Super \((a, d)\)-star-antimagic graphs

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Abstract

A simple graph \(G = (V, E)\) admitting an \(H\)-covering is said to be \((a, d)\)-\(H\)-antimagic if there exists a bijection \(f : V \cup E \rightarrow \{1, 2, \ldots, |V| + |E|\}\) such that, for all subgraphs \(H'\) of \(G\) isomorphic to \(H\), \(wt_f(H') = \sum_{v \in V(H')} f(v) + \sum_{e \in E(H')} f(e)\), form an arithmetic progression \(a, a + d, \ldots, a + (t - 1)d\), where \(a\) is the first term, \(d\) is the common difference and \(t\) is the number of subgraphs in the \(H\)-covering. Then \(f\) is called an \((a, d)\)-\(H\)-antimagic labeling. If \(f(V) = \{1, 2, \ldots, |V|\}\), then \(f\) is called super \((a, d)\)-\(H\)-antimagic labeling. In this paper we investigate the existence of super \((a, d)\)-star-antimagic labelings of a particular class of banana trees and construct a star-antimagic graph.

Mathematics Subject Classification (2010). 05C78, 05C70

Keywords. \(H\)-covering, (super) \((a, d)\)-\(H\)-antimagic labeling, star, banana tree

1. Introduction

Let \(G = (V, E)\) be a finite simple graph. A family of subgraphs \(H_1, H_2, \ldots, H_t\) of \(G\) is called an \(edge-covering\) of \(G\) if each edge of \(E\) belongs to at least one of the subgraphs \(H_i\), \(i = 1, 2, \ldots, t\). Then the graph \(G\) admitting an \(H\)-covering is \((a, d)\)-\(H\)-antimagic if there exists a bijection \(f : V \cup E \rightarrow \{1, 2, \ldots, |V| + |E|\}\) such that, for all subgraphs \(H'\) of \(G\) isomorphic to \(H\), the \(H'\)-weights,

\[
wt_f(H') = \sum_{v \in V(H')} f(v) + \sum_{e \in E(H')} f(e),
\]

form an arithmetic progression \(a, a + d, \ldots, a + (t - 1)d\), where \(a > 0\) is the first term, \(d \geq 0\) is the common difference and \(t\) is the number of subgraphs of \(G\) isomorphic to \(H\). Such a labeling is called super if \(f(V) = \{1, 2, \ldots, |V|\}\). For \(d = 0\) it is called \(H\)-magic and \(H\)-supermagic, respectively.

The notion of \(H\)-magic graphs was due to Gutiérrez and Lladó [3]. Inayah, Salman and Simanjuntak [4] introduced the concept of \((a, d)\)-\(H\)-antimagic labeling. In [5] they proved that there exists a super \((a, d)\)-\(H\)-antimagic total labeling for shackles of a connected graph.

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Received: 12.01.2017; Accepted: 04.12.2017
H. Semaničová-Feňovčíková, Bača, Lascsová, Miller and Ryan [8] proved that wheels $W_n$, $n \geq 3$ are super $(a,d)$-$C_k$-antimagic for every $k = 3, 4, \ldots, n-1, n+1$ and $d = 0, 1, 2$. For more information see [1,2] and [7].

2. Preliminaries and known results

We use the following notations. For two integers $a$, $b$, $a < b$, let $[a, b]$ denote the set of all integers from $a$ to $b$. For any set $S$, subset of integers $\mathbb{Z}$ we write, $\sum S = \sum_{x \in S} x$ and for an integer $k$, let $k + S = \{k + x : x \in S\}$. Thus $k + [a, b]$ is the set $\{x \in \mathbb{Z} : k + a \leq x \leq k + b\}$. It can be easily verified that $\sum (k + S) = k|S| + \sum S$.

If $\mathbb{P} = \{X_1, X_2, \ldots, X_n\}$ is a partition of a set $X$ of integers with the same cardinality then we say $\mathbb{P}$ is an $n$-equipartition of $X$. Also we denote the set of subsets sums of the parts of $\mathbb{P}$ by $\sum \mathbb{P} = \{\sum X_1, \sum X_2, \ldots, \sum X_n\}$.

**Lemma 2.1.** [3] Let $h$ and $k$ be two positive integers. For each integer $0 \leq t \leq \lfloor h/2 \rfloor$ there is a $k$-equipartition $\mathbb{P} = \{X_1, X_2, \ldots, X_k\}$ of $X = [1, hk]$ such that $\sum \mathbb{P}$ is an arithmetic progression of difference $d = h - 2t$.

**Lemma 2.2.** [6] If $h$ is even, then there exists a $k$-equipartition $\mathbb{P} = \{X_1, X_2, \ldots, X_k\}$ of $X = [1, hk]$ such that $\sum X_r = h(hk + 1)/2$ for $1 \leq r \leq k$.

