International Electronic Journal of Geometry
Volume 1 No. 2 pp. 15-32 (2008) ©IEJG

AFFINE PARTS OF ABELIAN SURFACES AS COMPLETE INTERSECTION OF THREE QUARTICS

A. LESFARI

(Communicated by E. Ballico)

Abstract. We consider an integrable system in five unknowns having three quartics invariants. We show that the complex affine variety defined by putting these invariants equal to generic constants, completes into an abelian surface; the jacobian of a genus two hyperelliptic curve. This system is algebraic completely integrable and it can be integrated in genus two hyperelliptic functions.

1. Introduction

The problem of finding and integrating hamiltonian systems, has attracted a considerable amount of attention in recent years. Beside the fact that many integrable hamiltonian systems have been on the subject of powerful and beautiful theories of mathematics, another motivation for its study is : the concepts of integrability have been applied to an increasing number of physical systems, biological phenomena, population dynamics, chemical rate equations, to mention only a few. However, it seems still hopeless to describe or even to recognize with any facility, those hamiltonian systems which are integrable, though they are quite exceptional. The resolution of the well known Korteweg-de-Vries equation has generated an enormous number of new ideas in the area of hamiltonian completely integrable systems. It has led to unexpected connections between mechanics, spectral theory, Lie algebra theory, algebraic geometry and even differential geometry. All these connections have generated renewed interest in the questions around complete integrability of finite and infinite dimensional systems, ordinary and partial differential equations. However given a hamiltonian system, it remains often hard to fit it into any of those general frameworks. But luckily, most of the problems possess the much richer structure of the so called algebraic complete integrability (concept introduced et systematized by Adler and van Moerbeke).
In this paper, we shall be concerned with finite dimensional algebraic completely integrable systems. A dynamical system is algebraic completely integrable in the sense of Adler and van Moerbeke [2] if it can be linearized on a complex algebraic torus $\mathbb{C}^n/lattice$ (=abelian variety). The invariants (often called first integrals or constants) of the motion are polynomials and the phase space coordinates (or some

2000 *Mathematics Subject Classification.* 70H06, 14H40, 14H70, 37J35, 14K20.
Key words and phrases. Abelian surfaces, Curves, Integrable systems, Kummer surfaces.

algebraic functions of these) restricted to a complex invariant variety defined by putting these invariants equals to generic constants, are meromorphic functions on an abelian variety. Moreover, in the coordinates of this abelian variety, the flows (run with complex time) generated by the constants of the motion are straight lines. Some results concerning geodesic flow on $SO(4)$ [2,10], Kowalewski's top [15], Hénon-Heiles system [17],...was obtained. However, besides the fact that many hamiltonian completely integrable systems posses this structure, another motivation for its study which sounds more modern is : algebraic completely integrable systems come up systematically whenever you study the isospectral deformation of some linear operator containing a rational indeterminate. Therefore there are hidden symmetries which have a group theoretical foundation. The concept of algebraic complete integrability is quite effective in small dimensions and has the advantage to lead to global results, unlike the existing criteria for real analytic integrability, which, at this stage are perturbation results. In fact, the overwhelming majority of dynamical systems, hamiltonian or not, are non-integrable and possess regimes of chaotic behavior in phase space.

In the present paper, we discuss another interesting interaction between complex geometry and dynamical systems. We present an integrable system (2.1) in five unknowns having three quartics invariants. It generalizes some well known integrable systems related to the Yang-Mills system for a field with gauge group $SU(2)$ (see remark 2.1 and remark 2.2). This system is algebraic completely integrable in $\mathbb{C}^5$, it can be integrated in genus 2 hyperelliptic functions. We show that the complex affine variety B(2.3) defined by putting these invariants equal to generic constants, is a double cover of a Kummer surface and the system (2.1) can be integrated in genus 2 hyperelliptic functions. We make a careful study of the algebraic geometric aspect of the affine variety B(2.3) of the system (2.1). We find via the Kowalewski-Painlevé analysis the principal balances of the hamiltonian field defined by the hamiltonian. To be more precise, we show that the system (2.1) possesses Laurent series solutions in t, which depend on 4 free parameters : α, β, γ and θ. These meromorphic solutions restricted to the surface $B(2.3)$ are parameterized by two isomorphic smooth hyperelliptic curves $\mathcal{H}_{\varepsilon=\pm i}(3.2)$ of genus 2 that intersect in only one point at which they are tangent to each other. The affine variety $B(2.3)$ is embedded into $\mathbb{P}^{15}$ and completes into an abelian variety $\widetilde{B}$ (the jacobian of a genus 2 curve) by adjoining a divisor $\mathcal{D} = \mathcal{H}_i + \mathcal{H}_{-i}$. The latter has geometric genus 5 and $\mathcal{S} = 2\mathcal{D}$ (very ample) has genus 17. The flow (2.1) evolves on $\widetilde{B}$ and is tangent to each hyperelliptic curve $\mathcal{H}_\varepsilon$ at the point of tangency between them. Consequently, the system (2.1) is algebraic integrable.

We mention here (see remark 2.1) a few open problems. What is the relationship between the 5-dimensional system (2.1) with $F_3 \neq 0$ and the system corresponding to the hamiltonian (2.9)? It also remains an open question to find a physical model that realizes the system (2.1).

Acknowledgements : I wish to express my thanks to the referee for stimulating questions, comments and suggestions which have helped to clarify the overall picture.

2. A five-dimensional integrable system

Let us consider the following system of five differential equations in the unknowns $z_1, \dots, z_5$:

$$
\begin{aligned}
\dot{z}_1 &= 2z_4, \\
\dot{z}_2 &= z_3, \\
\dot{z}_3 &= -4az_2 - 6z_1z_2 - 16z_2^3, \\
\dot{z}_4 &= -az_1 - z_1^2 - 8z_1z_2^2 + z_5, \\
\dot{z}_5 &= -8z_2^2z_4 - 2az_4 - 2z_1z_4 + 4z_1z_2z_3,
\end{aligned} \tag{2.1}
$$

where the dot denotes differentiation with respect to the time t.

Theorem 2.1. *The system (2.1) possesses three quartic invariants and is completely integrable in the sense of Liouville. The complex affine variety B(2.3) defined by putting these invariants equal to generic constants, is a double cover of a Kummer surface (2.4) and the system (2.1) can be integrated in genus 2 hyperelliptic functions.*

Proof. The following three quartics are constants of motion for this system

$$
\begin{aligned}
F_1 &= \frac{1}{2}z_5 + 2z_1z_2^2 + \frac{1}{2}z_3^2 + \frac{1}{2}az_1 + 2az_2^2 + \frac{1}{4}z_1^2 + 4z_2^4, \\
F_2 &= az_1z_2 + z_1^2z_2 + 4z_1z_2^3 - z_2z_5 + z_3z_4, \\
F_3 &= z_1z_5 - 2z_1^2z_2^2 - z_4^2.
\end{aligned} \tag{2.2}
$$

The system (2.1) can be written as a hamiltonian vector field

$$\dot{z} = J\frac{\partial H}{\partial z}, \quad z = (z_1, z_2, z_3, z_4, z_5)^\top,$$

where $H = F_1$. The hamiltonian structure is defined by the Poisson bracket

$$\{F, H\} = \left\langle \frac{\partial F}{\partial z}, J\frac{\partial H}{\partial z} \right\rangle = \sum_{k,l=1}^{5} J_{kl}\frac{\partial F}{\partial z_k}\frac{\partial H}{\partial z_l},$$

where

$$\frac{\partial H}{\partial z} = \left(\frac{\partial H}{\partial z_1}, \frac{\partial H}{\partial z_2}, \frac{\partial H}{\partial z_3}, \frac{\partial H}{\partial z_4}, \frac{\partial H}{\partial z_5}\right)^\top,$$

and

$$J = \begin{bmatrix} 0 & 0 & 0 & 2z_1 & 4z_4 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & -4z_1z_2 \\ -2z_1 & 0 & 0 & 0 & 2z_5 - 8z_1z_2^2 \\ -4z_4 & 0 & 4z_1z_2 & -2z_5 + 8z_1z_2^2 & 0 \end{bmatrix},$$

is a skew-symmetric matrix for which the corresponding Poisson bracket satisfies the Jacobi identities. The second flow commuting with the first is regulated by the equations

$$\dot{z} = J\frac{\partial F_2}{\partial z}, \quad z = (z_1, z_2, z_3, z_4, z_5)^\top,$$

and is written explicitly as

$$\begin{aligned}
\dot{z}_1 &= 2z_1z_3 - 4z_2z_4,\\
\dot{z}_2 &= z_4,\\
\dot{z}_3 &= z_5 - 8z_1z_2^2 - az_1 - z_1^2,\\
\dot{z}_4 &= -2az_1z_2 - 4z_1^2z_2 - 2z_2z_5,\\
\dot{z}_5 &= -4az_2z_4 - 4z_1z_2z_4 - 16z_2^3z_4 - 2z_3z_5 + 8z_1z_2^2z_3.
\end{aligned}$$

