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NULL CURVES IN MINKOWSKI 3-SPACE

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Abstract. The purpose of this report is to give a self-contained survey on null curves in Minkowski 3-space. This report consists of two parts. In Part I of this report, we give some characterizations of null helices in terms of their associated curves. Part I includes some new results on null curves.

In part II, we give some applications of null curve theory to surface geometry in Minkowski 3-space and anti de Sitter 3-space.

Introduction

In mathematical study of relativity theory, a lightlike particle is understood as a future-pointing *null geodesic* in a spacetime, *i.e.*, a connected and time-oriented 4-dimensional Lorentzian manifold (See *eg.*, [56]). Many of the classical results from Riemannian geometry have Lorentzian counterparts. However, the presence of null curves often causes important and interesting differences between Lorentzian and Riemannian geometry.

One of the purpose of this report is to give a self-contained survey on such "different aspects" of null curves.

In Part I of this report, we shall give some characterizations of null helices in terms of their associated curves.

In part II, we shall give some applications of null curve theory to surface geometry in Minkowski 3-space and anti de Sitter 3-space.

The fundamentals of classical string theory can be summarized as follows: A *closed string* is an object γ in the physical space, that is homeomorphic to S^1. Intuitively speaking, a string evolves in time while sweeping a surface Σ, called *world sheet*, in spacetime. For physical reasons Σ is supposed to be a timelike surface. The dynamical equations for a string are defined by a variational principle: The first area variation of Σ must vanish subject to the condition that the initial and final configuration of the string are kept fixed. Hence Σ will be a timelike minimal surface having two spacelike boundary components.

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These observations motivate us to study timelike minimal surfaces in terms of *null curves.*

Part II will be devoted to a representation formula of timelike minimal surfaces in Minkowski 3-space. Moreover we shall also study timelike surface of mean curvature 1 in the anti de Sitter 3-space.

In this report, we do not discuss variational problems (or optimal control problems) on null curves. For these problems, we refer to the works [28], [51], [52] by J. D. Grant, E. Musso and L. Nicolodi.

Part I Null curves

1. Framed null curves

1.1. **Minkowski 3-space.** Let $\mathbb{E}^3_1$ be a Minkowski 3-space with natural Lorentzian metric $\langle\cdot,\cdot\rangle = -\mathrm{d}\xi_1^2 + \mathrm{d}\xi_2^2 + \mathrm{d}\xi_3^2$. A vector $\boldsymbol{v} \in \mathbb{E}^3_1$ is said to be

- *spacelike* if $\langle \boldsymbol{v}, \boldsymbol{v}\rangle > 0$ or $\boldsymbol{v} = \boldsymbol{0}$,
- *timelike* if $\langle \boldsymbol{v}, \boldsymbol{v}\rangle < 0$,
- *null* if $\langle \boldsymbol{v}, \boldsymbol{v}\rangle = 0$ and $\boldsymbol{v} \neq \boldsymbol{0}$.

Null vectors are also called *lightlike vectors.* A vector $\boldsymbol{v}$ is called a *causal vector* if it is not spacelike.

The set Λ of all null vectors:

$$\Lambda = \{\boldsymbol{v} \in \mathbb{E}^3_1 \setminus \{\boldsymbol{0}\} \mid \langle \boldsymbol{v}, \boldsymbol{v}\rangle = 0\ \}$$

is called the *lightcone* of $\mathbb{E}^3_1$. Analogously, *lightcone* $\Lambda_{\boldsymbol{p}}$ with vertex $\boldsymbol{p}$ is a quadric defined by

$$\Lambda_{\boldsymbol{p}} = \{\boldsymbol{v} \in \mathbb{E}^3_1 \setminus \{\boldsymbol{p}\} \mid \langle \boldsymbol{v}-\boldsymbol{p}, \boldsymbol{v}-\boldsymbol{p}\rangle = 0\ \}.$$

The unit timelike vector $\boldsymbol{e}_1 = (1, 0, 0)$ time-orients $\mathbb{E}^3_1$. With respect to this time-orientation, the connected component

$$\Lambda_+ = \{\boldsymbol{v} \in \Lambda \mid \langle \boldsymbol{v}, \boldsymbol{e}_1\rangle < 0\ \}$$

is called the *future cone.* The another connected component is called the *past cone* and denoted by Λ_-.

One can easily check that two null vectors $\boldsymbol{p}$ and $\boldsymbol{q}$ are in the same connected component of Λ if and only if $\langle \boldsymbol{p}, \boldsymbol{q}\rangle < 0$.

Minkowski 3-space has two kinds of quadrics other than lightcone.

The *pseudosphere* $\mathbb{S}^2_1(\boldsymbol{m}; r)$ of radius $r > 0$ centered at $\boldsymbol{m} \in \mathbb{E}^3_1$ is a quadric:

$$\mathbb{S}^2_1(\boldsymbol{m}; r) = \{\boldsymbol{v} \in \mathbb{E}^3_1 \mid \langle \boldsymbol{v}-\boldsymbol{m}, \boldsymbol{v}-\boldsymbol{m}\rangle = r^2\ \}.$$

With respect to the Lorentzian metric induced from $\mathbb{E}^3_1$, $\mathbb{S}^2_1(\boldsymbol{m}; r)$ is a Lorentz 2-manifold of constant sectional curvature $1/r^2$. Note that pseudospheres are also called *Lorentz spheres.*

The pseudosphere of radius r centered at the origin is simply denoted by $\mathbb{S}^2_1(r)$. In particular, we denote $\mathbb{S}^2_1 = \mathbb{S}^2_1(1)$. Pseudospheres are called *de Sitter* 2*-spaces* in general relativity.

On the other hand, for $r > 0$, the quardric

$$\mathbb{H}^2_0(\boldsymbol{m}; r) = \{\boldsymbol{v} \in \mathbb{E}^3_1 \mid \langle \boldsymbol{v}-\boldsymbol{m}, \boldsymbol{v}-\boldsymbol{m}\rangle = -r^2\ \}$$

is a Riemannian 2-manifold of constant sectional curvature $-1/r^2$. This quardric has two connected components

$$\begin{aligned}\mathbb{H}^2_0(\boldsymbol{m};r)^+ &= \{\boldsymbol{v} \in \mathbb{H}^2_0(\boldsymbol{m};r) \mid \langle \boldsymbol{v}, \boldsymbol{e}_1\rangle < 0\},\\ \mathbb{H}^2_0(\boldsymbol{m};r)^- &= \{\boldsymbol{v} \in \mathbb{H}^2_0(\boldsymbol{m};r) \mid \langle \boldsymbol{v}, \boldsymbol{e}_1\rangle > 0\}.\end{aligned}$$

Each connected component is a simply connected Riemannian manifold of constant curvature $-1/r^2$ and hence identified with *hyperbolic* 2-*space* of curvature $-1/r^2$.

Let W be a 2-dimensional linear subspace of $\mathbb{E}^3_1$. Then there are three mutually exclusive possibilities for W:

(1) The restriction $\langle \cdot, \cdot \rangle|_W$ of the Lorentzian metric on W is positive definite. Then W is said to be *spacelike*.
(2) $\langle \cdot, \cdot \rangle|_W$ is Lorentzian, *i.e.*, nondegenerate and of signature $(1, 1)$. Then W is *timelike*.
(3) $\langle \cdot, \cdot \rangle|_W$ is degenerate. Then W is *lightlike* (or *null*).

The type into which W falls is called the *causal character*.