Arrange the numbers $X = [1, hk]$ in a matrix $A = (a_{i,j})_{h \times k}$, where $a_{i,j} = (i-1)k + j$ for $1 \leq i \leq h$ and $1 \leq j \leq k$. For $1 \leq r \leq k$, define $X_r = \{a_{i,r} : 1 \leq i \leq h/2\} \cup \{a_{i,k-r+1} : h/2 + 1 \leq i \leq h\}$. Then,

$$
\sum X_r = \sum_{i=1}^{h/2} a_{i,r} + \sum_{i=h/2+1}^{h} a_{i,k-r+1}
$$

$$
= \sum_{i=1}^{h/2} \{(i-1)k + r\} + \sum_{i=h/2+1}^{h} \{(i-1)k + k - r + 1\} = \frac{h(hk + 1)}{2}.
$$

**Lemma 2.3.** [6] Let $h$ and $k$ be two positive integers such that $h$ is even and $k \geq 3$ is odd. Then there exists a $k$-equipartition $\mathbb{P} = \{X_1, X_2, \ldots, X_k\}$ of $X = [1, hk]$ such that $\sum X_r = (h - 1)(hk + k + 1)/2 + r$ for $1 \leq r \leq k$. Thus, $\sum \mathbb{P}$ is a set of consecutive integers given by $\sum \mathbb{P} = (h - 1)(hk + k + 1)/2 + [1, k]$.

Let us arrange the set of integers $X = \{1, 2, 3, \ldots, hk\}$ in a matrix $A = (a_{i,j})_{h \times k}$, where $a_{i,j} = (i-1)k + j$ for $1 \leq i \leq h$ and $1 \leq j \leq k$. We construct a $k$-equipartition $Y_1, Y_2, \ldots, Y_k$ using the first $(h-1)$ rows of the matrix as $Y_r = \{a_{i,r} : 1 \leq i \leq h/2\} \cup \{a_{i,k-r+1} : h/2 + 1 \leq i \leq h\}$. For $1 \leq r \leq k$, we define $X_r = Y_{\sigma(r)} \cup \{(h - 1)k + \pi(r)\}$, where $\sigma$ and $\pi$ denote the permutations of $\{1, 2, \ldots, k\}$ given by

$$
\sigma(r) = \begin{cases} 
\frac{k - 2r + 1}{2} & \text{for } 1 \leq r \leq \frac{k-1}{2} \\
\frac{2k + 2r + 1}{2} & \text{for } \frac{k+1}{2} \leq r \leq k
\end{cases}
$$

and

$$
\pi(r) = \begin{cases} 
2r & \text{for } 1 \leq r \leq \frac{k-1}{2} \\
2r - k & \text{for } \frac{k+1}{2} \leq r \leq k.
\end{cases}
$$

Then it can be verified that $\sum X_r = (h - 1)(hk + k + 1)/2 + r$ for $1 \leq r \leq k$.

**Lemma 2.4.** [6] Let $h$ and $k$ be two positive integers and let $h$ be odd. Then there exists a $k$-equipartition $\mathbb{P} = \{X_1, X_2, \ldots, X_k\}$ of $X = [1, hk]$ such that $\sum X_r = (h - 1)(hk + k + 1)/2 + r$ for $1 \leq r \leq k$. Thus, $\sum \mathbb{P}$ is a set of consecutive integers given by $\sum \mathbb{P} = (h - 1)(hk + k + 1)/2 + [1, k]$. 

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Arrange the numbers $X = [1, hhk]$ in a matrix $A = (a_{i,j})_{h \times k}$, where $a_{i,j} = (i-1)k + j$ for $1 \leq i \leq h$ and $1 \leq j \leq k$. For $1 \leq r \leq k$, define $X_r = \{a_{i,r} : 1 \leq i \leq (h+1)/2\} \cup \{a_{i,k-r+1} : (h+3)2 \leq i \leq h\}$. Then $\sum X = (h-1)(hk + k + 1)/2 + [1, k]$. 

3. Super star-antimagic graphs

In this section we prove that a particular class of banana trees are super star-antimagic. The star $S_n$, $n \geq 1$ is a graph isomorphic to the complete bipartite graph $K_{1,n}$. A banana tree $Bt(n_1, n_2, \ldots, n_k)$ is the tree obtained by joining a vertex $v$ to one leaf vertex of each star in a family of disjoint stars $S_{n_1}, S_{n_2}, \ldots, S_{n_k}$. If $n_1 = n_2 = \cdots = n_k = n$ we will use the notation $Bt_n^k$ instead of $Bt(n, n, \ldots, n)$.

Theorem 3.1. The banana tree $Bt_n^k$, $k \geq 3$ admits a super $(a, d)$-antimagic labeling for $d \in \{0, 2, 4, \ldots, k+1\}$ if $k$ is odd and $d \in \{1, 3, 5, \ldots, k+1\}$ if $k$ is even.