These vector fields are in involution, i.e.,

$$\{F_1, F_2\} = \left\langle \frac{\partial F_1}{\partial z}, J\frac{\partial F_2}{\partial z}\right\rangle = 0,$$

and the remaining one is casimir, i.e.,

$$J\frac{\partial F_3}{\partial z} = 0.$$

Let $z \in \mathbb{C}^5$, $t \in \mathbb{C}$ and $\Delta \subset \mathbb{C}^5$ a non-empty Zariski open set. By the functional independence of the integrals F_1, F_2, F_3, the map

$$\varphi : (F_1, F_2, F_3) : \mathbb{C}^5 \longrightarrow \mathbb{C}^3,$$

is submersive, i.e., $dF_1(z), dF_2(z), dF_3(z)$ are linearly independent on Δ. Let

$$\begin{aligned}
\Omega &= \varphi\left(\mathbb{C}^5\backslash\Delta\right),\\
&= \left\{c \equiv (c_1, c_2, c_3) \in \mathbb{C}^3 : \exists z \in \varphi^{-1}(c)\text{with}\right.\\
&\qquad \left. dF_1(z), dF_2(z), dF_3(z)\text{linearly dependent}\right\},
\end{aligned}$$

be the set of critical values of φ. We denote by $\overline{\Omega}$ the Zariski closure of Ω in $\mathbb{C}^3$. The set $\left\{z \in \mathbb{C}^5 : \varphi(z) \in \mathbb{C}^3\backslash\overline{\Omega}\right\}$ is a non-empty Zariski open set in $\mathbb{C}^5$. Hence this set is everywhere dense in $\mathbb{C}^5$ for the usual topology. Indeed, since a polynomial map is continuous for the Zariski topology,

$$\left\{z \in \mathbb{C}^5 : \varphi(z) \in \mathbb{C}^3\backslash\overline{\Omega}\right\} = \varphi^{-1}\left(\mathbb{C}^3\backslash\overline{\Omega}\right),$$

is certainly a Zariski open set in $\mathbb{C}^5$. Suppose it empty, i.e., $\varphi\left(\mathbb{C}^5\right) \subset \overline{\Omega}$. By the functional independence of the integrals, the map φ is submersive on a non-empty Zariski open set $\Delta \subset \mathbb{C}^5$ and thus $\varphi(\Delta)$ is open in $\mathbb{C}^3$. Now, by Sard's lemma for varieties, $\mathbb{C}^3\backslash\overline{\Omega}$ is a non-empty Zariski open set (hence everywhere dense for the usual topology) in $\mathbb{C}^3$. So $\varphi(\Delta) \cap \left(\mathbb{C}^3\backslash\overline{\Omega}\right) \neq \emptyset$, a contradiction. Let B be the complex affine variety defined by

$$\begin{aligned}
B &= \varphi^{-1}(c),\\
&= \bigcap_{k=1}^{2}\{z : F_k(z) = c_k\} \subset \mathbb{C}^5.
\end{aligned} \tag{2.3}$$

For every $c \equiv (c_1, c_2, c_3) \in \mathbb{C}^3\backslash\overline{\Omega}$, the fibre B is a smooth[1] affine surface. Note that

$$\sigma : (z_1, z_2, z_3, z_4, z_5) \longmapsto (z_1, z_2, -z_3, -z_4, z_5),$$

is an involution on B. The quotient B/σ is a Kummer surface defined by

$$p\left(z_1, z_2\right) z_5^2 + q\left(z_1, z_2\right) z_5 + r\left(z_1, z_2\right) = 0, \tag{2.4}$$

[1]The smoothness of B play an important role in the construction of a compactification of B into an abelian surface (theorem 4.2).

where

$$\begin{aligned}
p(z_1,z_2) &= z_2^2+z_1,\\
q(z_1,z_2) &= \frac{1}{2}z_1^3+2az_1z_2^2+az_1^2-2c_1z_1+2c_2z_2-c_3,\\
r(z_1,z_2) &= -8c_3z_2^4+\left(a^2+4c_1\right)z_1^2z_2^2-8c_2z_1z_2^3-2c_2z_1^2z_2-4c_3z_1z_2^2\\
&\quad -\frac{1}{2}c_3z_1^2-4ac_3z_2^2-2ac_2z_1z_2-ac_3z_1+c_2^2+2c_1c_3.
\end{aligned}$$

Using $F_1=c_1$ (2.2), we have

$$z_5=2c_1-4z_1z_2^2-z_3^2-az_1-4az_2^2-\frac{1}{2}z_1^2-8z_2^4,$$

and substituting this into $F_2=c_2$, $F_3=c_3$, (2.2) yields

$$\begin{aligned}
&2az_1z_2+\frac{3}{2}z_1^2z_2+8z_1z_2^3-2c_1z_2+z_2z_3^2+4az_2^3+8z_2^5+z_3z_4=c_2,\\
&2c_1z_1-6z_1^2z_2^2-z_1z_3^2-az_1^2-4az_1z_2^2-\frac{1}{2}z_1^3-8z_1z_2^4-z_4^2=c_3. \qquad (2.5)
\end{aligned}$$

We introduce two coordinates s_1,s_2 as follows

$$\begin{aligned}
z_1 &= -4s_1s_2,\\
z_2 &= s_1+s_2,\\
z_3 &= \dot{s}_1+\dot{s}_2,\\
z_4 &= -2\left(\dot{s}_1s_2+s_1\dot{s}_2\right).
\end{aligned}$$

Upon substituting this parametrization, (2.5) turns into

$$\begin{aligned}
&(s_1-s_2)\left((\dot{s}_1)^2-(\dot{s}_2)^2\right)+8(s_1+s_2)\left(s_1^4+s_2^4+s_1^2s_2^2\right)\\
&+4a(s_1+s_2)\left(s_1^2+s_2^2\right)-2c_1(s_1+s_2)-c_2=0,\\
&(s_1-s_2)\left(s_2(\dot{s}_1)^2-s_1(\dot{s}_2)^2\right)+32s_1s_2\left(s_1^4+s_2^4+s_1^2s_2^2\right)\\
&+32s_1^2s_2^2\left(s_1^2+s_2^2\right)+16as_1s_2\left(s_1^2+s_2^2\right)+16as_1^2s_2^2-8c_1s_1s_2-c_3=0.
\end{aligned}$$

These equations are solved linearly for $\dot{s}_1^2$ and $\dot{s}_2^2$ as

$$\begin{aligned}
(\dot{s}_1)^2 &= \frac{-32s_1^6-16as_1^4+8c_1s_1^2+4c_2s_1-c_3}{4\left(s_2-s_1\right)^2}, \qquad (2.6)\\
(\dot{s}_2)^2 &= \frac{-32s_2^6-16as_2^4+8c_1s_2^2+4c_2s_2-c_3}{4\left(s_2-s_1\right)^2},
\end{aligned}$$

and can be integrated by means of the Abel transformation $\mathcal{H}\longrightarrow Jac(\mathcal{H})$, where the hyperelliptic curve $\mathcal{H}$ of genus 2 is given by an equation

$$w^2=-32s^6-16as^4+8c_1s^2+4c_2s-c_3.$$

Consequently, the equations (2.1) are integrated in terms of genus 2 hyperelliptic functions. This establishes the theorem.

Remark 2.1. Consider the case $F_3=0$, and the following change of variables

$$z_1=q_1^2,\ z_2=q_2,\ z_3=p_2,\ z_4=q_1p_1,\ z_5=2q_1^2q_2^2+p_1^2.$$

Substituting this into the constants of motion F_1, F_2, F_3 (2.2) leads obviously to the relations

$$\begin{aligned} H_1 &= \frac{1}{2}\left(p_1^2+p_2^2\right)+\frac{a}{2}\left(q_1^2+4q_2^2\right)+\frac{1}{4}q_1^4+4q_2^4+3q_1^2q_2^2, \\ H_2 &= aq_1^2q_2+q_1^4q_2+2q_1^2q_2^3-q_2p_1^2+q_1p_1p_2, \end{aligned} \tag{2.7}$$

whereas the last constant leads to an identity. Using the differential equations (2.1) combined with the transformation above leads to the system of differential equations

$$\begin{aligned} \dot{q}_1 &= p_1, \\ \dot{q}_2 &= p_2, \\ \dot{p}_1 &= -(a+q_1^2+6q_2^2)q_1, \\ \dot{p}_2 &= -2(2a+3q_1^2+8q_2^2)q_2, \end{aligned} \tag{2.8}$$

The last equation (2.1) for z_5 leads to an identity. The hamiltonian (2.7) is part of the well known family

$$H=\frac{1}{2}\left(p_1^2+p_2^2\right)+\frac{1}{2}\left(\Omega_1q_1^2+\Omega_2q_2^2\right)+Aq_1^4+Bq_1^2q_2^2+Cq_2^4+\frac{1}{2}\left(\frac{\alpha}{q_1^2}+\frac{\beta}{q_2^2}\right), \tag{2.9}$$

which describe the motion of particules interacting with a quartic potential $Aq_1^4+Bq_1^2q_2^2+Cq_2^4$ and perturbed by an inverse squared potential. There are four nontrivial known cases of complete integrability for the nonperturbed quartic potential and were all studied in detail from different points of view : (i) A:B:C=1:2:1, (ii) A:B:C=1:12:16, (iii) A:B:C=1:6:1, (iv) A:B:C=1:6:8. Cases (i), (ii) and (iii) are separable in ellipsoidal, paraboidal and cartesian coordinates respectively, while case (iv) is separable in the general sense (Ravoson, V., Ramani, A. and Grammaticos, B. [25]). The integrability of case (i) and separability in ellipsoidal coordinates was proved by Wojciechowski [30] (see also Kostov [14], Tondo [26]). The case (ii) appears as one of the entries in the polynomial hierarchy discussed in Eilbeck, J.C., Enolskii, V.Z., Kuznetsov, V.B. and Leykin, D.V. [7]. The cases (iii) and (iv) are proved to be canonically equivalent under the action of a Miura map restricted to the stationary coupled KdV systems associated with a fourth order Lax operator (Baker, S., Enolskii, V.Z. and Fordy, A.P. [3]). Moreover all the cases (i)-(iv) allow the deformation of the potential by linear combination of inverse squares and squares with certain limitations on the coefficients (see [7], [3]). There are also Lax representations known for all these cases which yield hyperelliptic algebraic curves in the cases (i) and (ii) and a 4-gonal curve in the cases (iii) and (iv). Various results concerning cases (i)-(iv) can be found in Hietarinta, J. [12,11], Grammaticos, B., Dorizzi, B. and Ramani, A. [8] and Perelomov, A.M. [23]. For a review of known results see also Conte, R., Musette, M. and Verhoeven, C. [6,29] and the references therein. Several questions remain unanswered : the 5-dimensional system (2.1) does not have an apparent physical interpretation, has this system any origin in physics? and it would be interesting to compare this system (with $F_3 = c_3 \neq 0$) and the system corresponding to the hamiltonian (2.9).

