tilde{\tilde{\in}}} (M, E)$, $e_H \mathrel{\tilde{\tilde{\in}}} (M, E) \mathrel{\tilde{\tilde{\subseteq}}} (S, E) \mathrel{\tilde{\tilde{\subseteq}}} (G, E)$. This implies $(G, E) \mathrel{\tilde{\tilde{\in}}} N_\tau(e_H)$. This completes the proof. $\square$

Definition 4.4. Let (U_1, U_2, τ, E) be a binary soft topological space over U_1, U_2 and (F, E) be a binary soft set over U_1, U_2. Then the binary soft closure of (F, E), denoted by $\widetilde{Cl}\ (F, E)$ is the intersection of all binary soft closed super sets of (F, E). Clearly $\widetilde{Cl}\ (F, E)$ is the smallest binary soft closed set over U_1, U_2 which contains (F, E), by Proposition 3.13.

Example 4.5. Let $U_1 = \{b_1, b_2, b_3\}, U_2 = \{t_1, t_2, t_3\}$, $E = \{e_1, e_2\}$ and $\tau = \{\tilde{\tilde{\Phi}}, \tilde{\tilde{E}}, (F_1, E), (F_2, E), (F_3, E), (F_4, E), (F_5, E)\}$, be binary soft topology defined over U_1, U_2, where $(F_1, E), (F_2, E), (F_3, E), (F_4, E), (F_5, E)$ are defined as follows:

$$\begin{array}{ll} F_1(e_1) = (\{b_2\}, \{t_2\}), & F_1(e_2) = (\{b_1\}, \{t_1\}), \\ F_2(e_1) = (\{b_2, b_3\}, \{t_2, t_3\}), & F_2(e_2) = (\{b_1, b_2\}, \{t_1, t_2\}), \\ F_3(e_1) = (\{b_1, b_2\}, \{t_1, t_2\}), & F_3(e_2) = (U_1, U_2), \\ F_4(e_1) = (\{b_1, b_2\}, \{t_1, t_2\}), & F_4(e_2) = (\{b_1, b_3\}, \{t_1, t_3\}), \\ F_5(e_1) = (\{b_2\}, \{t_2\}), & F_5(e_2) = (\{b_1, b_2\}, \{t_1, t_2\}) \end{array}$$

Clearly the binary soft closed sets are
$\{(e_1, (\{b_1, b_3\}, \{t_1, t_3\})), (e_2, (\{b_2, b_3\}, \{t_2, t_3\}))\}$, $\{(e_1, (\{b_1\}, \{t_1\})), (e_2, (\{b_3\}, \{t_3\}))\}$, $\{(e_1, (\{b_3\}, \{t_3\})), (e_2, (\Phi, \Phi))\}$, $\{(e_1, (\{b_3\}, \{t_3\})), (e_2, (\{b_2\}, \{t_2\}))\}$, $\{(e_1, (\{b_1, b_3\}, \{t_1, t_3\})), (e_2, (\{b_3\}, \{t_3\}))\}$, $\tilde{\tilde{\Phi}}$, $\tilde{\tilde{E}}$.
Now we consider the binary soft set (G, E) as:
$(G, E) = \{(e_1, (\{b_1\}, \{t_3\})), (e_2, (\{b_2\}, \{t_2\}))\}$. Clearly the binary soft closure of (G, E) is $\widetilde{Cl}(G, E) = \{(e_1, (\{b_1, b_3\}, \{t_1, t_3\})), (e_2, (\{b_2, b_3\}, \{t_2, t_3\}))\}$.

Theorem 4.6. *Let (U_1, U_2, τ, E) be a binary soft topological space over U_1, U_2, (F, E) and (G, E) are binary soft sets over U_1, U_2. Then*

(1) $\widetilde{Cl}\ (\tilde{\tilde{\Phi}}) \tilde{\tilde{=}} (\tilde{\tilde{\Phi}})$ *and* $\widetilde{Cl}\ (\tilde{\tilde{E}}) \tilde{\tilde{=}} (\tilde{\tilde{E}})$.

(2) $(F, E)\ \tilde{\tilde{\subseteq}}\ \widetilde{Cl}\ (F, E)$.

