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THE INTRINSIC GEOMETRY OF A JORDAN DOMAIN

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Abstract. For a Jordan domain in the plane the length metric space of points connected to an interior point by a curve of finite length is a CAT(0) space and Gromov hyperbolic. With respect to the cone topology, that space plus its boundary at infinity is topologically the same as the original Jordan domain.

1. Introduction

A Jordan domain J is the the homeomorphic image of a closed disk in $\mathbf{E}^2$. The image of the boundary circle is a Jordan curve, which by the Jordan Curve Theorem separates the plane into two open domains, one bounded, the other not, such that the curve is the boundary of each. A *local geodesic* is a curve γ such that for every non-end point p of γ there is an open subarc ρ of γ containing p such that the arc of ρ between any two points of ρ is a shortest curve in J connecting those two points. Bourgin and Renz [3] have analyzed the local geodesics in such domains, concluding the following:

- A Jordan domain is uniquely geodesic.
- At every non-end point of a geodesic, the geodesic is supported by a closed half-disk with center at the point and interior contained in the interior of the Jordan domain.
- Geodesics are characterized in terms of separation: a point p is not on the geodesic from q to r if and only if there is a straight line segment σ with ends on the boundary ∂J of J, but otherwise in the interior of J, such that p is in one connected component of $J \setminus \sigma$ and q, r are in the other.

Although geodesics may be infinitely long when an end is on the boundary, the unique determination of a geodesic by its ends is still true.

We assume throughout that geodesics are parametrized by an arclength parameter, and for a geodesic ray the parameter is 0 at the origin of the ray and positive elsewhere.

There has been a little previous related work concerning not quite the same domains: in the thesis of F.-E. Wolter [7] it is proved that a simply connected plane domain with locally rectifiable boundary has the unique geodesic property;

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in a recent text [2, p. 310] there is an outline of a proof that locally simply-connected plane domains are locally CAT(0). In the latter, although the outline is entirely plausible and at a level of rigor appropriate to a text at that level, essential technical details (the unique geodesic property and a consideration of triangles with "tails") are not mentioned. It is probable that the CAT(0) property, but certainly not Gromov hyperboliciy, can be extended to unbounded domains by exhausting them with their intersection with increasing disks; also extensions giving natural curvature bounds for similar domains in Riemannian surfaces are likely to be true; but it seems as though these extensions would not present any interesting challenges.

2. The CAT(0) space

The set X of *finite-distance* points of J consists of interior points and those points of ∂J which can be connected to any (and hence every) interior point by a curve of finite length. Note that every pair of points in the interior can be connected by polygonal curve in the interior, and so have finite distance apart. If a straight line segment is interior to J except for one end, then that end is also in X; in particular, only the ends of a geodesic can fail to be in X, since non-ends are either already interior to J or a radius of a supporting half-disk provides a segment to an interior point. The remaining subset of ∂J, points not in X, is called the *(metric) boundary* of X, denoted ∂X.

A *geodesic triangle* in a metric space consists of three shortest curves (its *sides*) connecting a triple of points (its *vertices*). A *comparison triangle* in $\mathbf{E}^2$ is a triangle with the same side-lengths, and by taking equal distances from the vertices we get comparison points on the sides. A geodesic triangle is *thin* if the distance between every pair of points in the sides is $\leq$ the distance between the comparison points. A CAT(0) *space* is a complete metric space such that every pair of points can be joined by a (finite-length) geodesic and every geodesic triangle is thin. It is easy to show that a CAT(0) space is uniquely geodesic and contractible [4]. Conversely, if a space is locally CAT(0) and simply connected, then it is CAT(0) [1].