Remark 2.2. Consider the Yang-Mills system for a field with gauge group $SU(2)$:

$$D_jF_{jk}=\partial_jF_{jk}+[A_j,F_{jk}]=0,$$

where $F_{jk}, A_j \in T_eSU(2), 1 \leq j, k \leq 4$ and $F_{jk} = \partial_j A_k - \partial_k A_j + [A_j, A_k]$. The self-dual Yang-Mills (SDYM) equations is a universal system for which some reductions include all classical tops from Euler to Kowalewski (0+1-dimensions), K-dV, Nonlinear Schrödinger, Sine-Gordon, Toda lattice and N-waves equations (1+1-dimensions), KP and D-S equations (2+1-dimensions), etc... In the case of homogeneous double-component field, we have

$$\partial_j A_k = 0, (j \neq 1), A_1 = A_2 = 0, A_3 = n_1 U_1 \in su(2), A_4 = n_2 U_2 \in su(2),$$

where n_i are $su(2)$-generators, i.e., they satisfy commutation relations : $n_1 = [n_2, [n_1, n_2]], n_2 = [n_1, [n_2, n_1]]$. The system becomes

$$\partial^2 U_1 + U_1 U_2^2 = 0, \qquad \partial^2 U_2 + U_2 U_1^2 = 0.$$

By setting $U_j = q_j$, $\frac{\partial U_j}{\partial t} = p_j$, $j = 1, 2$, Yang-Mills equations are reduced to hamiltonian system with the hamiltonian

$$H = \frac{1}{2}(p_1^2 + p_2^2 + q_1^2 q_2^2).$$

The symplectic transformation

$$p_1 \leftarrow \frac{\sqrt{2}}{2}(p_1 + p_2), \quad p_2 \leftarrow \frac{\sqrt{2}}{2}(p_1 - p_2),$$

$$q_1 \leftarrow \frac{1}{2}(\sqrt[4]{2})(q_1 + iq_2), \quad q_2 \leftarrow \frac{1}{2}(\sqrt[4]{2})(q_1 - iq_2),$$

takes this hamiltonian into

$$H = \frac{1}{2}(p_1^2 + p_2^2) + \frac{1}{4}q_1^4 + \frac{1}{4}q_2^4 + \frac{1}{2}q_1^2 q_2^2. \tag{2.10}$$

We start with the generalized Yang-Mills hamiltonian

$$H = \frac{1}{2}(p_1^2 + p_2^2 + a_1 q_1^2 + a_2 q_2^2) + \frac{1}{4}q_1^4 + \frac{1}{4}a_3 q_2^4 + \frac{1}{2}a_4 q_1^2 q_2^2.$$

Note that if $a_1 = a_2 = 0$ and $a_3 = a_4 = 1$, we obtain the hamiltonian (2.10). It has been shown [13] that if $a_2 = 4a_1 \equiv 4a, a_3 = 16, a_4 = 6$, we obtain the integrals H_1 and H_2(2.7).

3. **Laurent series solutions and algebraic curves**

The invariant variety B(2.3) is a smooth affine surface for generic values of c_1, c_2 and c_3. So, the question I address is how does one find the compactification of B into an abelian surface? Following the methods in Adler-van Moerbeke [2], the idea of the direct proof is closely related to the geometric spirit of the (real) Arnold-Liouville theorem [18]. Namely, a compact complex n-dimensional variety on which there exist n holomorphic commuting vector fields which are independent at every point is analytically isomorphic to a n-dimensional complex torus $\mathbb{C}^n/Lattice$ and the complex flows generated by the vector fields are straight lines on this complex torus. Now, the main problem will be to complete B(2.3) into a non singular compact complex algebraic variety $\widetilde{B} = B \cup \mathcal{D}$ in such a way that the vector fields X_{F_1} and X_{F_2} generated respectively by F_1 and F_2, extend holomorphically along a divisor $\mathcal{D}$ and remain independent there. If this is possible, $\widetilde{B}$ is an algebraic complex torus (an abelian variety) and the coordinates $z_1, \ldots, z_5$ restricted to B are abelian functions. A naive guess would be to take the natural compactification $\overline{B}$ of B by projectivizing the equations : $\overline{B} = \bigcap_{k=1}^{3}\{F_k(Z) = c_k Z_0^4\} \subset \mathbb{P}^5$. Indeed, this can never work for a general reason : an abelian variety $\widetilde{B}$ of dimension bigger or equal than two is never a complete smooth intersection, that is it can never be described in some projective space $\mathbb{P}^n$ by n-dim $\widetilde{B}$ global polynomial homogeneous equations. In other words, if B is to be the affine part of an abelian surface, $\overline{B}$ must have a singularity somewhere along the locus at infinity $\overline{B} \cap \{Z_0 = 0\}$. In fact, we shall show that the existence of meromorphic solutions to the differential equations (2.1) depending on 4 free parameters can be used to manufacture the tori, without ever going through the delicate procedure of blowing up and down. Information about the tori can then be gathered from the divisor.

Theorem 3.1. *The system (2.1) possesses Laurent series solutions which depend on 4 free parameters : α, β, γ and θ. These meromorphic solutions restricted to the surface B(2.3) are parameterized by two isomorphic smooth hyperelliptic curves $\mathcal{H}_{\varepsilon=\pm i}$(3.2) of genus 2.*

Proof. Consider points at infinity which are limit points of trajectories of the flow. To be precise, we search for the set of Laurent solutions which remain confined to the fixed affine invariant surface $B(2.3)$, related to specific values of c_1, c_2 and c_3. The first fact to observe is that if the system is to have Laurent solutions depending on 4 free parameters, the Laurent decomposition of such asymptotic solutions must have the following form

$$\begin{aligned}
z_1 &= \frac{1}{t}\left(z_1^{(0)} + z_1^{(1)}t + z_1^{(2)}t^2 + z_1^{(3)}t^3 + z_1^{(4)}t^4 + \cdots\right),\\
z_2 &= \frac{1}{t}\left(z_2^{(0)} + z_2^{(1)}t + z_2^{(2)}t^2 + z_2^{(3)}t^3 + z_2^{(4)}t^4 + \cdots\right),\\
z_3 &= \frac{1}{t^2}\left(-z_2^{(0)} + z_2^{(2)}t^2 + 2z_2^{(3)}t^3 + 3z_2^{(4)}t^4 + \cdots\right),\\
z_4 &= \frac{1}{2t^2}\left(-z_1^{(0)} + z_1^{(2)}t^2 + 2z_1^{(3t)}t^3 + 3z_1^{(4)}t^4 + \cdots\right),\\
z_5 &= \frac{1}{t^3}\left(z_5^{(0)} + z_5^{(1)}t + z_5^{(2)}t^2 + z_5^{(3)}t^3 + z_5^{(4)}t^4 + \cdots\right).
\end{aligned}$$