(3)(F, E) *is a binary soft closed set if and only if* $(F, E)\ \tilde{\tilde{=}}\ \widetilde{Cl}\ (F, E)$.

(4) $\widetilde{Cl}\ \left(\widetilde{Cl}\ (F, E)\right)\ \tilde{\tilde{=}}\ \widetilde{Cl}\ (F, E)$.

(5) $(F, E)\ \tilde{\tilde{\subseteq}}\ (G, E)$ *implies* $\widetilde{Cl}\ (F, E)\ \tilde{\tilde{\subseteq}}\ \widetilde{Cl}\ (G, E)$.

(6) $\widetilde{Cl}\ \left((F, E) \tilde{\tilde{\cup}} (G, E)\right)\ \tilde{\tilde{=}}\ \widetilde{Cl}\ (F, E)\ \tilde{\tilde{\cup}}\ \widetilde{Cl}\ (G, E)$.

(7) $\widetilde{Cl}\ \left((F, E) \tilde{\tilde{\cap}} (G, E)\right)\ \tilde{\tilde{\subseteq}}\ \widetilde{Cl}\ (F, E)\ \tilde{\tilde{\cap}}\ \widetilde{Cl}\ (G, E)$.

Proof. (1) and (2) are obvious.

(3) If (F, E) is a binary soft closed set over U_1, U_2, then (F, E) is itself a binary soft closed set over U_1, U_2 which contains (F, E). So (F, E) is the smallest binary soft closed set containing (F, E) and $(F, E) \tilde{\tilde{=}}\ \widetilde{Cl}(F, E)$.

Conversely, suppose that $(F, E)\ \tilde{\tilde{=}}\ \widetilde{Cl}(F, E)$. Since $\widetilde{Cl}(F, E)$ is a binary soft closed set, so (F, E) is a binary soft closed set over U_1, U_2.

(4) Since $\widetilde{Cl}(F, E)$ is a binary soft closed set, therefore by part (3) we have $\widetilde{Cl}\left(\widetilde{Cl}(F, E)\right) \tilde{\tilde{=}} \widetilde{Cl}(F, E)$.

(5) Suppose that $(F, E) \tilde{\tilde{\subseteq}} (G, E)$. Then every binary soft closed super set of (G, E) will also contain (F, E). This means every binary soft closed super set of (G, E) is also a binary soft closed super set of (F, E). Hence the intersection of binary soft closed super sets of (F, E) is contained in the binary soft intersection of binary soft closed super sets of (G, E). Thus $\widetilde{Cl}(F, E) \tilde{\tilde{\subseteq}}\ \widetilde{Cl}(G, E)$.

(6) Since $(F, E)\ \tilde{\tilde{\subseteq}} (F, E) \tilde{\tilde{\cup}} (G, E)$ and $(G, E) \tilde{\tilde{\subseteq}} (F, E) \tilde{\tilde{\cup}} (G, E)$. So by part (5), $\widetilde{Cl}(F, E) \tilde{\tilde{\subseteq}}\ \widetilde{Cl}\left((F, E) \tilde{\tilde{\cup}} (G, E)\right)$ and $\widetilde{Cl}(G, E) \tilde{\tilde{\subseteq}}\ \widetilde{Cl}\left((F, E) \tilde{\tilde{\cup}} (G, E)\right)$. Thus $\widetilde{Cl}(F, E)\ \tilde{\tilde{\cup}}\ \widetilde{Cl}(G, E) \tilde{\tilde{\subseteq}}\ \widetilde{Cl}\left((F, E) \tilde{\tilde{\cup}} (G, E)\right)$.

Conversely suppose that $(F,E)\tilde{\tilde{\subseteq}}\ \widetilde{Cl}(F,E)$ and $(G,E)\tilde{\tilde{\subseteq}}\widetilde{Cl}\ (G,E)$. So $(F,E)\tilde{\tilde{\cup}}(G,E)\tilde{\tilde{\subseteq}}$ $\widetilde{Cl}(F,E)\ \tilde{\tilde{\cup}}\ \widetilde{Cl}(G,E)$. By Proposition 3.13, $\widetilde{Cl}(F,E)\ \tilde{\tilde{\cup}}\ \widetilde{Cl}(G,E)$ is a binary soft closed set over U_1, U_2 being the union of two binary soft closed sets. Then $\widetilde{Cl}\left((F,E)\tilde{\tilde{\cup}}(G,E)\right)\tilde{\tilde{\subseteq}}\widetilde{Cl}(F,E)\ \tilde{\tilde{\cup}}\ \widetilde{Cl}(G,E)$. Thus $\widetilde{Cl}\left((F,E)\tilde{\tilde{\cup}}(G,E)\right) = \widetilde{Cl}(F,E)\ \tilde{\tilde{\cup}}$ $\widetilde{Cl}(G,E)$.