Proof. Let the vertex set and the edge set of the banana tree $Bt_n^k$ be

$V(Bt_n^k) = \{v, v_1, v_i^j : i = 1, 2, \ldots, k; j = 1, 2, \ldots, k\},$

$E(Bt_n^k) = \{v_1v_i^j : i = 1, 2, \ldots, k; j = 1, 2, \ldots, k\} \cup \{vv_i^j : j = 1, 2, \ldots, k\}.$

Evidently, the graph $Bt_n^k$ admits a $S_k$-covering consisting of $k + 1$ stars isomorphic to $S_k$. We denote the $k$-stars by the symbols $\{S_k^1, S_k^2, \ldots, S_k^k, S_{k}^{k+1}\}$ such that the vertex set of $S_k^i$, $i = 1, 2, \ldots, k$, is $V(S_k^i) = \{v_i^1, v_i^2, \ldots, v_i^k\}$ and its edge set is $E(S_k^i) = \{v_i^jv_i^k : j = 1, 2, \ldots, k\}$. For the star $S_k^{k+1}$ holds $V(S_k^{k+1}) = \{v_i^1, v_i^2, \ldots, v_i^k\}$ and $E(S_k^{k+1}) = \{v_i^1 : j = 1, 2, \ldots, k\}$.

Let us distinguish two cases.

Case 1: $k$ is odd.

Since $k+1$ is even, by Lemma 2.3 then there exists a $k$-equipartition $\mathbb{P} = \{X_1, X_2, \ldots, X_k\}$ of $X = [1, (k+1)k]$ such that

$$\sum X_i = \frac{k(k+1)^2}{2} + i \quad \text{for } 1 \leq i \leq k. \quad (3.1)$$

It can be easily verified by the definition of $X_i$ in Lemma 2.3 that for $1 \leq i \leq k$

$$(\frac{k+1}{2} - 1) k + \sigma(i) \in X_i,$$

where $\sigma$ is the permutation on $\{1, 2, \ldots, k\}$ given by

$$\sigma(i) = \begin{cases} \frac{k-2-i+1}{2} & \text{for } 1 \leq i \leq \frac{k+1}{2}, \\ \frac{3k-2-i+1}{2} & \text{for } \frac{k+1}{2} \leq i \leq k. \end{cases}$$

We construct a new set of integers $X_{k+1}$ by choosing one particular element from each $X_i$, $i = 1, 2, \ldots, k$, together with $k^2 + k + 1$ as follows:

$$X_{k+1} = \left\{(\frac{k+1}{2} - 1) k + \sigma(i) : 1 \leq i \leq k\right\} \cup \left\{k^2 + k + 1\right\}.$$

Then

$$\sum X_{k+1} = \sum_{i=1}^{k} \left[\left(\frac{k+1}{2} - 1\right) k + \sigma(i)\right] + k^2 + k + 1$$

$$= \frac{k^2(k-1)}{2} + \frac{k(k+1)}{2} + k^2 + k + 1 = \frac{k(k+1)^2}{2} + k + 1. \quad (3.2)$$

From (3.1) and (3.2) we have

$$\sum X_i = \frac{k(k+1)^2}{2} + i \quad \text{for } 1 \leq i \leq k + 1.$$
where $d = k - 2t$ and $\Delta_t = \sum R^t_i - d$. Hence we have a $(k+1)$-equipartition $Q^t = \{Y^t_1, Y^t_2, \ldots, Y^t_{k+1}\}$ of the set $[k^2 + k + 2, 2k^2 + 2k + 1]$ such that
\[
\sum Y^t_i = k(k^2 + k + 1) + \Delta_t + di \quad \text{for } 1 \leq i \leq k + 1.
\]
For each $0 \leq t \leq \lfloor k/2 \rfloor$ we define a total labeling $f_t$, $f_t : V(Bt^k_t) \cup E(Bt^k_t) \rightarrow [1, 2k^2 + 2k + 1]$ as follows:
\[
f_t(E(S^t_i)) = Y^t_i \quad \text{for } 1 \leq i \leq k + 1,
\]
\[
f_t(v) = k^2 + k + 1,
\]
\[
f_t(V(S^t_i)) = X_i \quad \text{for } 1 \leq i \leq k + 1,
\]
with the restriction that for $i = 1, 2, \ldots, k$ it holds
\[
f_t(v^t_i) = \left(\frac{k+1}{2} - 1\right) k + \sigma(i).
\]
Then for $1 \leq i \leq k + 1$ we get
\[
wt_{f_t}(S^t_i) = \sum f_t(V(S^t_i)) + \sum f_t(E(S^t_i)) = \sum X_i + \sum Y^t_i = \frac{k(k+1)^2}{2} + i + k(k^2 + k + 1) + \Delta_t + di = a_t + (d+1)i,
\]
where $a_t = k(k+1)^2/2 + k(k^2 + k + 1) + \Delta_t$.

Since $k$ is odd and $d = k - 2t$ then for $0 \leq t \leq \lfloor k/2 \rfloor$ we have $d \in \{1, 3, 5, \ldots, k\}$. Hence the banana tree $Bt^k_t$ admits super $(a, d^*)$-$S_k$-antimagic labeling for $d^* \in \{2, 4, 6, \ldots, k+1\}$.

