The Third Isomorphism Theorem on UP-Bialgebras

Daniel A. Romano
International Mathematical Virtual Institute, Banja Luka, Bosnia and Herzegovina

Abstract

The concept of UP-bialgebras was introduced and analyzed by Mosrijai and Iampan at the beginning of 2019. Theorem that we can look at as the First theorem on UP-biisomorphism between the UP-bialgebras is given in our forthcoming text [9]. In this article we construct a form of the third theorem on UP-biisomorphism between UP-bialgebras.

1. Introduction

The concept of UP-algebras developed by Iampan in [1]. Examining the substructures in this algebra are done for example in articles [2, 3]. This author took part in analyzing the properties of UP-algebras and their substructures, also [4]-[6]. Some forms of the isomorphism theorem between UP-algebras can be found in [2, 3, 5, 6].

The concept of bi-algebraic structures was studied by Vasantha Kandasamy in 2003 [7]. The concept of UP-bialgebras with the associated substructures and their mutual connections can be found in [8]. In the forthcoming article [9], this author offered one form the first theorem of the isomorphism between the UP-bialgebras.

In this article we expose a form of the second isomorphism theorem between UP-bialgebras.

2. Preliminaries

In this section, we will present the necessary previous concepts of UP-algebras, their substructures and UP-homomorphisms taken from texts [1, 2, 3, 8]. We will also expose their mutual relationships in the form of proclams necessary for our intention.

2.1. UP-algebras

In this subsection we will describe some elements of UP-algebras and their substructures necessary for our intentions in this text.

**Definition 2.1** ([1]). An algebra $L = (L, \cdot, 0)$ of type $(2, 0)$ is called a UP-algebra where $L$ is a nonempty set, $\cdot$ is a binary operation on $L$, and 0 is a fixed element of $L$ (i.e. a nullary operation) if it satisfies the following axioms:

- **(UP-1)** $(\forall x, y \in L)((y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) = 0)$,
- **(UP-2)** $(\forall x \in L)(0 \cdot x = x)$,
- **(UP-3)** $(\forall x \in L)(x \cdot 0 = 0)$, and
- **(UP-4)** $(\forall x, y \in L)((x \cdot y = 0 \land y \cdot x = 0) \implies x = y)$.

**Definition 2.2** ([1]). A nonempty subset $J$ of a UP-algebra $(L, \cdot, 0)$ is called

1. a UP-subalgebra of $L$ if $(\forall x, y \in J)(x \cdot y \in J)$.
2. a UP-ideal of $L$ if
   (i) $0 \in J$; and
   (ii) $(\forall x, y, z \in L)((x \cdot (y \cdot z) \in J \land y \in J) \implies x \cdot z \in J)$.
The set \( \{0\} \) is a trivial UP-subalgebra (trivial UP-ideal) of \( L \).

In the article [6], Theorem 3.3, it has been shown that the conditions (i) and (ii) in the preceding definition are equivalent to the following conditions:

(iii) \( \forall x, y \in L \)(\( x \cdot y \in J \land x \in J \implies y \in J \)),

(iv) \( \forall x, y \in L \)(\( y \in J \implies x \cdot y \in J \)).

Definition 2.3 ([1]). Let \( (L, \cdot, 0) \) and \( (M, \cdot, 0_m) \) be two UP-algebras. A mapping \( f : L \rightarrow M \) is called a UP-homomorphism if

\[
(f(x) \cdot y) = f(x) \cdot f(y).
\]

A UP-homomorphism \( f : L \rightarrow M \) is called

(3) a UP-epimorphism if \( f \) is surjective,

(4) a UP-monomorphism if \( f \) is injective, and

(5) a UP-isomorphism if \( f \) is bijective.

Let \( f \) be a mapping form UP-algebra \( L \) to UP-algebra \( M \), and let \( A \) and \( B \) be nonempty subsets of \( L \) and of \( M \), respectively. The set \( f(A) = \{ f(x) \mid x \in A \} \) is called the image of \( A \) under \( f \). In particular, \( f(L) \) which denoted by \( f(L) \) is called the image of \( f \). The dual set \( f^{-1}(B) = \{ x \in L \mid f(x) \in B \} \) is called the inverse image of \( B \). Especially, the set \( \text{Ker}(f) = f^{-1}(\{0_m\}) = \{ x \in L \mid f(x) = 0_m \} \) is called the kernel of \( f \).