(7) Since $(F,E)\tilde{\tilde{\cap}}(G,E)\tilde{\tilde{\subseteq}}(F,E)$ and $(F,E)\tilde{\tilde{\cap}}(G,E)\,\tilde{\tilde{\subseteq}}(G,E)$. So by part (5), $\widetilde{Cl}\left((F,E)\tilde{\tilde{\cap}}(G,E)\right)\tilde{\tilde{\subseteq}}\,\widetilde{Cl}(F,E)$ and $\widetilde{Cl}\left((F,E)\tilde{\tilde{\cap}}(G,E)\right)\tilde{\tilde{\subseteq}}\widetilde{Cl}(G,E)$. Thus $\widetilde{Cl}\left((F,E)\tilde{\tilde{\cap}}(G,E)\right)\tilde{\tilde{\subseteq}}\widetilde{Cl}(F,E)\tilde{\tilde{\cap}}\ \widetilde{Cl}(G,E)$. Hence the proof is completed. □

The following example shows that the equality in the above Theorem 4.6(7) does not hold in general.

Example 4.7. In above Example 4.5, let us consider another binary soft set (H,E) as:
$(H,E)=\{(e_1,(\{b_2\},\{t_3\})),(e_2,(\{b_1,b_2\},\{t_3\}))\}$. Then, $\widetilde{Cl}(H,E)\tilde{\tilde{=}}\tilde{\tilde{E}}$. Also
$\left((G,E)\tilde{\tilde{\cap}}(H,E)\right)=\{(e_1,(\{\ \},\{t_3\})),(e_2,(\{b_2\},\{\ \}))\}$ and
$\widetilde{Cl}\left((G,E)\tilde{\tilde{\cap}}(H,E)\right)=\{(e_1,(\{b_3\},\{t_3\})),(e_2,(\{b_2\},\{t_2\}))\}$
But $\widetilde{Cl}\ (G,E)\ \tilde{\tilde{\cap}}\ \widetilde{Cl}\ (H,E)=\{(e_1,(\{b_1,b_3\},\{t_1,t_3\})),(e_2,(\{b_2,b_3\},\{t_2,t_3\}))\}$.
Thus, $\widetilde{Cl}\ \left((G,E)\tilde{\tilde{\cap}}(H,E)\right)\ \tilde{\tilde{\subseteq}}\ \widetilde{Cl}\ (G,E)\ \tilde{\tilde{\cap}}\ \widetilde{Cl}\ (H,E)$.
But $\widetilde{Cl}\ (G,E)\ \tilde{\tilde{\cap}}\ \widetilde{Cl}\ (H,E)\ \tilde{\tilde{\nsubseteq}}\widetilde{Cl}\ \left((G,E)\tilde{\tilde{\cap}}(H,E)\right)$.

Definition 4.8. Let (U_1,U_2,τ,E) be a binary soft topological space over U_1,U_2, (G,E) a binary soft set over U_1,U_2 and e_F is a binary soft point over U_1,U_2. Then e_F is said to be a binary soft interior point of (G,E), if there exists a binary soft open set (F,E) such that $e_F\tilde{\tilde{\in}}\ (F,E)\tilde{\tilde{\subseteq}}(G,E)$.