Theorem 2.1. *The set X of finite-distance points of a Jordan domain J is a* CAT(0) *space.*

Proof. Consider three points $p, q, r \in X$. We suppose that the geodesic triangle with vertices p, q, r is nondegenerate, since degenerate triangles are trivially thin. Then the two sides starting from p must consist of a common part (which may be just p) ending at a bifurcation point $\bar{p}$; similarly, there are bifurcation points $\bar{q}, \bar{r}$ on the sides starting from q, r. Then $\triangle\bar{p}\bar{q}\bar{r}$ is a Jordan curve in X and $\triangle pqr$ consists of $\triangle\bar{p}\bar{q}\bar{r}$ with three "tails". Clearly if we show that $\triangle\bar{p}\bar{q}\bar{r}$ is thin, then $\triangle pqr$ is also thin.

The interior of $\triangle\bar{p}\bar{q}\bar{r}$ must lie in the interior of X, and at each point of a side which also lies in the interior of X, that side is locally a straight line segment. Hence at those points the side is supported by half-disks in the interior of X, both interior and exterior to $\triangle\bar{p}\bar{q}\bar{r}$. But at a point of a side which is in ∂X, a supporting half-disk can only lie entirely in the domain bounded by $\triangle\bar{p}\bar{q}\bar{r}$. Thus, a side of $\triangle\bar{p}\bar{q}\bar{r}$ is supported at every point except the ends by an open half-disk interior to $\triangle\bar{p}\bar{q}\bar{r}$. Therefore the sides are locally convex and have well-defined one-sided directions at every point and also a total curvature. At the ends the signed turning angle (choose an orientation of $\mathbf{E}^2$!) is at most π, while the signed total curvature of any

simple closed, locally convex curve must be 2π. The signed total curvatures of the sides must be ≤ 0, so together they add to at least $2\pi - 3\pi = -\pi$. In particular, each side must actually be convex, i.e., is on the boundary of its convex hull, and the remaining part of that boundary is a straight line segment. The triangle of these three segments has $\triangle\bar{p}\bar{q}\bar{r}$ in its interior, so that $\triangle\bar{p}\bar{q}\bar{r}$ must have angle sum $\leq 2\pi$. One of Alexandrov's criteria for a locally CAT(0) space is that locally angle sums of triangles are $\leq 2\pi$. Since X is simply connected, we conclude that it is also CAT(0).

It is also easy to see that X is complete. Indeed, distances in X are $\geq$ the corresponding Euclidean distances, so that a Cauchy sequence in X is also Cauchy in the plane. The limit in the plane is in J and the distance of the limit from any point of the sequence is finite. □

Let us call a triangle in X such as $\triangle\bar{p}\bar{q}\bar{r}$ a *Jordan triangle.* As a corollary to the above proof we have a uniform bound on the perimeter of Jordan triangles:

Corollary 2.1. *If D is the diameter of J in $\mathbf{E}^2$, then the perimeter of every Jordan triangle is bounded by $4D$.*

Proof. In $\mathbf{E}^2$ the length of a closed convex curve is monotone increasing with respect to the inclusion ordering of convex hulls. We can choose Euclidean segments from the vertices of $\triangle\bar{p}\bar{q}\bar{r}$ so that they don't intersect the sides and either are concurrent or meet in pairs to give a triangle $\triangle abc$ in the interior. The latter case is generic, pictured in Figure 1. The sides of the Jordan triangle are shorter than the smaller triangle in which they are contained; that is, $\ell(\tau) < d(\bar{p}, a) + d(a, \bar{r})$, etc. In the