Proposition 1.1. *Let W be a 2-dimensional lightlike linear subspace of $\mathbb{E}^3_1$. Then there exists a basis $\{\boldsymbol{u}, \boldsymbol{v}\}$ of W such that $\boldsymbol{u}$ is spacelike, $\boldsymbol{v}$ is null and $\langle \boldsymbol{u}, \boldsymbol{v}\rangle = 0$.*

The vector product operation $\times$ of $\mathbb{E}^3_1$ is defined by

$$\boldsymbol{a} \times \boldsymbol{b} := (a_3b_2 - a_2b_3, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1)$$

for $\boldsymbol{a} = (a_1, a_2, a_3)$ and $\boldsymbol{b} = (b_1, b_2, b_3)$.

Let us denote by $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ the natural basis of $\mathbb{E}^3_1$, *i.e.*,

$$\boldsymbol{e}_1 = (1, 0, 0),\ \boldsymbol{e}_2 = (0, 1, 0),\ \boldsymbol{e}_3 = (0, 0, 1).$$

Then we have

$$\boldsymbol{e}_1 \times \boldsymbol{e}_2 = \boldsymbol{e}_3,\ \boldsymbol{e}_2 \times \boldsymbol{e}_3 = -\boldsymbol{e}_1,\ \boldsymbol{e}_3 \times \boldsymbol{e}_1 = \boldsymbol{e}_2.$$

The vector product operation $\times$ is related to the determinant function by

$$\det(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) = \langle \boldsymbol{x} \times \boldsymbol{y}, \boldsymbol{z}\rangle.$$

Let us denote by $\mathrm{O}_1(3)$ the *semi-orthogonal group* of $\mathbb{E}^3_1$:

$$\mathrm{O}_1(3) = \{A \in \mathrm{GL}_3\mathbb{R} \mid {}^tA\ \varepsilon\ A = \varepsilon\},\ \varepsilon = \mathrm{diag}(-1, 1, 1).$$

The semi-orthogonal group $\mathrm{O}_1(3)$ has four connected components. (See [56, Chapter 9]). The identity component of $\mathrm{O}_1(3)$ is denoted by $\mathrm{SO}^+_1(3)$.

A *null frame* is an ordered triple of vectors:

$$L = (A, B, C),$$

where A and B are null vectors satisfying $\langle A, B\rangle = 1$, C is a unit spacelike vector orthogonal to the timelike plane spanned by A and B, and $\det L = \pm 1$.

To any null frame L, the *associated orthonormal frame* $\mathcal{F} = \mathcal{F}(L)$ is

$$\mathcal{F}(L) = (\frac{1}{\sqrt{2}}(A - B), \frac{1}{\sqrt{2}}(A + B), C).$$

A null frame L is called *proper* if its associated orthonormal frame $\mathcal{F} = \mathcal{F}(L)$ lies in $\mathrm{SO}^+_1(3)$.

A typical example of null frame

$$\mathcal{L} = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{1.1}$$

is associated to the natural orthonormal frame $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$. The following propositions are easily verified (See Graves [29]).

Proposition 1.2. *Let L be a null frame. Then its associated orthonormal frame is $\mathcal{F} = L \cdot \mathcal{L}^{-1}$.*

Proposition 1.3. *The set of all proper null frame is the right coset of* $\mathrm{SO}_1^+(3)$ *determined by $\mathcal{L}$.*

The Lie algebra $\mathfrak{so}_1^+(3)$ is given explicitly by

$$\mathfrak{so}_1^+(3) = \left\{ \begin{pmatrix} 0 & u & v \\ u & 0 & -w \\ v & w & 0 \end{pmatrix} \mid u, v, w \in \mathbb{R} \right\}.$$

1.2. **Null curves.** Take a curve $F = F(t)$ in $\mathrm{SO}_1^+(3)$. Then $F^{-1}F' \in \mathfrak{so}_1^+(3)$. Express $F^{-1}F'$ as

$$F^{-1}\frac{\mathrm{d}F}{\mathrm{d}t} = \begin{pmatrix} 0 & u & v \\ u & 0 & -w \\ v & w & 0 \end{pmatrix}.$$

Then $L = F \cdot \mathcal{L}$ satisfies the following formula:

$$L^{-1}\frac{\mathrm{d}L}{\mathrm{d}t} = \begin{pmatrix} k_1 & 0 & -k_3 \\ 0 & -k_1 & -k_2 \\ k_2 & k_3 & 0 \end{pmatrix}, \quad k_1 = u, \ k_2 = \frac{v+w}{\sqrt{2}}, \ k_3 = \frac{-v+w}{\sqrt{2}}.$$

Thus we have the following fundamental result:

Lemma 1.1. *Let $L = L(t)$ be a curve in* $\mathrm{GL}_3\mathbb{R}$. *Then* $L(t) \in \mathrm{SO}_1^+(3) \cdot \mathcal{L}$ *if and only if* $L(t_0) \in \mathrm{SO}_1^+(3) \cdot \mathcal{L}$ *for some t_0 and $L^{-1}dL/dt$ has the form:*

$$L^{-1}\frac{\mathrm{d}L}{\mathrm{d}t} = \begin{pmatrix} k_1 & 0 & -k_3 \\ 0 & -k_1 & -k_2 \\ k_2 & k_3 & 0 \end{pmatrix}.$$

A parametrized curve $\gamma = \gamma(t)$ is said to be a *null curve* or *lightlike curve* if its tangent vector field $\mathrm{d}\gamma/\mathrm{d}t$ is null.

A *null frame* for a null curve $\gamma(t)$ is a proper null frame field $L(t) = (A, B, C)(t)$ along γ such that $\mathrm{d}\gamma/\mathrm{d}t$ is a positive scalar multiple of $A(t)$. In such a case γ is said to be *framed by L*. A parametrized null curve (γ, L) together with a proper null frame is called a *framed null curve*.

Note that null frame fields for null curves are *not* unique. In fact, if γ is framed by $L = (A, B, C)$. Then for any positive real numbers λ, μ, $L_{\lambda,\mu} := (\lambda A, \lambda^{-1}B - \lambda\mu^2 A/2 - \mu C, C + \lambda\mu A)$ also frames γ. Therefore, the curve and a frame must be given together.

Now we describe the Frenet–Serret formula for a framed null curve.

Let $\gamma(t)$ be a framed null curve with proper frame $L = (A, B, C)$ such that $\mathrm{d}\gamma/\mathrm{d}t = k_0(t)A(t)$. Then by the Lemma 1.1, we have the following Frenet-Serret formula of (γ, L):

$$L^{-1}\frac{\mathrm{d}L}{\mathrm{d}t} = \begin{pmatrix} k_1(t) & 0 & -k_3(t) \\ 0 & -k_1(t) & -k_2(t) \\ k_2(t) & k_3(t) & 0 \end{pmatrix}. \tag{1.2}$$

Theorem 1.1. (Fundamental theorem of null curve theory)
If an initial data (p, k_0, k_1, k_2, k_3) is specified, then there exists a unique framed null curve $(\gamma(t), L(t))$ such that $\gamma(0) = p$, $\mathrm{d}\gamma/\mathrm{d}t = k_0(t)A(t)$.

Proposition 1.4. ([29, Proposition 2.5]) *A parametrized null curve $\gamma(t)$ starting at γ_0 lies in the inside of the lightcone with vertex γ_0.*

2. The pseudo-arc parameter

2.1. **Canonical parametrization.** Duggal and Bejancu [19] wrote that

"`The property` "`null`" `is unaffected by parameter changing, there are no canonical choice of the parameter –like as arclength parameter for spacelike curves or proper time parameter for timelike curve`".