Putting these expansions into

$$\begin{aligned}
\ddot{z}_1 &= -2az_1 - 2z_1^2 - 16z_1z_2^2 + 2z_5,\\
\ddot{z}_2 &= -4az_2 - 6z_1z_2 - 16z_2^3,\\
\dot{z}_5 &= -8z_2^2z_4 - 2az_4 - 2z_1z_4 + 4z_1z_2z_3,
\end{aligned}$$

deduced from (2.1), solving inductively for the $z_k^{(j)} (k = 1, 2, 5)$, one finds at the 0^{th} step (resp. 2^{th} step) a free parameter α (resp. β) and the two remaining ones γ, θ at the 4^{th} step. More precisely, we have

a) 0^{th} step :

$$\left\{\begin{array}{l}
z_2^{(0)}\left(1 + 8\left(z_2^{(0)}\right)^2\right) = 0,\\
z_1^{(0)} - z_5^{(0)} + 8z_1^{(0)}\left(z_2^{(0)}\right)^2 = 0,\\
z_5^{(0)} = 0.
\end{array}\right. \Longrightarrow \left\{\begin{array}{l}
z_1^{(0)} = \alpha = \text{free parameter},\\
z_2^{(0)} = \frac{\varepsilon\sqrt{2}}{4},\ \varepsilon \equiv \pm i,\\
z_5^{(0)} = 0.
\end{array}\right.$$

b) 1^{th} step :

$$\left\{\begin{array}{l} z_2^{(0)}\left(8z_2^{(0)}z_2^{(1)}+z_1^{(0)}\right)=0,\\ -z_5^{(1)}+8z_1^{(1)}\left(z_2^{(0)}\right)^2+\left(z_1^{(0)}\right)^2+16z_1^{(0)}z_2^{(0)}z_2^{(1)}=0,\\ 2z_5^{(1)}+4z_1^{(0)}z_2^{(0)}z_2^{(1)}-4z_1^{(1)}\left(z_2^{(0)}\right)^2+\left(z_1^{(0)}\right)^2=0.\end{array}\right. \Longrightarrow \left\{\begin{array}{l} z_1^{(1)}=-\alpha^2,\\ z_2^{(1)}=\frac{\varepsilon\sqrt{2}}{4}\alpha,\\ z_5^{(1)}=0.\end{array}\right.$$

c) 2^{th} step :

$$\left\{\begin{array}{l} 24z_2^{(0)}\left(z_2^{(1)}\right)^2+2az_2^{(0)}+3z_1^{(1)}z_2^{(0)}+3z_1^{(0)}z_2^{(1)}+24\left(z_2^{(0)}\right)^2z_2^{(2)}=0,\\ 16z_1^{(0)}z_2^{(0)}z_2^{(2)}-z_5^{(2)}+2z_1^{(0)}z_1^{(1)}+8z_1^{(2)}\left(z_2^{(0)}\right)^2+16z_1^{(1)}z_2^{(0)}z_2^{(1)}\\ +8z_1^{(0)}\left(z_2^{(1)}\right)^2+az_1^{(0)}=0,\\ z_5^{(2)}+az_1^{(0)}+z_1^{(1)}z_1^{(0)}-8\left(z_2^{(0)}\right)^2z_1^{(2)}+4\left(z_2^{(1)}\right)^2z_1^{(0)}\\ +8z_2^{(0)}z_2^{(2)}z_1^{(0)}-4z_1^{(1)}z_2^{(1)}z_2^{(0)}=0.\end{array}\right.$$

$$\Longrightarrow \left\{\begin{array}{l} z_1^{(2)}=\beta=\text{free parameter},\\ z_2^{(2)}=\frac{\varepsilon\sqrt{2}}{12}\left(-3\alpha^2+2a\right),\\ z_5^{(2)}=-\frac{1}{3}a\alpha+\alpha^3-\beta.\end{array}\right.$$

d) 3^{th} step :

$$\left\{\begin{array}{l} -6z_1^{(1)}z_2^{(1)}-4az_2^{(1)}-6z_1^{(0)}z_2^{(2)}-6z_1^{(2)}z_2^{(0)}-48\left(z_2^{(0)}\right)^2z_2^{(3)}\\ -16\left(z_2^{(1)}\right)^3-96z_2^{(0)}z_2^{(1)}z_2^{(2)}-2z_2^{(3)}=0,\\ -2z_1^{(3)}+2z_5^{(3)}-2\left(z_1^{(1)}\right)^2-2az_1^{(1)}-4z_1^{(0)}z_1^{(2)}-16z_1^{(1)}\left(z_2^{(1)}\right)^2\\ -16z_1^{(3)}\left(z_2^{(0)}\right)^2-32z_1^{(1)}z_2^{(0)}z_2^{(2)}-32z_1^{(2)}z_2^{(0)}z_2^{(1)}\\ -32z_1^{(0)}z_2^{(1)}z_2^{(2)}-32z_1^{(0)}z_2^{(0)}z_2^{(3)}=0,\\ \left(z_2^{(0)}\right)^2z_1^{(3)}-z_2^{(1)}z_2^{(2)}z_1^{(0)}-z_2^{(0)}z_2^{(3)}z_1^{(0)}+z_2^{(0)}z_2^{(1)}z_1^{(2)}=0.\end{array}\right.$$

$$\Longrightarrow \left\{\begin{array}{l} z_1^{(3)}=\frac{1}{6}\alpha\left(3\beta-9\alpha^3+4a\alpha\right),\\ z_2^{(3)}=\frac{\varepsilon\sqrt{2}}{8}\left(3\beta-\alpha^3\right),\\ z_5^{(3)}=3\alpha^4-a\alpha^2-3\alpha\beta.\end{array}\right.$$

e) 4^{th} step :

$$\left\{\begin{array}{l}
48z_2^{(0)}z_2^{(1)}z_2^{(3)}+24\left(z_2^{(0)}\right)^2z_2^{(4)}+3z_1^{(3)}z_2^{(0)}+3z_1^{(0)}z_2^{(3)}+2az_2^{(2)}+3z_1^{(1)}z_2^{(2)}\\
+3z_1^{(2)}z_2^{(1)}+24\left(z_2^{(1)}\right)^2z_2^{(2)}+24z_2^{(0)}\left(z_2^{(2)}\right)^2+3z_2^{(4)}=0,\\
16z_1^{(0)}z_2^{(0)}z_2^{(4)}+16z_1^{(0)}z_2^{(1)}z_2^{(3)}+3z_1^{(4)}-z_5^{(4)}+az_1^{(2)}+2z_1^{(1)}z_1^{(2)}+2z_1^{(0)}z_1^{(3)}\\
+8z_1^{(4)}\left(z_2^{(0)}\right)^2+16z_1^{(1)}z_2^{(1)}z_2^{(2)}+8z_1^{(2)}\left(z_2^{(1)}\right)^2+8z_1^{(0)}\left(z_2^{(2)}\right)^2\\
+16z_1^{(1)}z_2^{(0)}z_2^{(3)}+16z_1^{(2)}z_2^{(0)}z_2^{(2)}+16z_1^{(3)}z_2^{(0)}z_2^{(1)}=0,\\
-20z_2^{(0)}z_2^{(1)}z_1^{(3)}+16z_2^{(0)}z_2^{(4)}z_1^{(0)}-z_5^{(4)}-16\left(z_2^{(0)}\right)^2z_1^{(4)}-4\left(z_2^{(1)}\right)^2z_1^{(2)}\\
-z_1^{(0)}z_1^{(3)}-az_1^{(2)}-z_1^{(1)}z_1^{(2)}+8\left(z_2^{(2)}\right)^2z_1^{(0)}-8z_2^{(0)}z_2^{(2)}z_1^{(2)}\\
+4z_1^{(1)}z_2^{(3)}z_2^{(0)}+16z_2^{(1)}z_2^{(3)}z_1^{(0)}+4z_1^{(1)}z_2^{(1)}z_2^{(2)}=0.
\end{array}\right.$$

$$\Longrightarrow\left\{\begin{array}{l}
z_1^{(4)}=\gamma=\text{free parameter},\\
z_2^{(4)}=\theta,\\
z_5^{(4)}=\left(4\alpha\varepsilon\sqrt{2}\theta+2\gamma+\frac{8}{3}a\alpha^3-\frac{1}{3}a\beta-\alpha^2\beta-3\alpha^5-\frac{4}{9}a^2\alpha\right).
\end{array}\right.$$

The Laurent series solutions are explicitly given by

$$\begin{array}{rcl}
z_1&=&\dfrac{1}{t}\left(\alpha-\alpha^2t+\beta t^2+\dfrac{1}{6}\alpha(3\beta-9\alpha^3+4a\alpha)t^3+\gamma t^4+\cdots\right),\\
z_2&=&\dfrac{\varepsilon\sqrt{2}}{4t}\left(1+\alpha t+\dfrac{1}{3}(-3\alpha^2+2a)t^2+\dfrac{1}{2}(3\beta-\alpha^3)t^3-2\varepsilon\sqrt{2}\theta t^4+\cdots\right),\\
z_3&=&\dfrac{\varepsilon\sqrt{2}}{4t^2}\left(-1+\dfrac{1}{3}(-3\alpha^2+2a)t^2+(3\beta-\alpha^3)t^3-6\varepsilon\sqrt{2}\theta t^4+\cdots\right),\\
z_4&=&\dfrac{1}{2t^2}\left(-\alpha+\beta t^2+\dfrac{1}{3}\alpha(3\beta-9\alpha^3+4a\alpha)t^3+3\gamma t^4+\cdots\right),\\
z_5&=&\dfrac{1}{t}\left(-\dfrac{1}{3}a\alpha+\alpha^3-\beta+(3\alpha^4-a\alpha^2-3\alpha\beta)t\right.\\
&&\left.+\left(4\varepsilon\sqrt{2}\alpha\theta+2\gamma+\dfrac{8}{3}a\alpha^3-\dfrac{1}{3}a\beta-\alpha^2\beta-3\alpha^5-\dfrac{4}{9}a^2\alpha\right)t^2+\cdots\right),
\end{array}\tag{3.1}$$

with $\varepsilon=\pm i$. Using the majorant method, we can show that the formal Laurent series solutions are convergent. Substituting the solutions (3.1) into $F_1=c_1$, $F_2=c_2$ and $F_3=c_3$, and equating the t^0-terms yields

$$\begin{array}{rcl}
F_1&=&\dfrac{15}{8}\alpha^4-\dfrac{5}{6}a\alpha^2-\dfrac{5}{4}\alpha\beta-\dfrac{7}{36}a^2-\dfrac{5}{4}\varepsilon\sqrt{2}\theta=c_1,\\
F_2&=&\varepsilon\sqrt{2}\left(\dfrac{1}{4}\alpha^5-\gamma+\dfrac{\varepsilon\sqrt{2}}{2}\alpha\theta-\dfrac{2}{3}a\alpha^3+\dfrac{1}{3}a\beta+\dfrac{1}{6}a^2+\dfrac{1}{2}\alpha^2\beta\right)=c_2,\\
F_3&=&-\dfrac{11}{2}\alpha^6-\beta^2+4\alpha\gamma+3\alpha^2\varepsilon\sqrt{2}\theta+\alpha^3\beta-\dfrac{1}{3}a^2\alpha^2+\dfrac{10}{3}a\alpha^4=c_3.
\end{array}$$