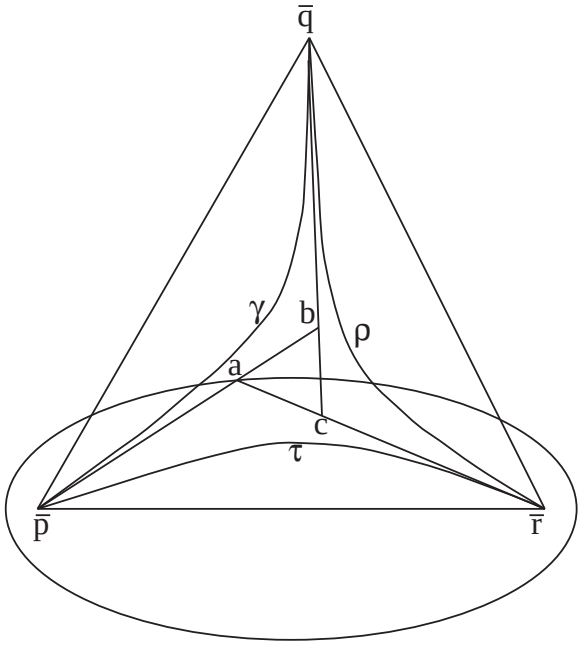

FIGURE 1. Jordan triangle

concurrent case, when point $a = b = c = \bar{m}$, then the segments $\bar{m}\bar{p}$, $\bar{m}\bar{q}$, $\bar{m}\bar{r}$ are in the interior of $\triangle\bar{p}\bar{q}\bar{r}$ except for the ends. The sum of the lengths of these segments from $\bar{m}$ is at most $2D$ (the sum of distances in $\mathbf{E}^2$ from several points is a convex function, so the maximum on a triangle must occur at a vertex), and hence twice that sum is greater than twice the sum of the two longest side-lengths of $\triangle\bar{p}\bar{q}\bar{r}$.

The generic case can be reduced to the concurrent case by showing that there is an interior point $\bar{m}$ of $\triangle abc$ such that the segments $\bar{m}\bar{p}$, $\bar{m}\bar{q}$, $\bar{m}\bar{r}$ again have pairwise sums of lengths which bound the lengths of τ, γ, ρ. Consider the ellipse with foci $\bar{p}, \bar{r}$ passing through a. For any point on that ellipse the sum of distances from $\bar{p}, \bar{r}$ is the same, hence greater than $\ell(\tau)$. For a point outside the ellipse the sum is even greater, so that is true for any point (except a) on the tangent line to the ellipse at a. But the rule for reflection in an ellipse tells us that that tangent line bisects $\angle bac$. Similarly, the angle bisectors of $\angle abc, \angle bca$ give points from which the sums of distances from $\bar{p}, \bar{q}$ and $\bar{q}, \bar{r}$ majorize the lengths of γ, ρ. Thus we can take $\bar{m}$ to be the incenter of $\triangle abc$. □

3. Gromov hyperbolicity

A geodesic metric space X is *δ-hyperbolic*, where $\delta \geq 0$, if for every triangle the distance from any point on one side of the triangle to the union of the other two sides is $\leq \delta$. If X is δ-hyperbolic for some δ, then it is *Gromov hyperbolic*.

When X is a Jordan domain, it is clear that it is δ-hyperbolic if the defining property is true for every Jordan triangle in X. Half the perimeter of the triangle obviously satisfies the defining property, so by Corollary 2.1 X is $2D$-hyperbolic. However, a better value for δ is obtained by observing that for a Euclidean triangle the extreme case is equilateral and if the side-lengths are D, then the maximum distance from a point to the other sides is half the altitude. Thus we obtain the sharp value for δ, realized by the case when X is bounded by an equilateral triangle of side D.

Corollary 3.1. *If X is a Jordan domain with Euclidean diameter D, then it is $\sqrt{3}D/4$-hyperbolic.*

4. The cone topology

For a CAT(0) space X the boundary at infinity ∂X consists of asymptote classes of geodesic rays; two rays are *asymptotic* if the distance between pairs at equal distance from their origins is bounded. The *cone topology* on $\bar{X} = X \cup \partial X$ is defined by specifying a neighborhood basis: for points of X we take the usual metric neighborhoods. The neighborhoods of ∂X are defined in terms of a fixed center point $p \in X$, and then it is proved that the resulting topology is independent of p. For an infinite ray γ starting at p and positive numbers C, ϵ we define a neighborhood $\mathcal{N}(\gamma, C, \epsilon)$ of the asymptote class of γ to be the points on geodesics starting at p which pass through the ball $B(\gamma(C), \epsilon)$ and have distance $> C$ from p, along with the asymptote classes of the extensions of those geodesics to rays whenever possible. (The definition was originally due to Eberlein and O'Neill for Riemannian manifolds [5].) For each asymptote class there is exactly one ray for a given origin [4].