We shall see in this section, one can take a canonical parameter for every null curve (See Lemma 2.1 and Remark 2.3).

Lemma 2.1. *Let $\gamma(t)$ be a parametrized null curve in $\mathbb{E}^3_1$. Then there exits a reparametrization $t = t(s)$ of γ so that $k_1 \equiv 0$.*

Proof. Let $\gamma(t)$ be a parametrized null curve framed by a proper frame (A, B, C). The null frame (A, B, C) satisfies the Frenet-Serret equation (1.2). Consider a reparametrization $t = t(s)$. Then

$$\tilde{A}(s) := \frac{\mathrm{d}\gamma}{\mathrm{d}s} = \frac{\mathrm{d}t}{\mathrm{d}s}k_0(t)A(t),\ \tilde{B}(s) := \frac{\mathrm{d}s}{\mathrm{d}t}B(t),\ \tilde{C}(s) := C(t)$$

is a null frame along $\gamma(s)$. Denote by $\{\tilde{k}_i\}$ the corresponding curvature functions of γ with respect to the new frame $(\tilde{A}, \tilde{B}, \tilde{C})$. Then we have

$$k_1(t)\frac{\mathrm{d}s}{\mathrm{d}t} = \tilde{k}_1(s)\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2 + \frac{\mathrm{d}^2 s}{\mathrm{d}t^2},$$

$$k_2(t) = \left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2 \tilde{k}_2(s), \tag{2.1}$$

$$k_3(t) = \tilde{k}_3(s)\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2. \tag{2.2}$$

Thus if we choose the reparametrization:

$$s(t) = c_1\int_{t_0}^{t}\exp\left(\int_{t_0}^{t}k_1(t)\mathrm{d}t\right)\mathrm{d}t + c_2, \quad c_1 \in \mathbb{R}^{\times},\ c_2 \in \mathbb{R}$$

then $\tilde{k}_1(s) = 0$. □

Duggal and Bejancu [19] called this parameter s a *distinguished parameter* of γ.

Remark 2.1. Graves [29] called parametrized null curves such that $k_3 = 0$ *generalized null cubics*. The transformation law (2.1)–(2.2) implies that vanishing of k_2 or k_3 is invariant under reparametrization. Thus the definition of "generalized null cubic" is independent of the choice of parametrization.

Generalized null cubics will be discussed again in Example 2.1.

Note that some authors call a null frame with $k_1 = 0$ a *Cartan frame* and a framed null curve, a *Cartan framed null curve*. (*eg.*, Bonnor [7], Duggal and Bejancu [19] and Ikawa [35]).

On the other hand, it is known that every parametrized null curve $\gamma = \gamma(t)$ is framed by a proper null frame such that the original parameter is the distinguished parameter of the framed curve. In fact, the following result is obtained by K. Honda and the first named author:

Proposition 2.1. ([32]) *Let $\gamma = \gamma(s)$ be a parametrized null curve which is not a geodesic. Then there exists uniquely a proper null frame $F = (A, B, C)$ such that*

$$F^{-1}\frac{\mathrm{d}F}{\mathrm{d}s} = \begin{pmatrix} 0 & 0 & -k_3 \\ 0 & 0 & -k_2 \\ k_2 & k_3 & 0 \end{pmatrix}, \quad A = \frac{\mathrm{d}\gamma}{\mathrm{d}s}.$$

Proof. Put $A(s) = \gamma'$. Then from the assumption, γ' and γ'' are linearly independent and hence γ'' is spacelike, *i.e.*, $\langle \gamma'', \gamma'' \rangle > 0$. Because $\langle \gamma', \gamma'' \rangle = 0$. Thus there exists a unique section (vector field) $B(s)$ of the orthogonal compliment $\gamma''(s)^{\perp}$ of $\gamma''(s)$ such that

$$\langle A(s), B(s) \rangle = 1, \quad \langle B(s), B(s) \rangle = 0.$$

We define the vector field $C(s)$ along γ by $C(s) = A(s) \times B(s)$. Then we have $\langle A(s), C(s) \rangle = \langle B(s), C(s) \rangle = 0$ and $\langle C(s), C(s) \rangle = 1$.

Moreover, there exist functions $k_2(s)$ and $k_3(s)$ which satisfy

$$A'(s) = \gamma''(s) = k_2(s)C(s), \quad B'(s) = k_3(s)C(s).$$

Uniqueness follows from the construction we have done.@□

Note that one can reparametrize γ further so that γ'' is a unit spacelike vector field. Later, this reparametrization will be discussed again.

Corollary 2.1. *Let γ be a null curve parametrized by the distinguished parameter s which is framed by $F = (A, B, C)$. Then*

$$F^{-1}\frac{\mathrm{d}F}{\mathrm{d}s} = \begin{pmatrix} 0 & 0 & -k_3 \\ 0 & 0 & -k_2 \\ k_2 & k_3 & 0 \end{pmatrix}.$$

We call the vector field B the *binormal vector field*. The vector field C will be called the *principal normal vector field* of γ.

Hereafter we call a framed null curve parametrized by the distinguished parameter, a *null Frenet curve*.

One can easily see that null curves parametrized by the distinguished parameter with nonzero constant k_2 and $k_3 = 0$ are cubic curves. Based on this fact, Graves called such curves generalized null cubics. As we mentioned before, the notion of generalized null cubic does not depend on parametrization.

Example 2.1. (*cf.* [53]) Let ϕ and ψ be functions satisfying $\phi' = (\psi')^2$. Then the curve

$$\gamma(s) = \left(\frac{1}{\sqrt{2}} \left(s + \frac{\phi(s)}{2} \right), \frac{1}{\sqrt{2}} \left(s - \frac{\phi(s)}{2} \right), \psi(s) \right)$$

is a null Frenet curve with frame

$$A = \left(\frac{1}{\sqrt{2}} \left(1 + \frac{\phi'(s)}{2} \right), \frac{1}{\sqrt{2}} \left(1 - \frac{\phi'(s)}{2} \right), \psi'(s) \right),$$

$$B = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right), \quad C = \left(\frac{\psi'(s)}{\sqrt{2}}, -\frac{\psi'(s)}{\sqrt{2}}, 1 \right).$$

Direct computation shows that the curvature and torsion of γ are

$$k_2(s) = \psi''(s), \ k_3 = 0.$$

Thus γ is a generalized null cubic. In particular, in case that k_2 is a nonzero constant, then

$$\phi(s) = \frac{k_2^2}{3} s^3 + ak_2 s^2 + a^2 s + c, \quad \psi(s) = \frac{k_2}{2} s^2 + as + b,$$

where a, b, c are constants.

In case that k_2 is constant, clearly γ is a cubic curve.

Remark 2.2. In p. 66 of [19], Duggal and Bejancu claimed that $\cdots$ "`locally, for any null curves of a 3-dimensional Lorentzian manifold we find a Cartan frame such that it is a generalized null cubic`".