To prove that the banana tree $Bt^k_t$ admits a super $S_k$-magic labeling also, we define a total labeling $g : V(Bt^k_t) \cup E(Bt^k_t) \rightarrow [1, 2k^2 + 2k + 1]$ as follows:
\[
g(E(S^t_i)) = \left\lfloor \frac{k}{2} \right\rfloor (E(S^t_i)) = Y^t_i \quad \text{for } 1 \leq i \leq k + 1,
\]
\[
g(v) = k^2 + k + 1,
\]
\[
g(V(S^t_i)) = X_{k+2-i} \quad \text{for } 1 \leq i \leq k + 1,
\]
with the restriction that for $i = 1, 2, \ldots, k$
\[
g(v^t_i) = \left(\frac{k+1}{2} - 1\right) k + \sigma(i).
\]
Then for $1 \leq i \leq k + 1$
\[
wt_{g}(S^t_i) = \sum g(V(S^t_i)) + \sum g(E(S^t_i)) = \sum X_{k+2-i} + \sum Y^t_i = \frac{k(k+1)^2}{2} + k + 2 - i + k(k^2 + k + 1) + \left\lfloor \frac{k}{2} \right\rfloor + i
\]
\[
= \frac{k(k+1)^2}{2} + k + 2 + k(k^2 + k + 1) + \left\lfloor \frac{k}{2} \right\rfloor.
\]

**Case ii:** $k$ is even.

Since $k + 1$ is odd, by Lemma 2.4, there exists a $k$-equipartition $P = \{X_1, X_2, \ldots, X_k\}$ of $X = [1, (k+1)k]$ such that
\[
\sum X_i = \frac{k(k+1)^2}{2} + i \quad \text{for } 1 \leq i \leq k.
\]

(3.3)

It can be easily verified by the definition of $X_i$ in Lemma 2.4 that for $1 \leq i \leq k/2$
\[
\left(\frac{k+2}{2} - 1\right) k + i \in X_i
\]
and for $k/2 + 1 \leq i \leq k$
\[
\left(\frac{k}{2} - 1\right) k + i \in X_i.
\]
We construct a new set of integers $X_{k+1}$ by choosing one particular element from each $X_i$ for $i = 1, 2, \ldots, k$ together with $k^2 + k + 1$ as follows:

$$X_{k+1} = \left\{ \left( \frac{k^2}{2} - 1 \right) k + j : 1 \leq j \leq \frac{k}{2} \right\} \cup \left\{ \left( \frac{k}{2} - 1 \right) k + j : \frac{k}{2} + 1 \leq j \leq k \right\} \cup \{k^2 + k + 1\}.$$ 

Then

$$\sum_{j=1}^{k} \left( \left( \frac{k^2}{2} - 1 \right) k + j \right) + \sum_{j=\frac{k}{2}+1}^{k} \left( \left( \frac{k}{2} - 1 \right) k + j \right) + k^2 + k + 1 = \frac{k^2(k-1)}{2} + \frac{k(k+1)^2}{2} + k + 1.$$ (3.4)

From (3.3) and (3.4) we have

$$\sum_{i=1}^{k} X_i = \frac{k(k+1)^2}{2} + i \quad \text{for} \quad 1 \leq i \leq k + 1.$$ 

By Lemma 2.1, for each integer $0 \leq t \leq \lfloor k/2 \rfloor$ there is a $(k + 1)$-equipartition $R^t = \{R^t_1, R^t_2, \ldots, R^t_{k+1}\}$ of $[1, k(k+1)]$ such that for $1 \leq i \leq k + 1$ holds

$$\sum_{i=1}^{k} R^t_i = \Delta_t + di,$$

where $d = k - 2t$ and $\Delta_t = \sum R^t_i - d$. Hence we have a $(k + 1)$-equipartition $Q^t = \{Y^t_1, Y^t_2, \ldots, Y^t_{k+1}\}$ of the set $[k^2 + k + 2, 2k^2 + 2k + 1]$ such that

$$\sum_{i=1}^{k} Y^t_i = (k^2 + k + 1) + \Delta_t + di \quad \text{for} \quad 1 \leq i \leq k + 1.$$ 

For each $0 \leq t \leq k/2$ we define a total labeling $f_t : V(B_t^k) \cup E(B_t^k) \rightarrow [1, 2k^2 + 2k + 1]$ as follows:

$$f_t(E(S^t_i)) = Y^t_i \quad \text{for} \quad 1 \leq i \leq k + 1,$$

$$f_t(v) = k^2 + k + 1,$$

$$f_t(V(S^t_i)) = X_i \quad \text{for} \quad 1 \leq i \leq k + 1,$$

with the restriction that

$$f_t(v^t_i) = \begin{cases} \left( \frac{k^2}{2} - 1 \right) k + i \\ \left( \frac{k}{2} - 1 \right) k + i \end{cases} \quad \text{for} \quad \frac{k}{2} + 1 \leq i \leq k.$$ 

Then for $1 \leq i \leq k + 1$ we get

$$\text{wt}_{f_t}(S^t_i) = \sum_{k} f_t(V(S^t_i)) + \sum_{k} f_t(E(S^t_i)) = \sum_{k} X_i + \sum_{k} Y^t_i = \frac{k(k+1)^2}{2} + i + k(k^2 + k + 1) + \Delta_t + di = a_t + (d+1)i,$$

where $a_t = k(k+1)^2/2 + k(k^2 + k + 1) + \Delta_t$.