Lemma 4.1. *For a Jordan domain J and the space of finite points X, two asymptotic rays in X must eventually coincide.*

Proof. We first prove this for two asymptotic rays γ, σ for which the origins are interior points p, q of J such that the geodesic connecting them is a straight line segment in the interior. Then any triangle $\triangle pqr$ must end in a tail at r as soon as $d(p,r) + d(q,r) > 4D$, since bifurcation of geodesics can only occur at boundary points, not at p, q (unless $\triangle pqr$ is degenerate and the following conclusion is trivial). Taking $r = \gamma(s)$, as soon as $s + d(q,r) > 4D$ the continuation of the segment from q to $\gamma(s)$ by $\gamma|_{[s,\infty)}$ is a geodesic ray asymptotic to γ. Since the origin is q, that ray must coincide with σ. So for close-by interior points "eventually" means as soon as the sum of distances from their origins exceeds $4D$.

Now for arbitrary origins in X we can connect them by a curve τ through the interior, and for any two interior points of τ, we conclude that beyond the distance $2D$ from τ, the rays in the given asymptote class coincide. The limit rays at the ends then must also coincide beyond that distance. □

Theorem 4.1. *For a Jordan domain J the asymptote classes of the space of finite points X can be indentified with the points of ∂J at infinite distance. The topology on J coincides under this identification with the cone topology on $X \cup \partial X$.*

Proof. A ray in X has a unique limit point in ∂J, which by Lemma 4.1 is the same for all rays in the same asymptote class.

Suppose that $q \in \partial J$ is the end of a ray γ from $p \in X$, so q is identified with the asymptote class of γ. Let $B(\gamma(C), \epsilon)$ specify a cone neighborhood $\mathcal{N}(\gamma, C, \epsilon)$ of

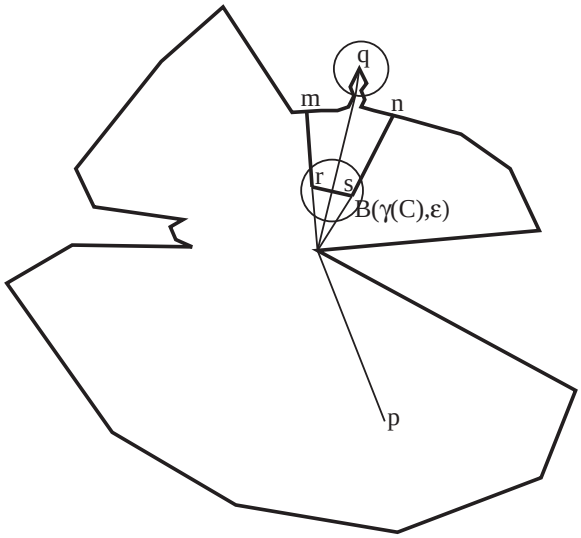

FIGURE 2. Disk inside cone neighborhood

q. Then there is a straight line segment rs contained in $B(\gamma(C), \epsilon)$, and containing $\gamma(C)$, transverse to γ. If $\gamma(C) \in \partial J$, then, say, take $r = \gamma(C)$ and rs to be a radius of a supporting half-disk at $\gamma(C)$. Otherwise rs can be in the interior of J. Let m, n be the first points in geodesic extensions of the geodesics pr, ps where those extensions intersect ∂J (so that if $r \in \partial J$ let $m = r$). Since m, n must be at finite distance from p, they are different from q. See Figure 2.