A relation of congruence on UP-algebras is introduced in [1] by Definition 3.1 and Proposition 3.5 on this way: If \( J \) is a UP-ideal of a UP-algebra \( L \), then the relation \( \sim_J \) defined by

\[
(\forall x, y \in L)(x \sim_J y \iff (x \cdot y \in J \land y \cdot x \in J))
\]

is a UP-congruence on \( L \). Further on, any relation of congruence on UP-algebras has this form according to the claim (1) of Theorem 3.6 and the claim (1) of Theorem 3.7 in [1]. In particular, if \( f : L \rightarrow M \) is a UP-homomorphism between UP-algebras, then the relation \( \sim_J \) determined by \( \text{Ker}(f) \) is a UP-congruence in \( L \). The factor-set \( L/\sim_J = \{ [x]_J \mid x \in L \} \) is a UP-algebra according to the claim (4) of Theorem 3.7 in [1]. We also use the following notion \( L/J = \{ [x]_J \mid x \in L \} \) to denote this factor algebra.

2.2. UP-bialgebras

The concept of UP-bialgebras and some of their substructures were introduced and analyzed by Mosrijai and Iampan in the recently published work [8]. In this subsection, taking into account their determinations, we describe the concept of UP-bialgebras and some notions connected with them. So, in this subsection, we will repeat the concept of UP-bialgebras and the notions of UP-bisubalgebras and UP-biideals of UP-bialgebras, and will expose some results related to substructures of such algebras.

Definition 2.4 ([8], Definition 3.1). An algebra \( L = (L, \cdot, +, 0) \) of type \((2, 2, 0)\) is called a UP-bialgebra where \( L \) is a nonempty set, \( \cdot \) and \( + \) are two binary internal operations on \( L \), and \( 0 \) is a fixed element of \( L \) if there exist two distinct proper subsets \( L_1 \) and \( L_2 \) of \( L \) with respect to \( \cdot \) and \( + \), respectively, such that

\[
\begin{align*}
\text{(UPB-1)} & \quad L = L_1 \uplus L_2; \\
\text{(UPB-2)} & \quad (L_1, 0) \text{ is a UP-algebra, and} \\
\text{(UPB-3)} & \quad (L_2, +, 0) \text{ is a UP-algebra.}
\end{align*}
\]

We will denote the UP-bialgebra by \( L = L_1 \uplus L_2 \). In case of \( L_1 \cap L_2 = \{0\} \), we call \( L \) zero disjoint.

Definition 2.5 ([8], Definition 3.7). A nonempty subset \( J \) of a UP-bialgebra \( L = L_1 \uplus L_2 \) is called a UP-biideal (UP-bisubalgebra) of \( L \) if there exist subsets \( J_1 \subseteq L_1 \) and \( J_2 \subseteq L_2 \) with respect to \( \cdot \) and \( + \), respectively, such that

\[
\begin{align*}
\text{(6)} & \quad J_1 \neq J_2; \\
\text{(7)} & \quad (J_1, 0) \text{ is a UP-ideal (UP-subalgebra) of } (L_1, 0), \\
\text{(8)} & \quad (J_2, +, 0) \text{ is a UP-ideal (UP-subalgebra) of } (L_2, +, 0).
\end{align*}
\]

In case of \( J_1 \cap J_2 = \{0\} \), we call \( J \) zero disjoint.

The important relationship between these notions is the following:

Proposition 2.6 ([9]). If \( J \supseteq \{0\} \) is a UP-subalgebra (resp., UP-ideal) of UP-algebra \( L_1 \) (of UP-algebra \( L_2 \), respectively), such that \( \{0\} \neq J \), then on \( J \) can be seen as a zero disjoint UP-bisubalgebra (resp., UP-biideal) of UP-bialgebra \( L = L_1 \uplus L_2 \).

2.3. UP-bihomomorphisms

Let \( f : L \rightarrow M \) be a function from a set \( L \) to a set \( M \) and \( C \subseteq L \). Then the restriction of \( f \) to \( C \) is the function \( f|_C : C \rightarrow M \).