Since $k$ is even and $d = k - 2t$ for $0 \leq t \leq k/2$ we have $d \in \{0, 2, 4, 6, \ldots, k\}$. Hence the banana tree $B_t^k$ admits a super $(a, d^*)$-$S_k$-antimagic labeling for $d^* \in \{1, 3, 5, \ldots, k + 1\}$. \hfill \Box

Figure 1 illustrates a super $(a, 6)$-$S_5$-antimagic labeling of the banana tree $B_t^5$ and a super $(a, 3)$-$S_6$-antimagic labeling of the banana tree $B_t^6$.

By the symbol $St^p_n$, $n \geq 1$, $p \geq 0$, we denote the graph obtained from the star $S_n$ by attaching $p$ pendant edges to every vertex of degree 1 in $S_n$. Let us denote the vertices and edges in $St^p_n$ such that

$$V(St^p_n) = \{v, v_1, v^*_i : i = 1, 2, \ldots, n; j = 1, 2, \ldots, p\},$$

$$E(St^p_n) = \{v_i v^*_j : i = 1, 2, \ldots, n; j = 1, 2, \ldots, p\} \cup \{v v_i : i = 1, 2, \ldots, n\}.$$
Evidently, the graph $S^n_p$ admits a $S_{p+1}$-covering. If $p \geq n$ then the $S_{p+1}$-covering of $S^n_p$ consists of $n$ stars isomorphic to $S_{p+1}$. Let us denote the $(p+1)$-stars by the symbols \( \{S^1_{p+1}, S^2_{p+1}, \ldots, S^n_{p+1}\} \) such that the edge set of \( S_{p+1}, i = 1, 2, \ldots, n \), is
\[
V(S^i_{p+1}) = \{v, v_i, v_i^j : j = 1, 2, \ldots, p\}
\]
and its edge set is
\[
E(S^i_{p+1}) = \{vv_i, v_i^j v_i^j : j = 1, 2, \ldots, p\}.
\]

If $p = n - 1$ then the graph $S^n_{p+1}$ contains $n + 1$ subgraphs isomorphic to $S_n$. We denote them \( \{S^1_n, S^2_n, \ldots, S^{n+1}_n\} \), where for $i = 1, 2, \ldots, n$, is \( V(S^n_i) = \{v, v_i, v_i^j : j = 1, 2, \ldots, n - 1\} \) and \( E(S^n_i) = \{vv_i, v_i^j v_i^j : j = 1, 2, \ldots, n - 1\} \). Moreover, \( V(S^{n+1}_n) = \{v, v_i : i = 1, 2, \ldots, n\} \) and \( E(S^{n+1}_n) = \{vv_i : i = 1, 2, \ldots, n\} \).

**Theorem 3.2.** Let $n, p$ be positive integers, $p \geq n$. Then the graph $S^n_p$ admits a super \((a, d)\)-star-antimagic labeling for $d \in \{0, 2, \ldots, 2(p+1)\}$. Moreover, if $n$ and $p$ are odd, $n \geq 3$, then differences $d \in \{1, 3, \ldots, p + 2\}$ are also feasible.

**Proof.** The graph $S^n_p$ has $(n(p + 1) + 1)$ vertices and $(n(p + 1))$ edges. If $p \geq n$ then the graph $S^n_p$ contains $n$ subgraphs isomorphic to $S_{p+1}$.

By Lemma 2.1, for each integer $0 \leq t \leq [(p + 1)/2]$ there is a $n$-equipartition $Q = \{Q^1_t, Q^2_t, \ldots, Q^{n+1}_t\}$ of $[1, n(p + 1)]$ such that $\sum Q^i_t = \Delta_t + di$, where $d = p + 1 - 2t$ and $\Delta_t = \sum Q^i_t - d$.

Hence we have a $n$-equipartition $Y^{t_1} = \{Y^{t_1}_1, Y^{t_1}_2, \ldots, Y^{t_1}_{n}\}$ of the set $[2, n(p + 1) + 1]$ such that
\[
\sum Y^{t_1}_i = n + \Delta_{t_1} + (p + 1 - 2t_1)i \quad \text{for } 1 \leq i \leq n,
\]
where $0 \leq t_1 \leq [(p + 1)/2]$.

Moreover, we have a $n$-equipartition $Z^{t_2} = \{Z^{t_2}_1, Z^{t_2}_2, \ldots, Z^{t_2}_{n}\}$ of the set $[n(p + 1) + 2, 2n(p + 1) + 1]$ such that
\[
\sum Z^{t_2}_i = n(n(p + 1) + 1) + \Delta_{t_2} + (p + 1 - 2t_2)i \quad \text{for } 1 \leq i \leq n,
\]
where $0 \leq t_2 \leq [(p + 1)/2]$. 

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**Figure 1.** A super \((a, 6)\)-star-antimagic labeling of the banana tree $B_{10}^5$ and a super \((a, 3)\)-star-antimagic labeling of the banana tree $B_{10}^6$. 