Proposition 4.9. *Let (U_1,U_2,τ,E) be a binary soft topological space over U_1,U_2, then we have the following:*

(1) *Each binary soft point e_F over U_1,U_2 has a binary soft neighborhood.*

(2) *If (F,E) and (G,E) are binary soft neighborhoods of some binary soft point e_F over U_1,U_2, then $(F,E)\ \tilde{\tilde{\cap}}(G,E)$ is also a binary soft neighborhood of binary soft point e_F over U_1,U_2.*

(3) *If (F,E) is a binary soft neighborhood of binary soft point e_F over U_1,U_2 and $(F,E)\ \tilde{\tilde{\subseteq}}\ (G,E)$, then (G,E) is also a binary soft neighborhood of binary soft point e_F over U_1,U_2.*

Proof. (1) For any binary soft point e_F over U_1,U_2, $e_F\tilde{\tilde{\in}}\tilde{\tilde{E}}$ and since $\tilde{\tilde{E}}\in\tau$. Thus $\tilde{\tilde{E}}$ is a binary soft neighborhood of e_F.

(2) Let (F,E) and (G,E) be the binary soft neighborhoods of binary soft point e_F over U_1,U_2, then there exist $(F_1,E),(F_2,E)\in\tau$ such that $e_F\tilde{\tilde{\in}}\ (F_1,E)\tilde{\tilde{\subseteq}}(F,E)$ and $e_F\tilde{\tilde{\in}}$ $(F_2,E)\tilde{\tilde{\subseteq}}(G,E)$. Now $e_F\tilde{\tilde{\in}}\ (F_1,E)$ and $e_F\tilde{\tilde{\in}}\ (F_2,E)$ implies that $e_F\tilde{\tilde{\in}}\ (F_1,E)\tilde{\tilde{\cap}}(F_2,E)$ and $(F_1,E)\tilde{\tilde{\cap}}(F_2,E)\ \in\ \tau$. So we have $e_F\tilde{\tilde{\in}}\ (F_1,E)\tilde{\tilde{\cap}}(F_2,E)\tilde{\tilde{\subseteq}}(F,E)\tilde{\tilde{\cap}}(G,E)$. Thus $(F,E)\tilde{\tilde{\cap}}(G,E)$ is a binary soft neighborhood of e_F.

(3) Let (F,E) be a binary soft neighborhood of binary soft point e_F over U_1,U_2 and $(F,E)\tilde{\tilde{\subseteq}}(G,E)$. By definition there exists a binary soft open set (F_1,E) such that $e_F\tilde{\tilde{\in}}$ $(F_1,E)\tilde{\tilde{\subseteq}}(F,E)\tilde{\tilde{\subseteq}}(G,E)$. Thus $e_F\tilde{\tilde{\in}}(F_1,E)\tilde{\tilde{\subseteq}}(G,E)$. Hence (G,E) is a binary soft neighborhood of e_F. This completes the proof. □

Definition 4.10. Let (U_1, U_2, τ, E) be a binary soft topological space over U_1, U_2 and V_1, V_2 be a non-empty subset of U_1, U_2. Then $\tau_{(V_1,V_2)} = \{({}^{(V_1,V_2)}F, E) | (F, E) \in \tau\}$ is said to be the binary soft relative topology over V_1, V_2 and $(V_1, V_2,\ \tau_{(V_1,V_2)}, E)$ is called a binary soft subspace of (U_1, U_2, τ, E). We can easily verify that $\tau_{(V_1,V_2)}$ is, in fact, a binary soft topology over V_1, V_2.

Example 4.11. Any binary soft subspace of a binary soft discrete topological space is a binary soft discrete topological space.

Example 4.12. Any binary soft subspace of a binary soft indiscrete topological space is a binary soft indiscrete topological space.

Definition 4.13. Let (U_1, U_2, τ, E) be a binary soft topological space over U_1, U_2, then binary soft interior of binary soft set (F, E) over U_1, U_2 is denoted by $\widetilde{Int}\ (F, E)$ and is defined as the union of all binary soft open sets contained in (F, E). Thus $\widetilde{Int}\ (F, E)$ is the largest binary soft open set contained in (F, E).