They considered the following procedure:

Define a new frame field $\bar{F} = (\bar{A}(s), \bar{B}(s), \bar{C}(s))$ by

$$\bar{A}(s) := A(s), \ \bar{B}(s) := -\frac{f(s)}{2} A(s) + B(s) + f(s)C(s), \ \bar{C}(s) := C(s) - f(s)A(s),$$

where f is a solution to

$$\frac{\mathrm{d}f}{\mathrm{d}s} + \frac{k_2(s)}{2} f(s)^2 + k_3(s) = 0.$$

Then the new frame has zero torsion $\overline{k}_3 = 0$. However this frame $\bar{F}$ is not a Cartan frame because of uniqueness of Cartan frame. We can check that $\bar{F}$ is not the Cartan frame by straightforward computation. In fact, the new frame satisfies

$$\frac{\mathrm{d}}{\mathrm{d}s}(\bar{A}, \bar{B}, \bar{C}) = (\bar{A}, \bar{B}, \bar{C}) \begin{pmatrix} \bar{k}_1 & 0 & -\bar{k}_3 \\ 0 & -\bar{k}_1 & -\bar{k}_2 \\ \bar{k}_2 & \bar{k}_3 & 0 \end{pmatrix}$$

with

$$\bar{k}_1 = f(s)k_2(s), \quad \bar{k}_2(s) = k_2(s), \quad \bar{k}_3 = 0.$$

Thus $(\gamma, \bar{F})$ is not a generalized null cubic.

For a null curve $\gamma(t)$ parametrized by the distinguished parameter t such that $\langle \gamma'', \gamma'' \rangle > 0$, we can reparametrize $t = t(s)$ so that $\langle \gamma_{ss}, \gamma_{ss} \rangle = 1$. Then γ is framed by $F = (A, B, C)$ such that

$$A = \gamma_s = \frac{\mathrm{d}\gamma}{\mathrm{d}s}, \quad C = \gamma_{ss} = \frac{\mathrm{d}^2\gamma}{\mathrm{d}s^2}.$$

With respect to this parameter and frame, γ obeys the following Frenet-Serret formula:

$$F^{-1}\frac{\mathrm{d}F}{\mathrm{d}s} = \begin{pmatrix} 0 & 0 & -k \\ 0 & 0 & -1 \\ 1 & k & 0 \end{pmatrix}. \tag{2.3}$$

The parameter s is called the *pseudo-arc parameter* [22]. We call the function k the *lightlike curvature* of γ. The lightlike curvature is Lorentzian invariant (See [22, Corollary 3.4]).

This reparametrization is most economical way for the study of null curves. In fact, null curves are completely determined only by its lightlike curvature up to Lorentz transformations. Thus this representation may be considered as a "canonical representation" for null curves.

Remark 2.3. The lightlike curvature k is called *null Cartan curvature* in [22]. Null Frenet curves parametrized by the pseudo-arc parameter are called *null Cartan curves* in [22]. Clearly, by definition, null geodesics can not be null Cartan curves in the sense of [22].

2.2. **Null helices.** A curve in Euclidean 3-space is said to be a *helix* if its curvature and torsion are non-zero constants.

More generally, a curve in $\mathbb{E}^3$ is said to be a *generalized helix* if the tangent vector field of γ has constant angle with a fixed direction (called the *axis* of the curve). Generalized helices are also called *curves of constant slope*, *cylindrical helices* ([55]) or *general helices*.

In classical differential geometry of spatial curves, the following result is known (see *eg.*,[20], [44], [66]):

Theorem 2.1. (Bertrand-Lancret-de Saint Venant) *A curve $\gamma(s)$ in Euclidean 3-space is a generalized helix if and only if its ratio of curvature and torsion is constant.*

For historical review of this classical result and related topics, we refer to [64] and [66].

Now we consider such curves in null curve geometry.

Definition 2.1. ([22, Definition 3.5]) A null Frenet curve parametrized by the pseudo-arc parameter is said to be a *null helix* if its lightlike curvature is constant.

Note that null Frenet curves with constant curvature and torsion have been called "null helices" by some authors ([7], [35] etc.). Moreover, lightlike analogue of generalized helices (or constant slope curves) is introduced and studied independently by H. Balgetir, M. Bektaş, M. Eugüt [3], [6] and B. Şahin, E. Kiliç, R. Güneş [62].

A null Frenet curve $\gamma(t)$ parametrized by the distinguished parameter t is said to be a null *generalized helix* or null *general helix* if there exists a vector $\boldsymbol{q} \in \mathbb{E}^3_1$ such that $\langle \gamma_t, \boldsymbol{q} \rangle$ is constant. The line spanned by $\boldsymbol{q}$ is called the *axis* of the generalized helix.

The above definition is a slight modification of that of generalized helix in $\mathbb{E}^3$.

Lemma 2.2. *A non-geodesic null Frenet curve is a null generalized helix if and only if its slope k_3/k_2 is constant.*

Proof. Let us express $\boldsymbol{q}$ as

$$\boldsymbol{q} = q_A A + q_B B + q_C C.$$

Then the coefficients are given by

$$q_A = \langle \boldsymbol{q}, B\rangle, \ q_B = \langle \boldsymbol{q}, A\rangle, \ q_C = \langle \boldsymbol{q}, C\rangle.$$

By the assumption, q_B is a constant. Differentiating these coefficients, we get

$$q_A' = k_3\ q_C, \quad q_B' = k_2\ q_C, \quad q_C' = -k_3\ q_B - k_2 > q_A.$$

From these, we notice that q_A is constant and so is the slope $k_3/k_2 = -q_A/q_B$.

Conversely, if k_3/k_2 is constant, then define a vector $\boldsymbol{q}$ by $\boldsymbol{q} := A + (k_3/k_2)B$. Then $\langle A, \boldsymbol{q}\rangle$ is constant. □

Remark 2.4. Lemma 2.2 was claimed in [6] and [62].

Note that, precisely speaking, the statement in [6] is inaccurate. In fact, the authors claimed that their results hold for null generalized helices in general Lorentzian 3-manifolds. But, obviously, their results hold only for null generalized helices in Minkowski 3-space. In fact, they used linear space structure of the ambient space.

Under the pseudo-arc parametrization, the following result is obtained:

Corollary 2.2. ([23, Proposition 6]) *A null Frenet curve parametrized by the pseudo-arc parameter is a null generalized helix if and only if it is a null helix.*

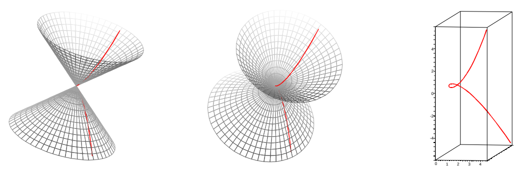
Example 2.2. ([23]) The null helices parametrized by pseudo-arc parameter are congruent to one of the following:

(1)
$$\left(-\frac{s}{\sigma}, \frac{1}{\sigma^2}\sin(\sigma s), \frac{1}{\sigma^2}\cos(\sigma s)\right), \ k = \sigma^2/2 > 0;$$

(2)
$$\left(\frac{1}{\sigma^2}\sinh(\sigma s), \frac{1}{\sigma^2}\cosh(\sigma s), -\frac{s}{\sigma}\right), \ k = -\sigma^2/2 > 0;$$

(3)
$$\left(\frac{s^3}{4} + \frac{s}{3}, \frac{s^2}{2}, \frac{s^3}{4} - \frac{s}{3}\right), \ k = 0;$$

Note that, under the pseudo-arc parametrization, generalized null cubics are represented as null helices of zero lightlike curvature. Moreover, such curves are unique up to Lorentz transformation. Null helices of zero lightlike curvature are cubic curves with respect to the pseudo-arc. We call null helices of zero lightlike curvature, *null cubics*.