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We define a total labeling \( f : V(S_{p+1}^i) \cup E(S_{p+1}^i) \to [1, 2n(p+1) + 1] \) such that

\[
\begin{align*}
    f(v) &= 1, \\
    f(V(S_{p+1}^i) \setminus \{v\}) &= Y_{i1} \\
    f(E(S_{p+1}^i)) &= Z_{i2}
\end{align*}
\]

for \( 1 \leq i \leq n \),

Then for \( 1 \leq i \leq n \) we get

\[
\begin{align*}
    wt_f(S_{p+1}^i) &= \sum f(V(S_{p+1}^i)) + \sum f(E(S_{p+1}^i)) \\
    &= f(v) + \sum f(V(S_{p+1}^i) \setminus \{v\}) + \sum f(E(S_{p+1}^i)) \\
    &= 1 + \sum Y_{i1} + \sum Z_{i2} = 1 + (n + \Delta t_1 + (p + 1 - 2t_1)i) \\
    &\quad + (n(n(p + 1) + 1) + \Delta t_2 + (p + 1 - 2t_2)i) \\
    &= a + (2p + 2 - 2t_1 - 2t_2)i,
\end{align*}
\]

where \( a = n^2(p + 1) + 2n + 1 + \Delta t_1 + \Delta t_2 \).

Thus, the weights of stars \( S_{p+1}^i \) for arithmetic sequence with difference \( d = (2p + 2 - 2t_1 - 2t_2) \). As \( 0 \leq t_1 \leq [(p + 1)/2] \) and \( 0 \leq t_2 \leq [(p + 1)/2] \) we get that for \( p \) odd

\[
d \in \{0, 2, \ldots, 2(p + 1)\}
\]

and for \( p \) even

\[
d \in \{2, 4, \ldots, 2(p + 1)\}.
\]

Moreover, for \( p \) even we define a total labeling \( f_0 : V(S_{p+1}^i) \cup E(S_{p+1}^i) \to [1, 2n(p+1) + 1] \) such that

\[
\begin{align*}
    f_0(v) &= 1, \\
    f_0(V(S_{p+1}^i) \setminus \{v\}) &= Y_{i0} \\
    f_0(E(S_{p+1}^i)) &= Z_{n+1-i}
\end{align*}
\]

for \( 1 \leq i \leq n \)

For \( 1 \leq i \leq n \) we get

\[
\begin{align*}
    wt_{f_0}(S_{p+1}^i) &= \sum f_0(V(S_{p+1}^i)) + \sum f_0(E(S_{p+1}^i)) \\
    &= f_0(v) + \sum f_0(V(S_{p+1}^i) \setminus \{v\}) + \sum f_0(E(S_{p+1}^i)) \\
    &= 1 + \sum Y_{i0} + \sum Z_{n+1-i} = 1 + (n + \Delta_0 + 2(p + 1)i) \\
    &\quad + (n(n(p + 1) + 1) + \Delta_0 + 2(p + 1)(n + 1 - i)) \\
    &= n^2(p + 1) + 2n + 1 + 2(p + 1)(n + 1).
\end{align*}
\]

Thus also for \( p \) even we have a \( S_{p+1} \)-supermagic labeling of \( S_{p}^i \).

Let \( n \) and \( p \) be odd, \( n \geq 3 \). Then according to Lemma 2.3 there exists an \( n \)-equipartition \( \mathbb{P} = \{P_1, P_2, \ldots, P_n\} \) of \( X = [1, n(p + 1)] \) such that \( \sum P_i = p((p+1)n + n + 1)/2 + i \) for \( 1 \leq i \leq n \).

Hence we have an \( n \)-equipartition \( \mathbb{T} = \{T_1, T_2, \ldots, T_n\} \) of the set \([2, n(p + 1) + 1]\) such that

\[
\sum T_i = n + \frac{p((p+1)n + n + 1)}{2} + i \quad \text{for} \quad 1 \leq i \leq n.
\]

We define a total labeling \( g : V(S_{p+1}^i) \cup E(S_{p+1}^i) \to [1, 2n(p+1) + 1] \) such that

\[
\begin{align*}
    g(v) &= 1, \\
    g(V(S_{p+1}^i) \setminus \{v\}) &= Y_{i1} \\
    g(E(S_{p+1}^i)) &= T_i
\end{align*}
\]

for \( 1 \leq i \leq n \).
The star weights of \( S_{p+1}^i \), \( 1 \leq i \leq n \), are

\[
wt_g(S_{p+1}^i) = \sum g(V(S_{p+1}^i)) + \sum g(E(S_{p+1}^i)) \\
= g(v) + \sum g(V(S_{p+1}^i) - \{v\}) + \sum g(E(S_{p+1}^i)) \\
= 1 + \sum T_i + (n + \Delta_i + (p + 1 - 2t_1)i) \\
+ \left( n + \frac{p((p+1)n+n+1)}{2} + i \right) \\
= \left( 2n + \frac{p((p+1)n+n+1)}{2} + \Delta t_1 + 1 \right) + (p + 2 - 2t_1)i.
\]

As \( 0 \leq t_1 \leq (p + 1)/2 \) we get that

\[ p + 2 - 2t_1 \in \{1, 3, \ldots, p + 2\}. \]

This concludes the proof. \( \square \)

**Theorem 3.3.** The graph \( St_n^{n-1} \) admits a super \((a, d)\)-\(S_n\)-antimagic labeling for \( d \in \{0, 2(n-2)\} \).