Eliminating γ and θ from these equations, leads to an equation connecting the two remaining parameters α and β :

$$\beta^2+\frac{2}{3}\left(3\alpha^2-2a\right)\alpha\beta-3\alpha^6+\frac{8}{3}a\alpha^4+\frac{4}{9}\left(a^2+9c_1\right)\alpha^2-2\varepsilon\sqrt{2}c_2\alpha+c_3=0,\tag{3.2}$$

this defines two isomorphic hyperelliptic curves $\mathcal{H}_\varepsilon$ $(\varepsilon = \pm i)$. Let $g(\mathcal{H}_\varepsilon)$ = genus of$\mathcal{H}_\varepsilon$, n = number of sheets and v = number of branch points. Then by the Riemann-Hurwitz's formula,

$$g = -n + 1 + \frac{v}{2} = -2 + 1 + \frac{6}{2} = 2$$

The pole solutions (3.1) restricted to the surface $B(2.3)$ are parameterized by two isomorphic smooth hyperelliptic curves $\mathcal{H}_{\varepsilon=\pm i}(3.2)$ of genus 2, which finishes the proof of the theorem.

4. **Affine part of an abelian surface as the jacobian of a genus two hyperelliptic curve**

In order to embed $\mathcal{H}_\varepsilon$ into some projective space, one of the key underlying principles used is the Kodaira embedding theorem, which states that a smooth complex manifold can be smoothly embedded into projective space $\mathbb{P}^N$ with the set of functions having a pole of order k along positive divisor on the manifold, provided k is large enough; fortunately, for abelian varieties, k need not be larger than three according to Lefshetz. These functions are easily constructed from the Laurent solutions (3.1) by looking for polynomials in the phase variables which in the expansions have at most a k-fold pole. The nature of the expansions and some algebraic proprieties of abelian varieties provide a recipe for when to terminate our search for such functions, thus making the procedure implementable. Precisely, we wish to find a set of polynomial functions $\{f_0, \ldots, f_N\}$, of increasing degree in the original variables $z_1, \ldots, z_5$, having the property that the embedding $\mathcal{D}$ of $\mathcal{H}_i + \mathcal{H}_{-i}$ into $\mathbb{P}^N$ via those functions satisfies the relation : geometric genus $(2\mathcal{D}) \equiv g(2\mathcal{D}) = N + 2$. A this point, it may be not so clear why the curve $\mathcal{D}$ must really live on an abelian surface. Let us say, for the moment, that the equations of the divisor $\mathcal{D}$ (i.e., the place where the solutions blow up), as a curve traced on the abelian surface $\widetilde{B}$ (to be constructed in theorem 4.2), must be understood as relations connecting the free parameters as they appear firstly in the expansions (3.1). In the present situation, this means that (3.2) must be understood as relations connecting α and β. Let

$$L^{(r)} = \left\{ \begin{array}{l} \text{polynomials} \quad f = f(z_, \ldots, z_5) \\ \text{of degre} \leq r, \quad \text{with at worst a} \\ \text{double pole along} \quad \mathcal{H}_i + \mathcal{H}_{-i} \\ \text{and with} \quad z_1, \ldots, z_5 \quad \text{as in(3.1)} \end{array} \right\} / [F_k = c_k, k = 1, 2, 3],$$

and let $(f_0, f_1, \ldots, f_{N_r})$ be a basis of $L^{(r)}$. We look for r such that :

$$g(2\mathcal{D}^{(r)}) = N_r + 2, \quad 2\mathcal{D}^{(r)} \subset \mathbb{P}^{N_r}.$$

We shall show (theorem 4.1) that it is unnecessary to go beyond r=4.

Lemma 4.1. *The spaces $L^{(r)}$, nested according to weighted degree, are generated as follows*

$$\begin{aligned} L^{(1)} &= \{f_0, f_1, f_2, f_3, f_4, f_5\}, \\ L^{(2)} &= L^{(1)} \oplus \{f_6, f_7, f_8, f_9, f_{10}, f_{11}, f_{12}\}, \\ L^{(3)} &= L^{(2)}, \\ L^{(4)} &= L^{(3)} \oplus \{f_{13}, f_{14}, f_{15}\}, \end{aligned} \tag{4.1}$$

where

$$f_0 = 1, \qquad f_1 = z_1 = \frac{\alpha}{t} + \dots,$$
$$f_2 = z_2 = \frac{\varepsilon\sqrt{2}}{t} + \dots, \qquad f_3 = z_3 = -\frac{\varepsilon\sqrt{2}}{4t^2} + \dots,$$
$$f_4 = z_4 = -\frac{\alpha}{2t^2} + \dots, \qquad f_5 = z_5 = -\frac{\eta}{3t} + \dots,$$
$$f_6 = z_1^2 = \frac{\alpha^2}{t^2} + \cdots, \qquad f_7 = z_2^2 = -\frac{1}{8t^2} + \cdots,$$
$$f_8 = z_5^2 = \frac{\eta^2}{9t^2} + \cdots, \qquad f_9 = z_1 z_2 = \frac{\varepsilon\sqrt{2}\alpha}{4t^2} + \cdots,$$
$$f_{10} = z_1 z_5 = -\frac{\alpha\eta}{3t^2} + \cdots, \qquad f_{11} = z_2 z_5 = -\frac{\varepsilon\sqrt{2}\eta}{12t^2} + \cdots,$$
$$f_{12} = [z_1, z_2] = -\frac{\varepsilon\sqrt{2}\alpha^2}{2t^2} + \cdots, \qquad f_{13} = [z_1, z_5] = \frac{4\alpha^2\eta}{3t^2} + \cdots,$$
$$f_{14} = [z_2, z_5] = \frac{\varepsilon\sqrt{2}\alpha\eta}{6t^2} + \cdots, \qquad f_{15} = \left(z_3 - 2\varepsilon\sqrt{2}z_2^2\right)^2 = -\frac{\alpha^2}{2t^2} + \cdots,$$

with $[z_j, z_k] = \dot{z}_j z_k - z_j \dot{z}_k$, *the wronskien of* z_k *and* z_j, *and* $\eta \equiv 3\beta - 3\alpha^3 + a\alpha$.

Proof. The proof of this lemma is straightforward and can be done by inspection of the expansions (3.1). Note also that the functions z_1, z_2, z_5 behave as $1/t$ and if we consider the derivatives of the ratios z_1/z_2, z_1/z_5, z_2/z_5, the wronskiens $[z_1, z_2]$, $[z_1, z_5]$, $[z_2, z_5]$, must behave as $1/t^2$ since z_2^2, z_5^2 behave as $1/t^2$.

Note that $dimL^{(1)} = 6, \quad dimL^{(2)} = dimL^{(3)} = 13, \quad dimL^{(4)} = 16$.

Theorem 4.1. *$L^{(4)}$ provides an embedding of $\mathcal{D}^{(4)}$ into projective space $\mathbb{P}^{15}$ and $\mathcal{D}^{(4)}$ (resp. $2\mathcal{D}^{(4)}$) has genus 5 (resp. 17).*

Proof. It turns out that neither $L^{(1)}$, nor $L^{(2)}$, nor $L^{(3)}$, yields a curve of the right genus; in fact

$$g(2\mathcal{D}^{(r)}) \neq \dim L^{(r)} + 1, \quad r = 1, 2, 3.$$

For instance, the embedding into $\mathbb{P}^5$ via $L^{(1)}$ does not separate the sheets, so we proceed to $L^{(2)}$ and we consider the corresponding embedding into $\mathbb{P}^{12}$. For finite values of α and β, the curves $\mathcal{H}_i$ and $\mathcal{H}_{-i}$ are disjoint; dividing the vector $(f_0, \dots, f_{12})$ by f_7 and taking the limit $t \to 0$, to yield

$$\left[0:0:0:2\varepsilon\sqrt{2}:4\alpha:0:-8\alpha^2:1:-\frac{8}{9}\eta^2:-2\varepsilon\sqrt{2}\alpha:\frac{8}{3}\alpha\eta:\frac{2\varepsilon\sqrt{2}}{3}\eta:4\varepsilon\sqrt{2}\alpha^2\right].$$

The curve (3.2) has two points covering $\alpha = \infty$, at which $\eta \equiv 3\beta - 3\alpha^3 + a\alpha$ behaves as follows :

$$\begin{aligned} \eta &= -6\alpha^3 + 3a\alpha \pm 3\sqrt{4\alpha^6 - 4a\alpha^4 - 4c_1\alpha^2 + 2\varepsilon\sqrt{2}c_2\alpha - c_3}, \\ &= \begin{cases} -\frac{3(a^2+4c_1)}{4\alpha} + & \text{lower order terms,} \quad \text{picking the + sign,} \\ -12\alpha^3 + & O(\alpha), \quad \text{picking the - sign.} \end{cases} \end{aligned}$$