Example 4.14. Let $U_1 = \{b_1, b_2, b_3\}, U_2 = \{t_1, t_2, t_3\}$, $E = \{e_1, e_2\}$ and $\tau = \{\tilde{\tilde{\Phi}}, \tilde{\tilde{E}}, (F_1, E), (F_2, E), (F_3, E), (F_4, E), (F_5, E)\}$, where $(F_1, E), (F_2, E), (F_3, E)$, (F_4, E) and $(F_5, E)$are binary soft sets over U_1, U_2, defined as follows:

$$\begin{array}{ll} F_1(e_1) = (\{b_2\}, \{t_2\}), & F_1(e_2) = (\{b_1\}, \{t_1\}), \\ F_2(e_1) = (\{b_2, b_3\}, \{t_2, t_3\}), & F_2(e_2) = (\{b_1, b_2\}, \{t_1, t_2\}), \\ F_3(e_1) = (\{b_1, b_2\}, \{t_1, t_2\}), & F_3(e_2) = (U_1, U_2), \\ F_4(e_1) = (\{b_1, b_2\}, \{t_1, t_2\}), & F_4(e_2) = (\{b_1, b_3\}, \{t_1, t_3\}), \\ F_5(e_1) = (\{b_2\}, \{t_2\}), & F_5(e_2) = (\{b_1, b_2\}, \{t_1, t_2\}). \end{array}$$

Then τ defines a binary soft topology over U_1, U_2 and hence (U_1, U_2, τ, E) is a binary soft topological space over U_1, U_2. Let us consider $(F, E) = \{(\{b_2\}, \{t_2\}), (\{b_1, b_3\}, \{t_1, t_3\})\}$ be binary soft set over U_1, U_2. Then $\widetilde{Int}\ (F, E) = \{(\{b_2\}, \{t_2\}), (\{b_1\}, \{t_1\})\}$.

Theorem 4.15. *Let (U_1, U_2, τ, E) be a binary soft topological space over U_1, U_2 and (F, E), (G, E) are binary soft sets over U_1, U_2. Then*

(1) $\widetilde{Int}\ (\tilde{\tilde{\Phi}}) \tilde{\tilde{=}} (\tilde{\tilde{\Phi}})$ *and* $\widetilde{Int}\ (\tilde{\tilde{E}}) \tilde{\tilde{=}} (\tilde{\tilde{E}})$.

(2) $\widetilde{Int}\ (F, E) \tilde{\tilde{\subseteq}} (F, E)$.

(3) $\widetilde{Int}\ \left(\widetilde{Int}\ (F, E)\right) \tilde{\tilde{=}} (F, E)$.

(4)(F, E) *is a binary soft open set if and only if* $\widetilde{Int}\ (F, E) \tilde{\tilde{=}} (F, E)$.

(5)$(F, E) \tilde{\tilde{\subseteq}} (G, E)$ *implies* $\widetilde{Int}\ (F, E) \tilde{\tilde{\subseteq}} \widetilde{Int}\ (G, E)$

(6) $\widetilde{Int}\ (F, E)\ \tilde{\tilde{\cap}} \widetilde{Int}\ (G, E) \tilde{\tilde{=}} \widetilde{Int}\ \left((F, E) \tilde{\tilde{\cap}}\ (G, E)\right)$.

(7) $\widetilde{Int}\ (F, E)\ \tilde{\tilde{\cup}} \widetilde{Int}\ (G, E) \tilde{\tilde{\subseteq}} \widetilde{Int}\ \left((F, E) \tilde{\tilde{\cup}}\ (G, E)\right)$.

Proof. (1) and(2) are obvious.

(3) Since $\widetilde{Int}\ (F, E)$ is binary soft open and $\widetilde{Int}\ \left(\widetilde{Int}\ (F, E)\right)$ is the union of all binary soft open subsets over U_1, U_2 contained in $\widetilde{Int}\ (F, E)$, then $\widetilde{Int}\ (F, E) \tilde{\tilde{\subseteq}} \widetilde{Int}\ \left(\widetilde{Int}\ (F, E)\right)$. But $\widetilde{Int}\ \left(\widetilde{Int}\ (F, E)\right) \tilde{\tilde{\subseteq}} \widetilde{Int}\ (F, E)$ by (2). Hence $\widetilde{Int}\ \left(\widetilde{Int}\ (F, E)\right) = (F, E)$.

(4) If (F, E) is a binary soft open sets over U_1, U_2, then (F, E) is itself a binary soft open set over U_1, U_2 which contains (F, E). So $\widetilde{Int}\ (F, E)$ is the largest binary soft open set contained in (F, E) and $(F, E) \tilde{\tilde{=}} \widetilde{Int}\ (F, E)$.