(a) (b) (c)

FIGURE 1. Null cubics

Remark 2.5. Two generalizations of generalized helices in $\mathbb{E}^3$ to general Riemannian 3-manifolds are known. Now let (M, g) be a Riemannian 3-manifold and $\gamma(s)$ a curve in M parametrized by arclength. Then γ is said to be a *Frenet curve* (of osculating order 3) if there exists an orthonormal frame field $\mathcal{E} = \{E_1 = \gamma', E_2, E_3\}$ along γ such that $\mathcal{E}$ satisfies the Frenet-Serret equation:

$$\nabla_{E_1}\mathcal{E} = \mathcal{E}\begin{pmatrix} 0 & -k_2 & 0 \\ k_2 & 0 & -k_3 \\ 0 & k_3 & 0 \end{pmatrix}$$

for some functions $k_2 \geq 0$ and k_3. The functions k_2 and k_3 are called the *curvature* and *torsion* of γ, respectively.

(1) A curve $\gamma(s)$ is said to be a *generalized helix* if there exists a parallel vector field ξ on M such that $g(\xi, \gamma')$ is a constant.
(2) A curve $\gamma(s)$ is said to be a *generalized helix* if there exists a Killing vector field ξ on M such that $g(\xi, \gamma')$ is a constant.

M. Barros [5] used the second idea to generalize Bertrand-Lancret-de Saint Venant Theorem to curves in Riemannian 3-space forms and Lorentzian 3-space forms. If the ambient space M is of constant curvature the following result is known.

Lemma 2.3 ([5]). *Let M be a complete and simply connected Riemannian 3-manifold of constant curvature and γ a curve in M parametrized by arclength. A vector field ξ along γ is a Killing vector field aling γ if and only if it extends to a Killing vector field of M.*

The generalized helices in Riemannian 3-space forms are classified as follows [5] (see also [2]):

Theorem 2.2. *A curve γ in hyperbolic 3-space $\mathbb{H}^3$ is a generalized helix if and only if either*

- *$k_3 = 0$ and γ is a curve in some hyperbolic 2-space, or*
- *γ is a helix in $\mathbb{H}^3$.*

Theorem 2.3. *A curve γ in the unit 3-sphere $\mathbb{S}^3$ is a generalized helix if and only if either*

- *$k_3 = 0$ and γ is a curve in some 2-sphere, or*
- *there exists a constant b such that $k_3 \pm 1 = bk_2$.*

Corresponding results for 3-dimensional Lorentzian space forms are obtained by A. Ferrández [21].

The unit 3-sphere $\mathbb{S}^3$ is a typical example of contact Riemannian 3-manifold, especially, a Sasakian 3-manifold. On a Sasakian 3-manifold M, the Reeb vector field ξ is a unit Killing vector field. One can obtain the following Bertrand-Lancret-de Saint Venant type theorem for Sasakian 3-manifolds (see also [11]).

Theorem 2.4. ([10]) *A non-geodesic curve in a Sasakian 3-manifold parametrized by arclength makes constant angle with the Reeb vector field if and only if the ratio of $k_3 \pm 1$ and k_2 is constant.*

J. T. Cho and J. E. Lee [12] studied generalized helices in pseduo-Hermitian geometry.

3. Associated curves

3.1. Let $\gamma = \gamma(s)$ be a null Frenet curve parametrized by the pseudo-arc parameter s with frame (A, B, C). Take three functions u, v, w and define a new curve $\bar{\gamma}(\bar{s})$ by

$$\bar{\gamma}(\bar{s}) := \gamma(s) + u(s)A(s) + v(s)B(s) + w(s)C(s).$$

Here $\bar{s} = \bar{s}(s)$ is a function of s such that the new curve is also a null Frenet curve parametrized by the pseudo-arc parameter $\bar{s}$. The new null Frenet curve $\bar{\gamma}$ is called an *associated null Frenet curve* of γ with reference coordinate (u, v, w). Differentiating $\bar{\gamma}$, we have

$$\frac{\mathrm{d}\bar{s}}{\mathrm{d}s}\bar{A}(\bar{s}) = (1 + u' - kw)A + (v' - w)B + (w' + u + kv)C. \tag{3.1}$$

Since $\bar{A}$ is null, we have

$$(w' + u + kv)^2 + 2(1 + u' - kw)(v' - w) = 0. \tag{3.2}$$

As applications of (3.1)-(3.2), one can study null Frenet curves which admit associated null Frenet curves.

Proposition 3.1. ([33]) *Let $\gamma(s)$ be a null Frenet curve parametrized by the pseudo-arc parameter s and u, v, w functions of s Then $\gamma(s) + u(s)A(s) + v(s)B(s) + w(s)C(s)$ is a constant vector if and only if*

$$\begin{aligned} 1 + u' - kw &= 0, \\ v' - w &= 0, \\ w' + u + kv &= 0. \end{aligned} \tag{3.3}$$

This condition is called the Cesàro's fixed point condition.

Associated curves to null Frenet curves are studied recently by Balgetir, Bektaş, K. Honda and the first named author.

3.2. First, let us consider a pair of null curves which possess common principal normal direction.

J. Bertrand studied curves in Euclidean 3-space $\mathbb{E}^3$ whose principal normals are the principal normals of another curves. Such a curve is nowadays called a *Bertrand curve*. Bertrand curves are characterized as follows (See [20, p. 41]):

Proposition 3.2. *Let γ be a curve in Euclidean 3-space parametrized by the arclength. Then γ is a Bertrand curve if and only if γ is a plane curve or curves whose curvature κ and torsion τ are in linear relation:*

$$\mu\ \kappa + \nu\ \tau = 1$$

for some constant μ and ν. The product of torsions of Bertrand pair is constant.

Remark 3.1. Bertrand mates in $\mathbb{E}^3$ are particular examples of offset curves used in computer-aided geometric design (CAGD). See [54]. Bertrand curves and their geodesic imbedding in surfaces are recently rediscovered and studied in the context of modern soliton theory by Schief [63].

Now we study Bertrand property for null Frenet curves.

Definition 3.1. Let $\gamma(s)$ and $\bar{\gamma}(\bar{s})$ be null Frenet curves parametrized by the pseudo-arc parameter. Then the pair $(\gamma, \bar{\gamma})$ is said to be a null *Bertrand pair* if their principal normal vector fields C and $\bar{C}$ are linearly dependent.

The curve $\bar{\gamma}$ is called a *Bertrand mate* of γ and vice versa. A null Frenet curve is said to be a null *Bertrand curve* if it admits a Bertrand mate.

By definition, for a null Bertrand pair $(\gamma, \overline{\gamma})$, there exists a functional relation $\bar{s} = \bar{s}(s)$ such that

$$\overline{C}(\bar{s}(s)) = \epsilon\ C(s),\ \epsilon = \pm 1.$$

While there exists many Bertrand curves in $\mathbb{E}^3$, there are no nontrivial null Bertrand pairs.

Theorem 3.1. *Let γ be a null Frenet curve parametrized by the pseudo-arc. If γ admits a Bertrand mate then γ and its Bertrand mate have the same nonzero constant lightlike curvatures. Moreover the Bertrand mate is congruent to the original curve.*

Proof. Let $(\gamma, \overline{\gamma})$ be a null Bertrand pair. Then $\overline{\gamma}$ can be expressed as

$$\overline{\gamma}(\bar{s}(s)) := \gamma(s) + r(s)C(s) \tag{3.4}$$

for some function $r(s) \neq 0$ and some parametrization $\bar{s} = \bar{s}(s)$ with respect to the pseudo-arc parameter s of γ. Differentiating (3.4) with respect to s,

$$\overline{A}\ \frac{\mathrm{d}\bar{s}}{\mathrm{d}s} = A + r'\ C + rC'. \tag{3.5}$$

Here $\bar{s}$ is the pseudo-arc parameter of $\overline{\alpha}$. By using the Frenet-Serret formula (2.3), we have

$$\overline{A}\ \frac{\mathrm{d}\bar{s}}{\mathrm{d}s} = (1 - rk)A - rB + r'\ C. \tag{3.6}$$

Since $\overline{A}$ is null,

$$(r')^2 = 2r(1 - rk). \tag{3.7}$$

Next, since $\overline{\gamma}$ is a Bertrand mate of γ, $\overline{C}$ and C are linearly dependent, thus $\langle \overline{A}, C \rangle = 0$, hence r is a constant.