Then by picking the - sign and by dividing the vector $(f_0, \dots, f_{12})$ by f_8, the corresponding point is mapped into the point

$$[0:0:0:0:0:0:0:0:1:0:0:0:0],$$

in $\mathbb{P}^{12}$ which is independent of ε, whereas picking the + sign leads to two different points, according to the sign of ε. Thus, adding at least 2 to the genus of each curve, so that

$$g(2\mathcal{D}^{(2)}) - 2 > 12, \qquad 2\mathcal{D}^{(2)} \subset \mathbb{P}^{12} \neq \mathbb{P}^{g-2},$$

which contradicts the fact that $N_r = g(2\mathcal{D}^{(2)})-2$. The embedding via $L^{(2)}$ (or $L^{(3)}$) is unacceptable as well. Consider now the embedding $2\mathcal{D}^{(4)}$ into $\mathbb{P}^{15}$ using the 16 functions $f_0, \ldots, f_{15}$ of $L^{(4)}$ (4.1). It is easily seen that these functions separate all points of the curve (except perhaps for the points at ∞) : The curves $\mathcal{H}_i$ and $\mathcal{H}_{-i}$ are disjoint for finite values of α and β; dividing the vector $(f_0, \ldots, f_{15})$ by f_7 and taking the limit $t \to 0$, to yield

$$\left[0:0:0:2\varepsilon\sqrt{2}:4\alpha:0:-8\alpha^2:1:-\frac{8}{9}\eta^2:-2\varepsilon\sqrt{2}\alpha:\frac{8}{3}\alpha\eta:\frac{2\varepsilon\sqrt{2}}{3}\eta:\right.$$
$$\left.4\varepsilon\sqrt{2}\alpha^2:-\frac{32}{3}\alpha^2\eta:-\frac{4\varepsilon\sqrt{2}}{3}\alpha\eta:4\alpha^2\right].$$

About the point $\alpha = \infty$, it is appropriate to divide by g_8; then by picking the sign - in η above, the corresponding point is mapped into the point

$$[0:0:0:0:0:0:0:0:1:0:0:0:0:0:0:0],$$

in $\mathbb{P}^{15}$ which is independent of ε, whereas picking the + sign leads to two different points, according to the sign of ε. Hence, the divisor $\mathcal{D}^{(4)}$ obtained in this way has genus 5 and thus $2\mathcal{D}^{(4)}$ has genus 17 and $2\mathcal{D}^{(4)} \subset \mathbb{P}^{15} = \mathbb{P}^{g-2}$, as desired. This ends the proof of the theorem.

Let $L = L^{(4)}$, $\mathcal{D} = \mathcal{D}^{(4)}$ and $\mathcal{S} = 2\mathcal{D}^{(4)} \subset \mathbb{P}^{15}$. Next we wish to construct a surface strip around $\mathcal{S}$ which will support the commuting vector fields. In fact, $\mathcal{S}$ has a good chance to be very ample divisor on an abelian surface, still to be constructed.

Theorem 4.2. *The variety B(2.3) generically is the affine part of an abelian surface $\widetilde{B}$, more precisely the jacobian of a genus 2 curve. The reduced divisor at infinity*

$$\widetilde{B}\backslash B = \mathcal{H}_i + \mathcal{H}_{-i},$$

consists of two smooth isomorphic genus 2 curves $\mathcal{H}_\varepsilon$(3.2). The system of differential equations (2.1) is algebraic complete integrable and the corresponding flows evolve on $\widetilde{B}$.

Proof. We need to attaches the affine part of the intersection of the three invariants (2.2) so as to obtain a smooth compact connected surface in $\mathbb{P}^{15}$. To be precise, the orbits of the vector field (2.1) running through $\mathcal{S}$ form a smooth surface Σ near $\mathcal{S}$ such that $\Sigma\backslash B \subseteq \widetilde{B}$ and the variety $\widetilde{B} = B \cup \Sigma$ is smooth, compact and connected. Indeed, let $\psi(t,p) = \{z(t) = (z_1(t), \ldots, z_5(t)) : t \in \mathbb{C}, 0 < |t| < \varepsilon\}$, be the orbit of the vector field (2.1) going through the point $p \in \mathcal{S}$. Let $\Sigma_p \subset \mathbb{P}^{15}$ be the surface element formed by the divisor $\mathcal{S}$ and the orbits going through p, and set $\Sigma \equiv \cup_{p\in\mathcal{S}}\Sigma_p$. Consider the curve $\mathcal{S}' = \mathcal{H} \cap \Sigma$ where $\mathcal{H} \subset \mathbb{P}^{15}$ is a hyperplane transversal to the direction of the flow. If $\mathcal{S}'$ is smooth, then using the implicit function theorem the surface Σ is smooth. But if $\mathcal{S}'$ is singular at 0, then Σ would be singular along the trajectory ($t-$axis) which go immediately into the affine part B. Hence, B would be singular which is a contradiction because B is the fibre of a morphism from $\mathbb{C}^5$ to $\mathbb{C}^2$ and so smooth for almost all the three constants of the motion c_k. Next, let $\overline{B}$ be the projective closure of B into $\mathbb{P}^5$, let $Z = [Z_0 : Z_1 : \ldots : Z_5] \in \mathbb{P}^5$ and let $I = \overline{B} \cap \{Z_0 = 0\}$ be the locus at infinity. Consider the map $\overline{B} \subseteq \mathbb{P}^5 \to \mathbb{P}^{15}$, $Z \mapsto f(Z)$, where $f = (f_0, f_1, ..., f_{15}) \in L(\mathcal{S})$ (4.1) and let $\widetilde{B} = f(\overline{B})$. In a neighbourhood $V(p) \subseteq \mathbb{P}^{15}$ of p, we have $\Sigma_p = \widetilde{B}$ and $\Sigma_p \backslash \mathcal{S} \subseteq B$. Otherwise there would exist an element of surface $\Sigma'_p \subseteq \widetilde{B}$ such that $\Sigma_p \cap \Sigma'_p = (t - axis)$, orbit $\psi(t,p) = (t - axis) \backslash\ p \subseteq B$, and hence B would be singular along the $t-$axis which is impossible. Since the variety $\overline{B} \cap \{Z_0 \neq 0\}$ is irreducible and since the generic hyperplane section $\mathcal{H}_{gen.}$ of $\overline{B}$ is also irreducible, all hyperplane sections are connected and hence I is also connected. Now, consider the graph $\Gamma_f \subseteq \mathbb{P}^5 \times \mathbb{P}^{15}$ of the map f, which is irreducible together with $\overline{B}$. It follows from the irreducibility of I that a generic hyperplane section $\Gamma_f \cap \left\{\mathcal{H}_{gen.} \times \mathbb{P}^{15}\right\}$ is irreducible, hence the special hyperplane section $\Gamma_f \cap \left\{\{Z_0 = 0\} \times \mathbb{P}^{15}\right\}$ is connected and therefore the projection map $proj_{\mathbb{P}^{15}} \left\{\Gamma_f \cap \{\{Z_0 = 0\} \times \mathbb{P}^{15}\}\right\} = f(I) \equiv \mathcal{S}$, is connected. Hence, the variety $B \cup \Sigma = \widetilde{B}$ is compact, connected and embeds smoothly into $\mathbb{P}^{15}$ via f. We wish to show that $\widetilde{B}$ is an abelian surface equipped with two everywhere independent commuting vector fields. For doing that, let ϕ^{τ_1} and ϕ^{τ_2} be the flows corresponding to vector fields X_{F_1} and X_{F_2}. The latter are generated respectively by F_1 and F_2. For $p \in \mathcal{S}$ and for small $\varepsilon > 0$, $\phi^{\tau_1}(p), \forall \tau_1, 0 < |\tau_1| < \varepsilon$, is well defined and $\phi^{\tau_1}(p) \in B$. Then we may define ϕ^{τ_2} on B by $\phi^{\tau_2}(q) = \phi^{-\tau_1}\phi^{\tau_2}\phi^{\tau_1}(q), q \in U(p) = \phi^{-\tau_1}\left(U\left(\phi^{\tau_1}(p)\right)\right)$, where $U(p)$ is a neighbourhood of p. By commutativity one can see that ϕ^{τ_2} is independent of τ_1; $\phi^{-\tau_1-\varepsilon_1}\phi^{\tau_2}\phi^{\tau_1+\varepsilon_1}(q) = \phi^{-\tau_1}\phi^{-\varepsilon_1}\phi^{\tau_2}\phi^{\tau_1}\phi^{\varepsilon_1} = \phi^{-\tau_1}\phi^{\tau_2}\phi^{\tau_1}(q)$. We affirm that $\phi^{\tau_2}(q)$ is holomorphic away from $\mathcal{S}$. This because $\phi^{\tau_2}\phi^{\tau_1}(q)$ is holomorphic away from $\mathcal{S}$ and that ϕ^{τ_1} is holomorphic in $U(p)$ and maps bi-holomorphically $U(p)$ onto $U\left(\phi^{\tau_1}(p)\right)$. Now, since the flows ϕ^{τ_1} and ϕ^{τ_2} are holomorphic and independent on $\mathcal{S}$, we can show along the same lines as in the Arnold-Liouville theorem [2,18] that $\widetilde{B}$ is a complex torus $\mathbb{C}^2/lattice$ and so in particular $\widetilde{B}$ is a Kähler variety. And that will done, by considering the local diffeomorphism $\mathbb{C}^2 \to \widetilde{B}, (\tau_1, \tau_2) \mapsto \phi^{\tau_1}\phi^{\tau_2}(p)$, for a fixed origin $p \in B$. The additive subgroup $\left\{(\tau_1, \tau_2) \in \mathbb{C}^2 : \phi^{\tau_1}\phi^{\tau_2}(p) = p\right\}$ is a lattice of $\mathbb{C}^2$, hence $\mathbb{C}^2/lattice \to \widetilde{B}$ is a biholomorphic diffeomorphism and $\widetilde{B}$ is a Kähler variety with Kähler metric given by $d\tau_1 \otimes d\overline{\tau}_1 + d\tau_2 \otimes d\overline{\tau}_2$. Now, a compact complex Kähler variety having the required number as (its dimension) of independent meromorphic functions is a projective variety [22]. In fact, here we have $\widetilde{B} \subseteq \mathbb{P}^{15}$. Thus $\widetilde{B}$ is both a projective variety and a complex torus $\mathbb{C}^2/lattice$ and hence an abelian surface as a consequence of Chow theorem. By the classification theory of ample line bundles on abelian varieties, $\widetilde{B} \simeq \mathbb{C}^2/L_\Omega$ with period lattice given by the columns of the matrix