Conversely, suppose that $(F, E) \tilde{\tilde{=}} \widetilde{Int}\ (F, E)$. Since $\widetilde{Int}\ (F, E)$ is a binary soft open set, so (F, E) is a binary soft open set over U_1, U_2.

(5) Suppose that $(F,E)\ \tilde{\tilde{\subseteq}}(G,E)$. Since $\widetilde{Int}\ \ (F,E)\ \ \tilde{\tilde{\subseteq}}\ (F,E)\tilde{\tilde{\subseteq}}(G,E)$. $\widetilde{Int}\ (F,E)$ is a binary soft open subset of (G,E), so by definition of $\widetilde{Int}\ (G,E)$, $\widetilde{Int}\ \ (F,E)\tilde{\tilde{\subseteq}}\ \widetilde{Int}\ (G,E)$.

(6) From (5), we have $\Big((F,E)\tilde{\tilde{\cap}}(G,E)\Big)\tilde{\tilde{\subseteq}}(F,E)$, $\Big((F,E)\tilde{\tilde{\cap}}(G,E)\Big)\tilde{\tilde{\subseteq}}(G,E)$ implies $\widetilde{Int}\ \Big((F,E)\tilde{\tilde{\cap}}(G,E)\Big)\tilde{\tilde{\subseteq}}\ \ \widetilde{Int}\ (F,E)$, $\widetilde{Int}\ \Big((F,E)\tilde{\tilde{\cap}}(G,E)\Big)\ \ \tilde{\tilde{\subseteq}}\ \ \widetilde{Int}\ (G,E)$. So that $\widetilde{Int}\ \Big((F,E)\tilde{\tilde{\cap}}\ (G,E)\Big)\ \ \tilde{\tilde{\subseteq}}\ \ \widetilde{Int}\ (F,E)\tilde{\tilde{\cap}}\widetilde{Int}\ (G,E)$. Also, $\widetilde{Int}\ (F,E)\tilde{\tilde{\subseteq}}(F,E)$ and $\widetilde{Int}\ (G,E)\tilde{\tilde{\subseteq}}(G,E)$ implies $\widetilde{Int}\ (F,E)\tilde{\tilde{\cap}}\widetilde{Int}\ (G,E)\tilde{\tilde{\subseteq}}\Big((F,E)\tilde{\tilde{\cap}}(G,E)\Big)$. Therefore $\widetilde{Int}\ (F,E)\tilde{\tilde{\cap}}\widetilde{Int}\ (G,E)$ is a binary soft open subset of $\Big((F,E)\tilde{\tilde{\cap}}(G,E)\Big)$. Hence $\widetilde{Int}\ (F,E)\tilde{\tilde{\cap}}\widetilde{Int}\ (G,E)\tilde{\tilde{\subseteq}}\widetilde{Int}\ \Big((F,E)\tilde{\tilde{\cap}}(G,E)\Big)$. Thus $\widetilde{Int}\ \ \ (F,E)\tilde{\tilde{\cap}}\widetilde{Int}\ (G,E) = \widetilde{Int}\ \Big((F,E)\tilde{\tilde{\cap}}\ (G,E)\Big)$.

(7) Since $(F,E)\tilde{\tilde{\subseteq}}\Big((F,E)\tilde{\tilde{\cup}}(G,E)\Big)$ and $(G,E)\tilde{\tilde{\subseteq}}\Big((F,E)\ \tilde{\tilde{\cup}}(G,E)\Big)$. So by (5), $\widetilde{Int}\ (F,E)\tilde{\tilde{\subseteq}}\widetilde{Int}\Big((F,E)\ \tilde{\tilde{\cup}}\ (G,E)\Big)$ and $\widetilde{Int}(G,E)\tilde{\tilde{\subseteq}}\widetilde{Int}\Big((F,E)\tilde{\tilde{\cup}}\ (G,E)\Big)$. Thus $\widetilde{Int}(F,E)\ \tilde{\tilde{\cup}}\ \widetilde{Int}(G,E)\tilde{\tilde{\subseteq}}\ \widetilde{Int}\ \Big((F,E)\tilde{\tilde{\cup}}(G,E)\Big)$, since $\widetilde{Int}\ (F,E)\ \tilde{\tilde{\cup}}\ \widetilde{Int}(G,E)$ is a binary soft open set. Hence the proof is completed. □

The following example shows that the equality does not hold in above Theorem 4.15(7).