Then there is a Jordan curve obtained by chaining together geodesics mr, rs, sn with the arc of ∂J from n to m which does not contain q. A sufficiently small disk centered at q will not intersect that new Jordan curve, and any geodesic from p to a point of that disk must cross rs, so the point in the disk must be in $\mathcal{N}(\gamma, C, \epsilon)$. Thus there are neighborhoods of the induced topology on J in a cone topology neighborhood.

Conversely, suppose that U is the intersection of J with a disk D_1 centered at q, so that U is a neighborhood of the induced topology. We may assume that U does not contain p. Then there is a largest arc of ∂J containing q and contained in U. Let m, n be the ends of that arc. Since q is at infinite distance, it cannot be on the geodesic from m to n. Hence there must be a straight line segment with ends $r, s \in \partial J$ separating mn from q. See Figure 3. The arcs of ∂J in U from m to q

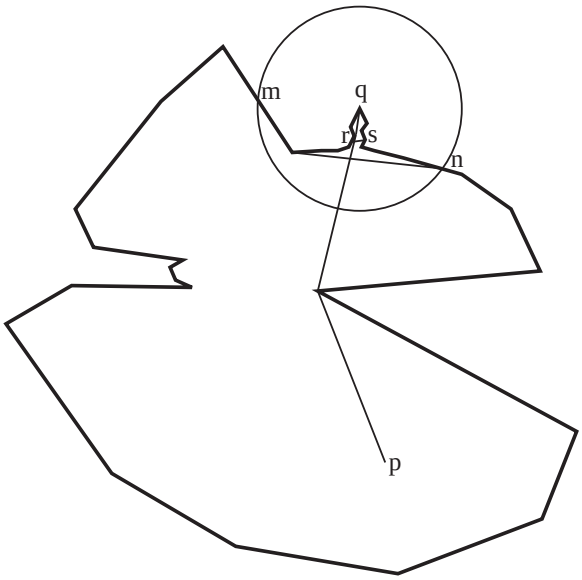

FIGURE 3. Cone neighborhood inside disk

and from n to q each must contain only one of r, s, since both of those arcs connect q to $\{m, n\}$ and cross rs. The geodesic mn together with the arc of ∂J with ends m, n not containing q is a Jordan curve having p inside and q, rs outside; then the Jordan curve consisting of rs and the arc of ∂J from r to s through q must bound a domain D_2 inside U. Any geodesic from p to a point of D_2 must cross rs. Let $\gamma(C)$ be the point where pq crosses rs. Then for sufficiently small ϵ the ball $B(\gamma(C), \epsilon)$ will be separated by rs and contained in U; we can take $\epsilon <$ the distance in $\mathbf{E}^2$ from $\gamma(C)$ to ∂D_1. Then any point of $\mathcal{N}(\gamma, C, \epsilon)$ will either be in $B(\gamma(C), \epsilon)$ or D_2, and hence in U. □

Remark 4.1. The continuity of geodesics in J as a function of their endpoints has also been addressed by Fabel ([6]). He proves that when the endpoints of a sequence converge in J, then the geodesics converge uniformly with respect to the Euclidean metric. There is also a convergence interpretation of Theorem 4.1.

Namely, it says that the convergence of endpoints is equivalent to convergence of the geodesics uniformly on bounded sets in the length metric. A related fact for CAT(0) spaces, which provides a link between the two kinds of convergence in Jordan domains, is that the distance $d(\gamma(t), \sigma(at))$ between two geodesics is a convex function of t for any $a > 0$. On a bounded interval of $\mathbf{R}$ the convergence of the end values of a convex function implies the uniform convergence on the interval. Thus Euclidean convergence of the infinite endpoints controls the unbounded ends of geodesics uniformly with respect to the Euclidean metric, and the convexity of the length metric (which majorizes the Euclidean metric) controls the bounded part. This provides the kernel of the proof that the two kinds of convergence are equivalent. However, Fabel goes on to prove a much stronger theorem, namely, the Euclidean-uniform continuity of geodesics not only as functions of the ends, but also as functions of the domain J with respect to the Hausdorff distance.

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