From (3.7), we conclude that $k = 1/r =$ nonzero constant. Hence γ is a null helix of nonzero lightlike curvature.

We investigate the Bertrand mate of γ in more detail. By (3.5), we notice that

$$\overline{A}(\bar{s}(s)) = \mu\ B(s), \quad \mu(s) = -r\ \frac{\mathrm{d}s}{\mathrm{d}\bar{s}} \neq 0. \tag{3.8}$$

This equation implies that

$$\overline{B}(\bar{s}(s)) = \mu(s)^{-1}A(s),\ \overline{C}(\bar{s}(s)) = -C(s).$$

Differentiating (3.8) with respect to s and using Frenet-Serret formula again, we obtain

$$\frac{\mathrm{d}\bar{s}}{\mathrm{d}s}\ \frac{\mathrm{d}\overline{A}}{\mathrm{d}\bar{s}} = \frac{\mathrm{d}\mu}{\mathrm{d}s}B + \mu\ k\ C.$$

This formula implies μ is constant and $d\bar{s}/ds = \pm 1$. Hence $\mu = \pm r$ and hence the Frenet frame of $\overline{\gamma}$ is given by $(\pm rB, \pm r^{-1}A, -C)$. Thus $\overline{\gamma}$ has constant lightlike curvature $1/r$ and hence congruent to γ.

Conversely let γ be a null Frenet curve with constant lightlike curvature $k = 1/r \neq 0$. Then

$$\overline{\gamma}(s) := \gamma(s) + rC(s)$$

is a null Frenet curve with pseudo-arc parameter s and framed by

$$\overline{A}(s) = -rB(s),\ \overline{B}(s) = -r^{-1}A(s),\ \overline{C}(s) = -C(s).$$

Thus $(\gamma, \overline{\gamma})$ is a Bertrand pair and $\overline{\gamma}$ has constant lightlike curvature $1/r$.

This completes the proof. $\square$

Theorem 3.1 implies that the Bertrand property characterizes null helix of nonzero lightlike curvature. Theorem 3.1 was essentially stated in [4].

Here we confirm that the Bertrand mate of a null helix of nonzero lightlike curvature is congruent to the original curve.

Example 3.1. Let $\gamma(s)$ be a null helix of lightlike curvature $1/2$:

$$\gamma(s) = (s, \cos s, \sin s)$$

with Frenet frame

$$A = (1, -\sin s, \cos s),\ B = (-1, -\sin s, \cos s)/2,\ C = (0, -\cos s, -\sin s).$$

Define $\bar{\gamma}$ by $\bar{\gamma} = \gamma + 2C$ with $\bar{s} = s$. Then

$$\bar{\gamma}(s) = (s, -\cos s, -\sin s).$$

Obviously $\bar{\gamma}$ is congruent to the original curve γ.

The case $k > 0$ can be checked in much the same way.

3.3. Next, we study pairs of null Frenet curves which possess common binormal direction. In $\mathbb{E}^3$, the following characterization theorem is classical.

Proposition 3.3. (*cf.* [30, p. 161, Ex. 14]) *Let $\gamma(s)$ be a curve parametrized by arc length parameter in $\mathbb{E}^3$. Assume that there exists a curve $\bar{\gamma}$ parametrized by arc length parameter such that the binormal direction of $\bar{\gamma}$ coincides with that of γ. Then both curves are plane curves.*

In null curve case, the following result is obtained.

Theorem 3.2. ([32])

(1) *Let $\gamma(s)$ be a null Frenet curve parametrized by pseudo-arc parameter. Assume that there exists a null Frenet curve $\bar{\gamma}$ parametrized by pseudo-arc parameter such that the binormal direction of $\bar{\gamma}$ coincides with that of γ. Then at the corresponding point, the lightlike curvatures of γ and $\bar{\gamma}$ coincide.*
(2) *Let $\gamma(s)$ be a null Frenet curve parametrized by pseudo-arc. Then there exists a null Frenet curve $\bar{\gamma}(\bar{s})$ parametrized by pseudo-arc such that the binormal directions of γ and $\bar{\gamma}$ coincide and $k(s) = \bar{k}(\bar{s})$.*

Proof.

(1) Let $\gamma(s)$ be a null Frenet curve parametrized by the pseudo-arc parameter s. Assume that there exists a null Frenet curve $\bar{\gamma}(\bar{s})$ parametrized by the pseudo-arc parameter $\bar{s}$ such that the binormal directions γ coincides with that of $\bar{\gamma}$. Then $\bar{\gamma}$ can be parametrized as

$$\bar{\gamma}(\bar{s}(s)) = \gamma(s) + v(s)B(s) \tag{3.9}$$

for some function $v(s) \neq 0$ and some parametrization $\bar{s} = \bar{s}(s)$. Hence $\bar{\gamma}$ is an associated curve to γ with reference coordinates $(0, v(s), 0)$.

Without loss of generality, we may assume that

$$\overline{B}(\bar{s}(s)) = a(s)B(s), \tag{3.10}$$

for some function $a(s) \neq 0$. Then by (3.1), we have

$$\frac{\mathrm{d}\bar{s}}{\mathrm{d}s}\bar{A}(\bar{s}) = A(s) + v'(s)B(s) + k(s)v(s)C(s). \tag{3.11}$$

Next, (3.2) together with $\langle \overline{A}, \overline{B} \rangle = 1$ implies

$$2v'(s) + \{v(s)k(s)\}^2 = 0 \tag{3.12}$$

and

$$a(s) = \frac{\mathrm{d}\bar{s}}{\mathrm{d}s}.$$

Thus, from (3.12), the function $v(s)$ is completely determined by the lightlike curvature;

$$\frac{1}{v(s)} = \frac{1}{2}\int k(s)^2 \mathrm{d}s + c$$

for some constant c. Note that $C \times B = -B$. The principal normal $\overline{C}$ of $\bar{\gamma}$ is given by

$$\overline{C}(\bar{s}(s)) = \overline{A}(\bar{s}(s)) \times \overline{B}(\bar{s}(s)) = C(s) - v(s)k(s)B(s). \tag{3.13}$$

Differentiating (3.10) with respect to s and using (3.13), we obtain

$$a(s)\bar{k}(\bar{s}(s))\{C(s) - v(s)k(s)B(s)\} = \frac{\mathrm{d}a}{\mathrm{d}s}(s)B(s) + a(s)k(s)C(s).$$

Comparing the both sides of this equation, we obtain

$$\bar{k}(\bar{s}(s)) = k(s), \ \frac{\mathrm{d}a}{\mathrm{d}s} = -a(s)v(s)k(s)^2. \tag{3.14}$$

From this we have

$$a(s) = a_0 v(s)^2, \ a_0 \in \mathbb{R}^{\times}.$$

Thus the lightlike curvature at the corresponding points coincide.