Example 4.16. Let us consider the binary soft topological space (U_1,U_2,τ,E) over U_1,U_2 in Example 4.14 and the binary soft set (F,E) and (G,E) over U_1,U_2 defined as follows:
$(F,E)=\{(\{b_2\},\{t_2\}),(\{b_1,b_3\},\{t_1,t_3\})\}$,
$(G,E)=\{(\{b_1,b_3\},\{t_1,t_3\}),(\{b_1,b_2,b_3\},\{t_1,t_2,t_3\})\}$.
Then $\widetilde{Int}\ (F,E)=\{(\{b_2\},\{t_2\}),(\{b_1\},\{t_1\})\}$ and $\widetilde{Int}\ (G,E)=\tilde{\tilde{\Phi}}$.

$$\begin{aligned}(F,E)\tilde{\tilde{\cup}}(G,E)\ &=\{(\{b_2\},\{t_2\}),(\{b_1,b_3\},\{t_1,t_3\})\}\\ &\tilde{\tilde{\cup}}\{(\{b_1,b_3\},\{t_1,t_3\}),(\{b_1,b_2,b_3\},\{t_1,t_2,t_3\})\}\\ &=\{(\{b_1,b_2,b_3\},\{t_1,t_2,t_3\}),(\{b_1,b_2,b_3\},\{t_1,t_2,t_3\})\}\\ &=\{(U_1,U_2),(U_1,U_2)\}\\ &=\tilde{\tilde{E}}\end{aligned}$$

Now $\widetilde{Int}\ \Big((F,E)\ \tilde{\tilde{\cup}}\ (G,E)\Big)=\ \widetilde{Int}\ \Big(\tilde{\tilde{E}}\Big)=\tilde{\tilde{E}}$, and

$\widetilde{Int}\ \ (F,E)\ \tilde{\tilde{\cup}}\ \widetilde{Int}\ (G,E)=\{(\{b_2\},\{t_2\}),(\{b_1\},\{t_1\})\}\ \tilde{\tilde{\cup}}\ \tilde{\tilde{\Phi}}=\{(\{b_2\},\{t_2\}),(\{b_1\},\{t_1\})\}$.
So that $\widetilde{Int}\ (F,E)\ \tilde{\tilde{\cup}}\widetilde{Int}\ (G,E)\tilde{\tilde{\subseteq}}\widetilde{Int}\ \Big((F,E)\tilde{\tilde{\cup}}(G,E)\Big)$.
But $\widetilde{Int}\ \Big((F,E)\ \tilde{\tilde{\cup}}\ (G,E)\Big)\ \tilde{\tilde{\nsubseteq}}\widetilde{Int}\ \ (F,E)\ \tilde{\tilde{\cup}}\ \widetilde{Int}\ (G,E)$.

The following theorem gives us the relationship between binary soft closure and binary soft interior in terms of binary soft relative complement of binary soft sets.

Theorem 4.17. *Let (F,E) be a binary soft set of binary soft topological space over U_1,U_2. Then*
(1)$\widetilde{Int}\ (F,E)'\tilde{\tilde{=}}(\widetilde{Cl}\ (F,E))'$.
(2)$\widetilde{Int}\ (F,E)\tilde{\tilde{=}}(\widetilde{Cl}\ (F,E)')'$.
(3)$\widetilde{Cl}\ (F,E)'\tilde{\tilde{=}}(\widetilde{Int}\ (F,E))'$.

The following example shows that the equalities hold in the above theorem.

Example 4.18. Let $U_1=\{b_1,b_2,b_3\},U_2=\{t_1,t_2,t_3\},,\ E=\{e_1,e_2\}$ and
$\tau=\{\tilde{\tilde{\Phi}},\tilde{\tilde{E}},(F_1,E),(F_2,E),(F_3,E),(F_4,E),(F_5,E),(F_6,E),(F_7,E)\}$,

where $(F_1,E),(F_2,E),(F_3,E),(F_4,E),(F_5,E),(F_6,E)$ and (F_7,E) are binary soft sets over U_1,U_2, defined as