**Proof.** The graph \( St_n^{n-1} \) has \((n^2 + 1)\) vertices and \( n^2 \) edges and contains \( n + 1 \) subgraphs isomorphic to \( S_n \).

We define a total labeling \( h, f : V(St_n^{n-1}) \cup E(St_n^{n-1}) \to \{1, 2, \ldots, 2n^2 + 1\} \) such that

\[
h(v) = 1, \quad \{h(u) : u \in V(St_n^{n-1}) - \{v\}\} = \{2, 3, \ldots, n^2 + 1\}, \]

\[
h(vv_i) = 2n^2 + 3 - h(v_i) \quad \text{for} \ 1 \leq i \leq n, \]

\[
h(v_i v_i') = 2n^2 + 3 - h(v'_i) \quad \text{for} \ 1 \leq i \leq n, \ 1 \leq j \leq n - 1.
\]

For the weights of the stars \( S_n^i \), \( 1 \leq i \leq n \) we get

\[
wt_h(S_n^i) = h(v_i) + h(v) + \sum_{j=1}^{n-1} h(v^j_i) + \sum_{j=1}^{n-1} h(v_i v^j_i) + h(vv_i) \\
= h(v_i) + 1 + \sum_{j=1}^{n-1} h(v^j_i) + \sum_{j=1}^{n-1} \left( 2n^2 + 3 - h(v^j_i) \right) \\
+ (2n^2 + 3 - h(v_i)) = n(2n^2 + 3) + 1. \quad (3.5)
\]

Moreover,

\[
wt_h(S_n^{n+1}) = h(v) + \sum_{i=1}^{n} h(v_i) + \sum_{i=1}^{n} h(vv_i) \\
= 1 + \sum_{i=1}^{n} h(v_i) + \sum_{i=1}^{n} \left( 2n^2 + 3 - h(v_i) \right) = n(2n^2 + 3) + 1. \quad (3.6)
\]

Using (3.5) and (3.6) we proved that \( h \) is a \( S_n \)-supermagic labeling of \( St_n^{n-1} \).

We distinguish two cases to obtain the difference \( 2(n - 2) \).

**Case i:** \( n \) is odd.

We define a total labeling \( f \) of \( St_n^{n-1} \) as follows:

\[
f(v) = 1, \quad f(v_i) = i + 1 \quad \text{for} \ 1 \leq i \leq n, \]

\[
f(v^j_i) = \begin{cases} 
  nj + 1 + i & \text{for} \ 1 \leq i \leq n, \ 1 \leq j \leq n - 2, \\
  n^2 + 2 - i & \text{for} \ 1 \leq i \leq n, \ j = n - 1, 
\end{cases}
\]
We define a total labeling $nS$ from $(n \leq i \leq n,\frac{n-1}{2})$, for $i \neq \frac{n+3}{2}, 2 \leq j \leq n-1$.

$$f(v_i v_j) = \begin{cases} n(n-1) + nj + 1 + i & \text{for } 1 \leq i \leq n, \\ 2n^2 - \frac{n-1}{2} & \text{for } i = \frac{n+3}{2}, j = 1, \end{cases}$$

$$f(vv_i) = \begin{cases} 2n^2 + 2i & \text{for } 1 \leq i \leq n, \ i \neq \frac{n+3}{2}, \\ n^2 + \frac{n-1}{2} + 3 & \text{for } i = \frac{n+3}{2}. \end{cases}$$

We find the $S_n$-weights of the stars in the covering.

$$wt_f(S_n^{n+1}) = f(v) + \sum_{i=1}^{n} f(v_i) + \sum_{i=1}^{n} f(vv_i)$$

$$= 1 + \sum_{i=1}^{n} [i + 1] + \sum_{i=1}^{n} \left[2n^2 + 2 - i\right] + n^2 + \frac{n-1}{2} + 3$$

$$= 2n^3 - n^2 + 4n + 3.$$ (3.7)

For $1 \leq i \leq n, \ i \neq (n + 3)/2 + 2$ is

$$wt_f(S_n^i) = f(v_i) + f(v) + \sum_{j=1}^{n-1} f(v_i v_j^i) + \sum_{j=1}^{n-1} f(v_i v_j) + f(vv_i)$$

$$= (i + 1) + 1 + \left(\sum_{j=1}^{n-2} [nj + 1 + i] + n^2 + 2 - i\right)$$

$$+ \left(\sum_{j=1}^{n} [n(n - 1) + nj + 1 + i]\right) + \left(2n^2 + 2 - i\right)$$

$$= 2n^3 - n^2 + 4n + 3 + 2(n - 2)i.$$ (3.8)

and

$$wt_f \left(S_{n+1}^{n+3}\right) = f(v_{n+3}) + f(v) + \sum_{j=1}^{n-1} f\left(v_{n+3}^j\right) + \sum_{j=1}^{n-1} f\left(v_{n+3} v_{n+3}^j\right)$$

$$+ f\left(vv_{n+3}\right) = \left(\frac{n-1}{2} + 2 + 1\right) + 1$$

$$+ \left(\sum_{j=1}^{n-2} [nj + 1 + \frac{n-1}{2} + 2]\right) + \left(2n^2 + 2 - \frac{n-1}{2} - 2\right)$$

$$+ \left(2\sum_{j=2}^{n-1} [n(n - 1) + nj + 1 + \frac{n-1}{2} + 2]\right) + \left(2n^2 - \frac{n-1}{2}\right)$$

$$+ \left(n^2 + \frac{n-1}{2} + 3\right) = 2n^3 - n^2 + 4n + 3 + 2(n - 2)\frac{n+3}{2}.$$ (3.9)

From (3.7), (3.8) and (3.9) we proved that the graph $St_n^{n-1}$ admits a super $(a, 2(n - 2))$-$S_n$-antimagic labeling when $n$ is odd.