$$\begin{pmatrix} \delta_1 & 0 & a & c \\ 0 & \delta_2 & c & b \end{pmatrix}, \qquad Im\begin{pmatrix} a & c \\ c & b \end{pmatrix} > 0,$$

and $\delta_1\delta_2 = g(\mathcal{H}_\varepsilon) - 1 = 1$, implying $\delta_1 = \delta_2 = 1$. Thus $\widetilde{B}$ is principally polarized and it is the jacobian of the hyperelliptic curve $\mathcal{H}_\varepsilon$. This completes the proof of the theorem.

Remark 4.1. We have seen that the reflection σ on the affine variety B amounts to the flip $\sigma : (z_1, z_2, z_3, z_4, z_5) \mapsto (z_1, z_2, -z_3, -z_4, z_5)$, changing the direction of the commuting vector fields. It can be extended to the (-Id)-involution about the origin of $\mathbb{C}^2$ to the time flip $(t_1,t_2)\mapsto(-t_1,-t_2)$ on $\widetilde{B}$, where t_1 and t_2 are the time coordinates of each of the flows X_{F_1} and X_{F_2}. The involution σ acts on the parameters of the Laurent solution (3.1) as follows

$$\sigma:(t,\alpha,\beta,\gamma,\theta,\varepsilon)\longmapsto(-t,-\alpha,-\beta,-\gamma,-\theta,-\varepsilon),$$

interchanges the curves $\mathcal{H}_{\varepsilon=\pm i}$(3.2) and the linear space L can be split into a direct sum of even and odd functions. Geometrically, this involution interchanges $\mathcal{H}_i$ and $\mathcal{H}_{-i}$, i.e., $\mathcal{H}_{-i}=\sigma\mathcal{H}_i$.

Remark 4.2. Consider on $\widetilde{B}$ the holomorphic 1-forms dt_1 and dt_2 defined by $dt_i(X_{F_j})=\delta_{ij}$, where X_{F_1} and X_{F_2} are the vector fields generated respectively by F_1 and F_2. Taking the differentials of $\zeta=1/z_1$ and $\xi=z_1/z_2$ viewed as functions of t_1 and t_2, using the vector fields and the Laurent series (3.1) and solving linearly for dt_1 and dt_2, we obtain as expected the hyperelliptic holomorphic differentials

$$\begin{aligned}\omega_1 &= dt_1|_{\mathcal{H}_\varepsilon}=\frac{1}{\triangle}\left(\frac{\partial\xi}{\partial t_2}d\zeta-\frac{\partial\zeta}{\partial t_2}d\xi\right)\Big|_{\mathcal{H}_\varepsilon}=\frac{\alpha d\alpha}{\sqrt{P(\alpha)}},\\ \omega_2 &= dt_2|_{\mathcal{H}_\varepsilon}=\frac{1}{\triangle}\left(\frac{-\partial\xi}{\partial t_1}d\zeta-\frac{\partial\zeta}{\partial t_1}d\xi\right)\Big|_{\mathcal{H}_\varepsilon}=\frac{\sqrt{2}d\alpha}{2\sqrt{P(\alpha)}},\end{aligned}$$

with $P(\alpha)\equiv 4\alpha^6-4a\alpha^4-4c_1\alpha^2+2\varepsilon\sqrt{2}c_2\alpha-c_3$ and $\Delta\equiv\frac{\partial\zeta}{\partial t_1}\frac{\partial\xi}{\partial t_2}-\frac{\partial\zeta}{\partial t_2}\frac{\partial\xi}{\partial t_1}$. The zeroes of ω_2 provide the points of tangency of the vector field X_{F_1} to $\mathcal{H}_\varepsilon$. We have $\frac{\omega_1}{\omega_2}=-\varepsilon\sqrt{2}\alpha$, and X_{F_1} is (doubly) tangent to $\mathcal{H}_\varepsilon$ at the point covering $\alpha=\infty$, i.e., where both the curves touch.

Remark 4.3. There are many examples of differential equations which have the weak Painlevé property that all movable singularities of the general solution have only a finite number of branches and some interesting integrable systems appear as coverings of algebraic completely integrable systems. The invariant varieties are coverings of abelian varieties and these systems are called algebraic completely integrable in the generalized sense. These systems are Liouville integrable and by the Arnold-Liouville theorem, the compact connected manifolds invariant by the real flows are tori; the real parts of complex affine coverings of abelian varieties. Most of these systems of differential equations possess solutions which are Laurent series of $t^{1/n}$ (t being complex time) and whose coefficients depend rationally on certain algebraic parameters. It was shown in series of publications of Vanhaecke [27,28], Abenda and Fedorov [1] and others that θ-divisor can serve as a carrier of integrability. Let $\mathcal{H}$ be a hyperelliptic curve of genus g and $\text{Jac}(\mathcal{H})=\mathbb{C}^g/\Lambda$ its jacobian variety where Λ is a lattice of maximal rank in $\mathbb{C}^g$. Let

$$\mathcal{A}_k:\text{Sym}^k(\mathcal{H})\to\text{Jac}(\mathcal{H}),(P_1,...,P_k)\mapsto\sum_{j=1}^{k}\int_{\infty}^{P_j}(\omega_1,...,\omega_g)\text{mod}.\Lambda, 0\le k\le g,$$

be the Abel map where $(\omega_1,...,\omega_g)$ is a canonical basis of the space of differentials of the first kind on $\mathcal{H}$. The theta divisor Θ is a subvariety of $\text{Jac}(\mathcal{H})$ defined as $\Theta\equiv\mathcal{A}\left[\text{Sym}^{g-1}(\mathcal{H})\right]/\Lambda$. By Θ_k we will denote the subvariety (called strata) of $\text{Jac}(\mathcal{H})$ defined by $\Theta_k\equiv\mathcal{A}_k\left[\text{Sym}^k(\mathcal{H})\right]/\Lambda$ and we have the following stratification

$$\{O\}\subset\Theta_0\subset\Theta_1\subset\Theta_2\subset...\subset\Theta_{g-1}\subset\Theta_g=\text{Jac}(\mathcal{H}),$$

where O is the origin of Jac($\mathcal{H}$). Vanhaecke [27] showed that these stratifications of the jacobian are connected with stratifications of the Sato grassmannian, via an extension of Krichever's map. He discuss the relation between Laurent solutions for the Master systems and stratifications of the jacobian of a hyperelliptic curve. In [28], the author studied Lie-Poisson structure in the jacobian and showed that invariant manifolds associated with Poisson brackets can be identified with these strata. Some problems were considered in [28] and [1], where a connection was established with the flows on these strata. Such varieties or their open subsets often appear as coverings of complex invariants manifolds of finite dimensional integrable systems (Hénon-Heiles and Neumann systems). In [21], we discuss another interesting interaction between algebraic geometry and dynamics : the system of differential equations (2.8). We solve this system in terms of genus two hyperelliptic functions; we set

$$q_2 = s_1 + s_2, \quad q_1^2 = -4s_1s_2, \quad p_2 = \dot{s}_1 + \dot{s}_2, \quad q_1p_1 = -2(\dot{s}_1s_2 + s_1\dot{s}_2).$$