$$\begin{array}{ll} F_1(e_1)=(\{b_1,b_2\},\{t_1,t_2\}), & F_1(e_2)=(\{b_1,b_2\},\{t_1,t_2\}), \\ F_2(e_1)=(\{b_2\},\{t_2\}), & F_2(e_2)=(\{b_1,b_3\},\{t_1,t_3\}), \\ F_3(e_1)=(\{b_2,b_3\},\{t_2,t_3\}), & F_3(e_2)=(\{b_1\},\{t_1\}), \\ F_4(e_1)=(\{b_2\},\{t_2\}), & F_4(e_2)=(\{b_1\},\{t_1\}), \\ F_5(e_1)=(\{b_1,b_2\},\{t_1,t_2\}), & F_5(e_2)=(U_1,U_2), \\ F_6(e_1)=(U_1,U_2), & F_6(e_2)=(\{b_1,b_2\},\{t_1,t_2\}), \\ F_7(e_1)=(\{b_2,b_3\},\{t_2,t_3\}), & F_7(e_2)=(\{b_1,b_3\},\{t_1,t_3\}). \end{array}$$

Then τ defines a binary soft topology over U_1,U_2 and hence (U_1,U_2,τ,E) is a binary soft topological space over U_1,U_2.
Clearly the binary soft closed sets are $\tilde{\tilde{E}},\tilde{\tilde{\Phi}},\{(\{b_3\},\{t_3\}),(\{b_3\},\{t_3\})\}$, $\{(\{b_1,b_3\},\{t_1,t_3\}),(\{b_2\},\{t_2\})\},\{(\{b_1\},\{t_1\}),(\{b_2,b_3\},\{t_2,t_3\})\}$, $\{(\{b_1,b_3\},\{t_1,t_3\}),(\{b_2,b_3\},\{t_2,t_3\})\}$, $\{(\{b_3\},\{t_3\}),(\Phi,\Phi)\},\{(\Phi,\Phi),(\{b_3\},\{t_3\})\}$ and $\{(\{b_1\},\{t_1\}),(\{b_2\},\{t_2\})\}$,
Let us take $(F,E)=\{(\{b_2\},\{t_2\}),(\{b_1,b_2\},\{t_1,t_2\})\}$. Then $(F,E)'=\{(\{b_1,b_3\},\{t_1,t_3\}),(\{b_3\},\{t_3\})\}$, $\widetilde{Int}\,(F,E)'=\tilde{\tilde{\Phi}}$, $(\widetilde{Int}\,(F,E)')'=\tilde{\tilde{E}}$. Also $\widetilde{Cl}\,(F,E)=\tilde{\tilde{E}}$, which implies $(\widetilde{Cl}\,(F,E))'=\tilde{\tilde{\Phi}}$.
Moreover, $\widetilde{Cl}\,(F,E)'=\{(\{b_1,b_3\},\{t_1,t_3\}),(\{b_2,b_3\},\{t_2,t_3\})\}$,
$(\widetilde{Cl}\,(F,E)')'=\{(\{b_2\},\{t_2\}),(\{b_1\},\{t_1\})\}$,
$\widetilde{Int}\,(F,E)=\{(\{b_2\},\{t_2\}),(\{b_1\},\{t_1\})\}$,
$\widetilde{Int}\,(F,E)'=\{(\{b_1,b_3\},\{t_1,t_3\}),(\{b_2,b_3\},\{t_2,t_3\})\}$.

5. Conculusion

Nowadays, many researchers worked on the findings of structures of soft sets theory initiated by Molodtsov and applied to many problems having uncertainties. Soft topological spaces based on soft set theory, which is a collection of information granules is the mathematical formulation of approximate reasoning about information systems. In the present work, we initiated binary soft topological spaces which are, in fact, a generalization of soft topological spaces in broader sense and are defined over two initial universes U_1 and U_2 with fixed set of parameters. The concepts of binary soft open set, binary soft closed set, binary soft neighborhood, binary soft closure and binary soft interior are studied and their basic properties are explored. We hope that the findings in this paper will help the researchers to enhance and promote the further study on binary soft topology to carry out general framework for the applications in practical life. In future, we will find out further properties of binary soft topological spaces and will work out on the applications of binary soft sets in information science, decision making problems as well as in medical diagnosis problems.

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