**Case ii:** $n$ is even.

We define a total labeling $g$ of $St_n^{n-1}$ as follows:

$$g(v) = 1,$$

$$g(v_i) = i + 1 \text{ for } 1 \leq i \leq n,$$

$$g(v_i^j) = \begin{cases} nj + 1 + i & \text{for } 1 \leq i \leq n, \ 1 \leq j \leq n - 2, \\ n^2 + 2 - i & \text{for } 1 \leq i \leq n, \ j = n - 1, \end{cases}$$
\[ g(v_i v_i') = n(n - 1) + nj + 1 + i \quad \text{for} \ 1 \leq i \leq n, \ 1 \leq j \leq n - 2, \]
\[ g(v_n v_n^{n-1}) = 2n(n - 1) + 1 + \frac{n}{2}, \]
\[ g(v_n v_n^{n-1}) = (2n - 1)n + 1, \]
\[ g(v_i v_i^{n-1}) = n(n - 1) + nj + 1 + i \quad \text{for} \ 1 \leq i \leq n - 1, \ i \neq \frac{n}{2}, \]
\[ g(v_n v_n) = 2n^2 + 2 - \frac{n}{2}, \]
\[ g(v_n) = 2n^2 + 2 - n, \]
\[ g(v_n) = 2(n(n - 1) + 1 + i) \quad \text{for} \ 1 \leq i \leq n - 1, \ i \neq \frac{n}{2}. \]

We find the \( S_n \)-weights of the stars in the covering.
\[
wt_g(S_n^{n+1}) = g(v) + \sum_{i=1}^{n} g(v_i) + \sum_{i=1}^{n} g(v v_i) = 1 + \sum_{i=1}^{n} [i + 1] \\
+ \sum_{i=1}^{n} [2(n(n - 1) + 1 + i) + 2n^2 + 2 - \frac{n}{2} + 2n^2 + 2 - n \\
= 2n^3 - n^2 + 4n + 3. \quad (3.10)
\]

For \( 1 \leq i \leq n - 1, \ i \neq n/2 \) we have
\[
wt_g(S_n^n) = g(v_i) + g(v) + \sum_{j=1}^{n-1} g(v_j') + \sum_{j=1}^{n-1} g(v_i v_j') + g(v v_i) \\
= (i + 1) + 1 + \left( \sum_{j=1}^{n-2} [nj + 1 + i] + n^2 + 2 - i \right) \\
+ \left( \sum_{j=1}^{n-2} [n(n - 1) + nj + 1 + i] + 2n^2 + 2 - i \right) \\
+ (2n(n - 1) + 1 + i) = 2n^3 - n^2 + 4n + 3 + 2(n - 2)i \quad (3.11)
\]

and
\[
wt_g(S_n^n) = g(v_n') + g(v) + \sum_{j=1}^{n-1} g(v_j') + \sum_{j=1}^{n-1} g(v_n v_j') + g(v v_n) \\
= (\frac{n}{2} + 1) + 1 + \left( \sum_{j=1}^{n-2} [nj + 1 + \frac{n}{2}] + n^2 + 2 - \frac{n}{2} \right) \\
+ \left( \sum_{j=1}^{n-2} [n(n - 1) + nj + 1 + \frac{n}{2}] + 2n(n - 1) + 1 + \frac{n}{2} \right) \\
+ \left( 2n^2 + 2 - \frac{n}{2} \right) = 2n^3 - n^2 + 4n + 3 + n(n - 2). \quad (3.12)
\]

\[
wt_g(S_n^n) = g(v_n) + g(v) + \sum_{j=1}^{n-1} g(v_j) + \sum_{j=1}^{n-1} g(v_n v_j) + g(v v_n) \\
= (n + 1) + 1 + \left( \sum_{j=1}^{n-2} [nj + 1 + n] + n^2 + 2 - n + 2 \right) \\
+ \left( \sum_{j=1}^{n-2} [n(n - 1) + nj + 1 + n] + 2n(n - 1) + 1 + n \right)
\]
According to (3.10) - (3.13) we have that for \( n \) even the graph \( S_{n-1}^n \) is super \((a, 2(n-2))-S_n\)-antimagic.

Figure 2 illustrates a super \((248, 6)\)-\(S_5\)-antimagic labeling of graph \( S_{10}^4 \).

**Figure 2.** A super \((248, 6)\)-\(S_5\)-antimagic labeling of graph \( S_{10}^4 \).

**Acknowledgment.** This work was supported by the Slovak Science and Technology Assistance Agency under the contract No. APVV-15-0116 and by VEGA 1/0233/18.

**References**