Definition 2.7 ([8], Definition 4.1). Let \( L = L_1 \uplus L_2 \) be a UP-bialgebra with two binary operations \( \cdot \) and \( + \), and let \( M = M_1 \uplus M_2 \) be a UP-bialgebra with two binary operations \( \cdot' \) and \( +' \). A mapping \( f \) from \( L = L_1 \uplus L_2 \) to \( M = M_1 \uplus M_2 \) is called a UP-bihomomorphism if it satisfies the following properties:

\[
\begin{align*}
\text{(9)} & \quad f|_{L_1} : L_1 \rightarrow M_1 \text{ is a UP-homomorphism, and} \\
\text{(10)} & \quad f|_{L_2} : L_2 \rightarrow M_2 \text{ is a UP-homomorphism.}
\end{align*}
\]

We say that these restrictions are natural restrictions. A UP-bihomomorphism \( f : L \rightarrow M \) is called

- a UP-biepimorphism if the natural restriction are UP-epimorphisms,
- a UP-bimonomorphism if the natural restriction are UP-monomorphisms, and
- a UP-biisomorphism if the natural restriction are UP-isomorphisms.
Proposition 2.8 ([8]). Let $f : L_1 \cong L_2 \longrightarrow M_1 \cong M_2$ be a UP-bihomomorphism. Then the following statements hold:

11. $f(0_L) = 0_M$, and
12. $\text{Ker}(f) = \{0_L\}$ if and only if $f$ is an injective mapping;
13. If $f$ is a UP-bisubalgebra of $L$, then the image $f(J)$ is a UP-bisubalgebra of $B$;
14. If $J = J_1 \cup J_2$ is a UP-biideal of $L$, and $J_1$ and $J_2$ are subsets of $L_1$ and of $L_2$, respectively, with $\text{Ker}(f) \subseteq J_1 \cap J_2$, then the image $f(J)$ is a UP-biideal of $M$;
15. If $D$ is a UP-bisubalgebra of $M$, then the inverse image $f^{-1}(D)$ is a UP-bisubalgebra of $L$; and
16. If $D$ is a UP-biideal of $M$, then the inverse image $f^{-1}(D)$ is a UP-biideal of $L$.

3. The main results

In our forthcoming article [9], we formulated and proved a form of the first isomorphism theorem between UP-bialgebras. To this direction, we used the following lemma.

Lemma 3.1 ([9]). Let $L = L_1 \cong L_2$ and $M = M_1 \cong M_2$ be two UP-bialgebras and let $f : L \longrightarrow M$ be a UP-bihomomorphism. Then the set $\text{Ker}(f_{L_1}) \cup \text{Ker}(f_{L_2})$ is a UP-biideal of $L$ and $\text{Ker}(f) = \text{Ker}(f_{L_1}) \cap \text{Ker}(f_{L_2})$ holds.

Let $L = L_1 \cong L_2$ be a UP-bialgebra with two binary operations $\cdot$ and $*$, and let $M = M_1 \cong M_2$ be a UP-bialgebra with two binary operations $\cdot'$ and $*'$ and let $f : L \longrightarrow M$ be a UP-homomorphism. Let $\sim_1$ be the congruence on $L_1$ generated by the UP-ideal $\text{Ker}(f_{L_1})$ and $\sim_2$ be the congruence on $L_2$ generated by the UP-ideal $\text{Ker}(f_{L_2})$.

$$\forall x, y \in L_1 \left( x \sim_1 y \iff (x \cdot y \in \text{Ker}(f_{L_1}) \land y \cdot x \in \text{Ker}(f_{L_1})) \right)$$

and

$$\forall x, y \in L_2 \left( x \sim_2 y \iff (x \cdot y \in \text{Ker}(f_{L_2}) \land y \cdot x \in \text{Ker}(f_{L_2})) \right).$$

Then we can construct the factor-UP-algebra $L_1/\sim_1$ and the factor-UP-algebra $L_2/\sim_2$. So, $L_1/\sim_1 \cong L_2/\sim_2$ is a UP-bialgebra with two binary operation $\sim_1 \odot L_1$ and $\sim_2 \odot L_2$ defined by

$$\forall [x]_{\sim_1}, [y]_{\sim_1} \in L_1/\sim_1 \left( [x]_{\sim_1} \odot [y]_{\sim_1} = [x \cdot y]_{\sim_1} \right)$$

and

$$\forall [x]_{\sim_2}, [y]_{\sim_2} \in L_2/\sim_2 \left( [x]_{\sim_2} \odot [y]_{\sim_2} = [x \cdot y]_{\sim_2} \right).$$