Straightforward calculation shows that equations (2.8) take the following form

$$\begin{aligned} \frac{ds_1}{\sqrt{P_6(s_1}} - \frac{ds_2}{\sqrt{P_6(s_2}} &= 0, \\ \frac{s_1ds_1}{\sqrt{P_6(s_1}} - \frac{s_2ds_2}{\sqrt{P_6(s_2}} &= dt, \end{aligned}$$

where $P_6(s) = s(-8s^5 - 4as^3 + 2b_1s + b_2)$. These equations are integrated in terms of genus 2 hyperelliptic functions. Also we show that the system (2.8) is algebraic completely integrable in the generalized sense. To be more precise, when one examines all possible singularities of the system (2.8), one finds that it possible for the variable q_1 to contain square root terms of the type $t^{1/2}$, which are strictly not allowed by the so called Painlevé test (i.e. the general solutions should have no movable singularities other than poles in the complex plane). Let A be the affine variety defined by

$$A = \bigcap_{k=1}^{2} \{z \in \mathbb{C}^4 : H_k(z) = b_k\}, \tag{4.2}$$

where $(b_1, b_2) \in \mathbb{C}^2$. Since A is the fibre of a morphism from $\mathbb{C}^4$ to $\mathbb{C}^2$ over $(b_1, b_2) \in \mathbb{C}^2$, for almost all b_1, b_2, therefore A is a smooth affine surface. We show that the system (2.8) admits 3-dimensional family of Laurent solutions in $t^{1/2}$, depending on three free parameters : u, v and w. There are precisely two such families, labeled by $\varepsilon = \pm i$, and they are explicitly given as follows

$$
\begin{aligned}
q_1 &= \frac{1}{\sqrt{t}}\left(u - \frac{1}{2}u^3 t + vt^2 + u^2\left(-\frac{11}{16}u^5 + \frac{1}{3}au + v\right)t^3\right.\\
&\left. + \frac{u}{4}(\frac{41}{32}u^8 - au^4 + \frac{3}{2}u^3 v + \frac{1}{6}a^2 - \frac{3\varepsilon\sqrt{2}}{2}w)t^4 + \cdots\right), \qquad (4.3)\\
q_2 &= \frac{\varepsilon\sqrt{2}}{4t}\left(1 + u^2 t + \frac{1}{3}(2a - 3u^4)t^2 + \frac{1}{8}u(24v - u^5)t^3 - 2\varepsilon\sqrt{2}wt^4 + \cdots\right),\\
p_1 &= \frac{1}{t\sqrt{t}}\left(-\frac{1}{2}u - \frac{1}{4}u^3 t + \frac{3}{2}vt^2 + \frac{5}{2}u^2\left(-\frac{11}{16}u^5 + \frac{1}{3}au + v\right)t^3\right.\\
&\left. + \frac{7u}{8}\left(\frac{41}{32}u^8 - au^4 + \frac{3}{2}u^3 v + \frac{1}{6}a^2 - \frac{3\varepsilon\sqrt{2}}{2}w\right)t^4 + \cdots\right),\\
p_2 &= \frac{\varepsilon\sqrt{2}}{4t^2}\left(-1 + \frac{1}{3}(2a - 3u^4)t^2 + \frac{1}{4}u(24v - u^5)t^3 - 6\varepsilon\sqrt{2}wt^4 + \cdots\right).
\end{aligned}
$$

These formal series solutions are convergent as a consequence of the majorant method. The pole solutions (4.3) restricted to the surface $A(4.2)$ are parameterized by two smooth curves $\mathcal{C}_{\varepsilon=\pm i}$ of genus 4 :

$$2v^2 + \frac{1}{6}(15u^4 - 8a)uv - \frac{39}{32}u^{10} + \frac{7}{6}au^6 + \frac{2}{9}(a^2 + 9b_1)u^2 - \varepsilon\sqrt{2}b_2 = 0. \qquad (4.4)$$

Applying the method explained in Piovan [24], we show that the invariant variety $A(4.2)$ can be completed as a cyclic double cover $\overline{A}$ of the jacobian of a genus curve, ramified along a divisor $\mathcal{H}_i + \mathcal{H}_{-i}$ where $\mathcal{H}_i$ and $\mathcal{H}_{-i}$ are two isomorphic hyperelliptic curves (3.2) of genus 2 that intersect in only one point at which they are tangent to each other. Moreover, $\overline{A}$ is smooth except at the point lying over the singularity (of type A_3) of $\mathcal{H}_i + \mathcal{H}_{-i}$ and the resolution $\widetilde{A}$ of $\overline{A}$ is a surface of general type with invariants : Euler characteristic of $\widetilde{A} = 1$ and geometric genus of $\widetilde{A} = 2$. Consequently, the system (2.8) is algebraic completely integrable in the generalized sense. The asymptotic solution (4.3) can be read off from (3.1) and the change of variable : $q_1 = \sqrt{z_1}, q_2 = z_2, p_1 = z_4/q_1, p_2 = z_3$. The function z_1 has a simple pole along the divisor $\mathcal{H}_i + \mathcal{H}_{-i}$ and a double zero along a hyperelliptic curve of genus 2 defining a double cover of $\widetilde{B}$ ramified along $\mathcal{H}_i + \mathcal{H}_{-i}$.

References

[1] Abenda, S. and Fedorov, Yu., On the weak Kowalewski-Painlevé Property for hyperelliptically separable systems, Acta Appl. Math., 60(2000) 137-178.

[2] Adler, M. and van Moerbeke, A., The complex geometry of the Kowalewski-Painlevé analysis, Invent. Math., 97(1989) 3-51.

[3] Baker, S., Enolskii, V.Z. and Fordy, A.P., Integrable quartic potentials and coupled KdV equations, Phys. Lett., 201A(1995) 167-174.

[4] Belokolos, A.I., Bobenko, V.Z., Enol'skii, V.Z., Its, A.R. and Matveev. V.B., Algebro-Geometric approach to nonlinear integrable equations, Springer-Verlag 1994.

[5] Christiansen, P. L., Eilbeck, J. C., Enolskii, V. Z. and Kostov, N. A., Quasi-periodic and periodic solutions for coupled nonlinear Schrdinger equations of Manakov type, Proc. R. Soc., A 456(2000) 2263-2281.

[6] Conte, R., Musette, M. and Verhoeven, C., Completeness of the cubic and quartic Hénon-Heiles hamiltonians, Theor. Math. Phys., 144(2005) 888-898.

[7] Eilbeck, J.C., Enolskii, V.Z., Kuznetsov, V.B. and Leykin, D.V., Linear r-matrix algebra for systems separable in parabolic coordinates, Phys. Lett., 180A(1993) 208-214.
[8] Grammaticos, B. Dorozzi, B., Ramani, A., Integrability of hamiltonians with third and fourth-degree polynomial potentials, J. Math. Phys., 24(1983) 2289-2295.
[9] Griffiths, P.A. and Harris, J., Principles of algebraic geometry, Wiley-Interscience 1978.
[10] Haine, L., Geodesic flow on $SO(4)$ and Abelian surfaces. Math. Ann., 263(1983) 435-472.
[11] Hietarinta, J., Classical versus quantum integrability. J. Math. Phys., 25(1984) 1833-1840.
[12] Hietarinta, J., Direct methods for the search of the second invariant. Phys. Rep., 147(1987) 87-154.
[13] Kasperczuk, S., Integrability of the Yang-Mills hamiltonian system, Celes. Mech. and Dyn. Astr., 58(1994), 387-391. Erratum Celes. Mech. and Dyn. Astr., 60(1994), 289.
[14] Kostov, N.A., Quasi-periodical solutions of the integrable dynamical systems related to Hill's equation. Lett. Math. Phys., 17(1989) 95-104.
[15] Lesfari, A., Abelian surfaces and Kowalewski's top, Ann. Scient. École Norm. Sup. Paris, 21(1988), sr. 4, 193-223.
[16] Lesfari., A., Completely integrable systems : Jacobi's heritage, J. Geom. Phys., 31(1999) 265-286.
[17] Lesfari, A., Le système différentiel de Hénon-Heiles et les variétés Prym, Pacific J. Math., 212(2003), N1, 125-132.
[18] Lesfari, A., Le théorème d'Arnold-Liouville et ses conséquences, Elem. Math., 58 (2003) 6-20.
[19] Lesfari, A., Analyse des singularités de quelques systèmes intégrables, C.R. Acad. Sci. Paris, 341(2005), Ser. I, 85-88.
[20] Lesfari, A., Abelian varieties, surfaces of general type and integrable systems, Beiträge Algebra Geom., 1(2007), Vol.48, 95-114.
[21] Lesfari, A., The Yang-Mills system and cyclic covering of abelian varieties, to appear Int. J. Geom. Methods Mod. Phys., Vol. 5(2008), N8.
[22] Moishezon., B.G., On n-dimensional compact varieties with n algebraically independent meromorphic functions, Amer. Math. Soc. Transl., 63(1967), 51-177.
[23] Perelomov, A.M., Integrable system of classical mechanics and Lie algebras, Birkhuser 1994.
[24] Piovan, L., Cyclic coverings of abelian varieties and the Goryachev-Chaplygin top, Math. Ann., 294(1992), 755-764.
[25] Ravoson, V., Ramani, A. and Grammaticos, B., Generalized separability for a hamiltonian with nonseparable quartic potential, Phys. Lett., 191A(1994), 91-95.
[26] Tondo, G., On the integrability of stationary and restricted flows of the KdV hierarchy, J. Phys. A : Mat. Gen., 28(1995), 5097-5115.
[27] Vanhaecke, P., Stratifications of hyperelliptic jacobians and the Sato Grassmannian, Acta Appl. Math., 40(1995), 143-172.
[28] Vanhaecke, P., Integrable systems and symmetric products of curves, Math. Z., 227(1998), 93-127.
[29] Verhoeven, C., Musette, M. and Conte, R., General solution of hamiltonians withs extend cubic and quartic potentials, Theor. Math. Phys., 134(2003), 128-138.
[30] Wojciechowski, S., Integrability of one particle in a perturbed central quartic potential, Physica Scripta, 31(1985), 433-438.

Department of Mathematics, Faculty of Sciences, University of Chouaïb Doukkali, B.P. 20, El-Jadida, Morocco
E-mail address: Lesfariahmed@yahoo.fr, Lesfari@ucd.ac.ma