Differentiating (3.13), we get

$$a(s)^2 = v(s)k'(s).$$

(2) For every null Frenet curve γ parametrized by pseudo-arc, define a new curve $\bar{\gamma}(\bar{s})$ by

$$\bar{\gamma}(\bar{s}(s)) := \gamma(s) + v(s)B(s), \quad 1/v(s) = \frac{1}{2}\int k(s)^2 \ \mathrm{d}s + c. \tag{3.15}$$

where c is a nonzero constant. Take a nonzero constant a_0 and define a function $\bar{s}$ by

$$\bar{s}(s) := a_0 \int v(s)^2 \ \mathrm{d}s.$$

Then $\bar{\gamma}$ is a null Frenet curve parametrized by the pseudo-arc $\bar{s}$ framed by

$$\overline{A} = \frac{\mathrm{d}}{\mathrm{d}\bar{s}}\bar{\gamma}, \ \overline{B} = \frac{\mathrm{d}\bar{s}}{\mathrm{d}s}B, \ \overline{C} = \overline{A} \times \overline{B}.$$

Clearly these two curves have common binormal directions. $\square$

Theorem 3.2 implies the following characterization of null cubics (null helices of zero lightlike curvature).

Corollary 3.1. *Let γ be a null Frenet curve parametrized by pseudo-arc parameter. Assume that there exists a null Frenet curve $\bar{\gamma}(\bar{s})$ parametrized by pseudo-arc such that the binormal directions of γ and $\bar{\gamma}$ coincide and $\langle \bar{\gamma} - \gamma, A \rangle$ is a constant. Then γ is a null cubic.*

Remark 3.2. Theorem 3.2 seems to be better comparing with Bäcklund transformations for constant torsion curves:

Theorem 3.3. ([8]) *Let $\gamma(s)$ be a unit speed curve in $\mathbb{E}^3$ with non-zero constant torsion τ and Frenet frame (T, N, B). Then, for any constant λ and a solution β to the ordinary differential equation:*

$$\frac{\mathrm{d}\beta}{\mathrm{d}s} = \lambda \sin\beta - \kappa,$$

the curve $\bar{\gamma}(s)$ defined by

$$\bar{\gamma}(s) := \gamma(s) + \frac{2\lambda}{\lambda^2 + \tau^2}(\cos\beta T + \sin\beta N)$$

is a curve of same constant torsion with arclength parameter s.

The new curve $\bar{\gamma}$ is called the *Bäcklund transform* of γ. Note that nonzero constant torsion curves are typical examples of Bertrand curves. Bäcklund transformations of nonzero constant torsion curves can be generalized to Bertrand curves. See [63].

3.4. Finally we study Cesàro's fixed point condition. Here we recall the following classical results.

Proposition 3.4. (*cf.* [9], [57]) *Let $\gamma(s)$ be a curve parametrized by the arclength parameter s in Euclidean 3-space $\mathbb{E}^3$, which is not a straight line. If there exists a point of $\mathbb{E}^3$ such that all the rectifying planes pass through the point. Then the slope τ/κ of γ is a linear function of s.*

Such curves are called *rectifying curves* by B. Y. Chen [9].

Proposition 3.5. (*cf.* [57]) *Let $\gamma(s)$ be a curve in $\mathbb{E}^3$ parametrized by arclength, which is not a straight line. If there exists a point of $\mathbb{E}^3$ such that all the osculating planes pass through the point. Then the curve is planar.*

Proposition 3.6. (*cf.* [57]) *Let $\gamma(s)$ be a curve in $\mathbb{E}^3$ parametrized by arclength, which is not a straight line. If there exists a point of $\mathbb{E}^3$ such that all the normal planes pass through the point, then the curve is spherical.*

In the case of null Frenet curves, the following result is obtained for rectifying curves.

Proposition 3.7. ([33]) *Let γ be a null Frenet curve parametrized by pseudo-arc parameter s. Assume that all the rectifying planes of γ pass through a fixed point. Then the lightlike curvature k of γ is a linear function of s.*

Proof. Let $\gamma(s)$ be a null Frenet curve with frame (A, B, C) parametrized by pseudo-arc and P a fixed point which all the rectifying planes of γ pass through. Then the position vector $\boldsymbol{p}$ of P (with respect to the origin) is represented as

$$\boldsymbol{p} = \gamma(s) + u(s)A(s) + v(s)B(s). \tag{3.16}$$

for some functions $u(s) \neq 0$ and $v(s)$.

Since $\boldsymbol{p}$ is a constant vector, the coefficient functions satisfy the Cesàro's fixed point condition:

$$u' = -1, \quad v' = 0, \quad u + kv = 0$$

From these, we notice that v is a constant, say b. Next, u is a linear function and expressed as $u(s) = -s + a$, where a is a constant. Hence the lightlike curvature must satisfy

$$(-s + a) + bk(s) = 0.$$

This implies that $b \neq 0$ and

$$k(s) = \frac{1}{b}s - \frac{a}{b}.$$

Thus k is a linear function of s.

Conversely, assume that k is a linear function of s, say, $k(s) = s/b - a/b, \ b \neq 0$. Then $\bar{\gamma} = \gamma + (-s + a)A + bB$ is a constant vector. □

Remark 3.3. Recently Chen [9] obtained the following result:

All the rectifying curves in Euclidean 3*-space are congruent to*

$$\gamma(t) = a(\sec t)\alpha(t), \ a > 0$$

where $\alpha(t)$ is a unit speed curve in the unit 2*-sphere $\mathbb{S}^2$ centered at the origin and a is a constant.*

It is easy to establish corresponding results for timelike curves or spacelike curves with non-null principal normal in Minkowski 3-space. For timelike case, α is a unit speed curve in the hyperbolic 2-space $\mathbb{H}^2$. Similarly, α is a unit speed curve in pseudosphere $\mathbb{S}^2_1$ for spacelike rectifying curves.

Here we give a new result on rectifying null Frenet curves.

Proposition 3.8. *Let γ be a null Frenet curve parametrized by pseudo-arc parameter s. If γ is a rectifying curve, then*

1. *$\langle \gamma, \gamma \rangle = c_1 s + c_2$ for some constants c_1 and c_2.*
2. *The binormal coefficient $\langle \gamma, A \rangle$ of the position vector of γ is a nonzero constant.*
3. *The tangential coefficient $\langle \gamma, B \rangle$ of the position vector of γ is $s + b$ for some constant b.*

   Conversely, if γ satisfies any of (1), (2) or (3), then it is a rectifying curve.

Proof. Let γ be a rectifying null Frenet curve. Then γ is written as

$$\gamma(s) = u(s)A(s) + v(s)B(s).$$

Hence $\gamma' = u'A + v'B + (u + vk)C$ and $\gamma' = A$, where k is the lightlike curvature. Thus

$$u' = 1, \quad v' = 0, \quad u + vk = 0.$$

Namely, we have

$$u = s + b, \quad v = a, \quad k = -\frac{s}{a}$$

for some constants $a \neq 0$ and b. Hence

$$\gamma(s) = (s + b)A(s) + aB(s).$$

From this equation, we get

$$\langle \gamma, \gamma \rangle = 2a(s + b).$$

Thus (1) is satisfied. The tangential coefficient u and binormal coefficient v of γ are

$$u(s) = \langle \gamma', B \rangle = s + b, \quad v(s) = \langle \gamma', A \rangle = a.$$

Thus γ satisfies (2) and (3).