Previous analysis enables us to introduce the following determination: Let $L = L_1 \cong L_2$ be a UP-bialgebra. For a pair $\sim_1, \sim_2$ of congruences on $L_1$ and $\sim_2$ on $L_2$ we write $L_1 \cong L_2/\sim_1 \sim_2$ instead of $L_1/\sim_1 \cong L_2/\sim_2$. If $f_1 : L_1 \longrightarrow M_1/\sim_1$ and $f_2 : L_2 \longrightarrow M_2/\sim_2$ are canonical UP-epimorphisms, then there is a unique canonical UP-epimorphism $f : L_1 \cong L_2/\sim_1 \sim_2$ such that $f_1 = f_2$. Particularly, there is a unique UP-epimorphism $f : L_1 \cong L_2 \longrightarrow (L_1 \cong L_2)/(\text{Ker}(f_{L_1}), \text{Ker}(f_{L_2}))$. The first theorem of isomorphism between UP-bialgebras has the form in which for simplicity we write $A(Ker(f_{L_1}), Ker(f_{L_2}))$.

Theorem 3.2 ([9]). Let $f : L \longrightarrow M$ be a UP-bihomomorphism. Then there exists the unique UP-bihomomorphism $g : L/\text{Ker}(f) \longrightarrow M$ such that $f = g \circ \pi$. In addition, for the UPB-subalgebra $f(L)$ of $M$ holds $L/\text{Ker}(f) \cong f(L)$.

Let us analyze now the following situation:

Let $J$ and $K$ be UP-biideals of a UP-bialgebra $L$ such that $J \subseteq K$. Then there exist UP-ideals $J_1$ and $K_1$ of the UP-algebra $L_1$ and there exist UP-ideals $J_2$ and $K_2$ of the UP-algebra $L_2$ such that $J_1 \neq J_2$ and $J = J_1 \cup J_2$, and $K_1 \neq K_2$ and $K = K_1 \cup K_2$, by Definition 2.5. If $J_1 \subseteq J_2$ and $K_2 \subseteq K_1$, then $J_1/J_2$ is a UP-ideal of UP-algebra $L_1/J_1$ and $K_2/J_2$ is a UP-ideal of UP-algebra $L_2/J_2$. From here follows $L_1/K_1 \cong (L_1/J_1)/(K_1/J_1)$ according to Theorem 3.10 in [6]. We also have $L_2/K_2 \cong (L_2/J_2)/(K_2/J_2)$ according to same theorem. So, the set $K_1/J_1 \oplus K_2/J_2$ is a UP-biideal of UP-algebra $L_1/J_1 \oplus L_2/J_2$. Thus, the mapping $g_1 : L_1/J_1 \longrightarrow L_1/K_1$ has $\text{Ker}(g_1) = K_1/J_1$. Analogously, the mapping $g_2 : L_2/J_2 \longrightarrow L_2/K_2$ has $\text{Ker}(g_2) = K_2/J_2$ as core. Therefore, the homomorphism $g : L/(J_1, J_2) \longrightarrow L/(K_1, K_2)$, determined by $g_{|L_1/J_1} = g_1$ and $g_{|L_2/J_2} = g_2$ has the core exactly $K_1/J_1 \oplus K_2/J_2$.

The previous analysis is a motivation for the following theorem we can be seen as the Third isomorphism theorem between UP-bialgebras.

Theorem 3.3. Let $L = L_1 \cong L_2$ be a UP-bialgebra and let $J = J_1 \cup J_2$ and $K = K_1 \cup K_2$ be UP-biideals such that $J_1 \subseteq K_1$ and $J_2 \subseteq K_2$. Then

$$L/(K_1, K_2) \cong (L/(J_1, J_2))/(K_1/J_1, K_2/J_2)$$

holds.

Final Observation

The concept of UP-algebras introduced and first results on them given by Lampan 2017 [1]. This author took part in analyzing the properties of UP-algebras and their substructures, also [4, 5, 6]. Algebraic bi-structure was analyzed by Vasantha Kandasamy in 2003 [7]. The concept of UP-bialgebras introduced and the first results were given by Mosrijai and lampan at the beginning of 2019 [8]. Using by the concept of UP-bihomomorphisms, introduced in [8], in this article we formulated and proved the theorem (Theorem 3.3), which can be viewed as the Third isomorphism theorem between UP-bialgebras.

Of course, there remains an open possibility of formulating and trying to prove other forms of these two isomorphism theorems between the UP-bialgebra.
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References