Conversely, assume that γ satisfies the condition (2), *i.e.*, $\langle \gamma, A \rangle = c$ for some constant c. Then $\langle \gamma, A' \rangle = 0$. This implies that γ is a rectifying curve since $A' = C$. Note that the condition (1) implies (2). Now, assume that γ satisfies the condition (3), *i.e.*, $\langle \gamma, B \rangle = s + b$ for some constant b. Then $\langle \gamma, B' \rangle = 0$. This also implies that γ is a rectifying curve, since $B' = \kappa C$. Here, κ is the lightlike curvature of γ. □

Theorem 3.4. *Let γ be a rectifying null Frenet curve in $\mathbb{E}^3_1$ parametrized by pseudo-arc parameter s. If γ satisfies $\langle \gamma, \gamma \rangle > 0$, then γ is given by*

$$\gamma(s) = \sqrt{2as}\ \varphi(s),$$

where $\varphi(s)$ is a timelike curve in the pseudosphere $\mathbb{S}^2_1$. By the reparametrization by the proper time t, γ is rewritten as

$$\gamma(t) = \sqrt{2a}\ e^t\ \hat{\varphi}(t),$$

where $\hat{\varphi}(t) = \varphi(e^t)$ is a unit speed timelike curve in $\mathbb{S}^2_1$.

Conversely, let γ be a null Frenet curve defined by

$$\gamma(s) = \sqrt{2as}\ \varphi(s)$$

for a positive number a and a timelike curve φ in $\mathbb{S}^2_1$ with $\langle \varphi', \varphi' \rangle = -1/(4s^2)$ and $\langle \varphi'', \varphi'' \rangle = 1/(2as)$. Then γ is a rectifying null Frenet curve parametrized by the pseudo-arc parameter s with $\langle \gamma, \gamma \rangle > 0$.

Proof. Assume that γ is a null rectifying curve in $\mathbb{E}^3_1$. Then, by Proposition 3.8,

$$\gamma(s) = (s + b)A + aB$$

for some constants $a \neq 0$ and b. Without loss of generality, we may assume that $b = 0$, *i.e.*,

$$\gamma(s) = sA + aB.$$

Hereafter we may study only the case $a > 0$. In fact, the other case $a < 0$ can be discussed in much the same way. Under this assumption, the curve have two components. $\gamma^- : (-\beta, 0) \to \mathbb{E}^3_1$ and $\gamma^+ : (0, \beta) \to \mathbb{E}^3_1$. The component γ_+ satisfies $\langle \gamma_+(s), \gamma(s)_+ \rangle = 2as > 0$. Define $\varphi_+ : (0, \beta) \longrightarrow \mathbb{E}^3_1$ by

$$\varphi_+(s) := \frac{\gamma(s)}{\sqrt{2as}}.$$

Then $\langle \varphi_+, \varphi_+ \rangle = 1$. The new curve φ_+ is a timelike curve, in fact, $\gamma'_+(s) = \frac{a}{\sqrt{2as}}\ \varphi_+(s) + \sqrt{2as}\varphi'_+(s)$ and $\langle \varphi'_+, \varphi_+ \rangle = 0$, so

$$\langle \gamma'_+, \varphi'_+ \rangle = -\frac{a}{2s\sqrt{2as}}.$$

Hence,

$$\langle \varphi'_+, \varphi'_+ \rangle = \frac{1}{\sqrt{2as}} \langle \gamma'_+, \varphi'_+ \rangle = -\frac{1}{4s^2}.$$

Thus $\varphi_+(s)$ is a timelike curve in $\mathbb{S}^2_1$. Now let t be the proper time and $\frac{\mathrm{d}t}{\mathrm{d}s} := \frac{1}{2s}$. Then

$$t = \frac{1}{2}\int_1^2 \frac{\mathrm{d}u}{u} = \frac{1}{2}\ln s.$$

Let $\hat{\varphi}_+(t) = \varphi_+(e^t)$. Then

$$\gamma_+(t) = \sqrt{2a}e^t\hat{\varphi}_+(t)$$

and $|\hat{\varphi}'_+(t)| = \langle \hat{\varphi}'_+(t), \hat{\varphi}'_+(t)\rangle = -1$.

For the other component, γ_-, we have similar represnttaion

$$\gamma^-(s) = \sqrt{-2as}\phi_-(s)$$

for some curve ϕ_- in $\mathbb{H}^2_0$.

Conversely, let γ be a null Frenet curve defined by

$$\gamma(s) = \sqrt{2as}\varphi(s)$$

for a positive number $a > 0$ and a timelike curve $\varphi(s)$ in the pseudosphere $\mathbb{S}^2_1$ with $|\varphi'(s)| = -\frac{1}{4s^2}$ and $|\varphi''(s)| = \frac{1}{2as}$.

By a direct computation with $|\varphi'(s)| = -\frac{1}{4s^2}$ and $|\varphi''(s)| = \frac{1}{2as}$, we see that $\langle\gamma'', \gamma''\rangle = 1$, *i.e.*, s is pseudo-arc parameter.

$$\gamma'(s) = \sqrt{2as}\left(\frac{1}{2s}\varphi + \varphi'\right)$$

and so

$$\langle\gamma, A\rangle = a,$$

where $A = \gamma'$. Since the tangential component of γ is a nonzero constant, by Proposition 3.8 γ is a rectifying curve in $\mathbb{E}^3_1$.

Analogously, let γ be a null Frenet curve defined by

$$\gamma(s) = \sqrt{2as}\psi(s)$$

for a positive number $a > 0$ and a curve $\psi(s)$ in the hyperbolic 2-space $\mathbb{H}^2_0$ with $|\varphi'(s)| = \frac{1}{4s^2}$. Then $\gamma(s)$ is a rectifying null Frenet curve with pseudo-arc parameter s. □

Next, we study null Frenet curves which admit a point such that all of the osculating planes pass through the point.

Proposition 3.9. ([33]) *Let γ be a null Frenet curve parametrized by pseudo-arc. Then there does not exist any point of $\mathbb{E}^3_1$ such that all the osculating planes pass through the point.*

Proof. Assume that $\boldsymbol{p} = \gamma(s) + u(s)A(s) + w(s)C(s)$ is constant, then from the Cesàro's fixed point condition:

$$1 + u' - kw = 0, \quad w = 0, \; w' + u = 0.$$

From the second and third equations, we have $u = w = 0$. But this contradicts to the first equation. □

This result is consistent with the fact the only null curves in Minkowski plane are null lines.

Theorem 3.5. ([33]) *Let $\gamma(s)$ be a null Frenet curve parametrized by pseudo-arc. Then there exists a point of $\mathbb{E}^3_1$ such that all the normal planes of γ pass through it if and only if the lightlike curvature k is a solution to*

$$k^3 = k''k - 3(k')^2, \quad k \neq 0.$$

Proof. If $\boldsymbol{p} = \gamma + vB + wC$ is constant, then by Cesàro's fixed point condition, we have

$$1 - kw = 0, \quad v' - w = 0, \quad w' + kv = 0.$$

From these we have

$$v = \frac{k'}{k^3}, \quad w = \frac{1}{k}, \tag{3.17}$$

$$k^3 = k''k - 3(k')^2. \tag{3.18}$$

Conversely, take a solution k to (3.18) and γ be a null Frenet curve with lightlike curvature k. Define two functions v and w by (3.17), then $\gamma+vB+wC$ is a constant vector. $\square$

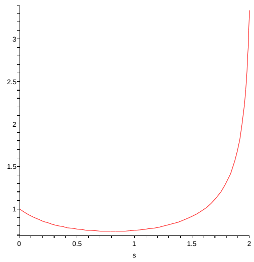


FIGURE 2. The solution to $k^3 = k''k-3(k')^2$ with initial condition $k(0) = 1$ and $k'(0) = -1$.

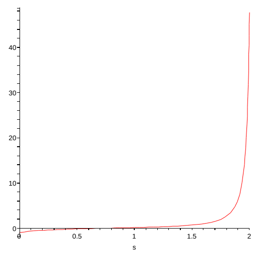
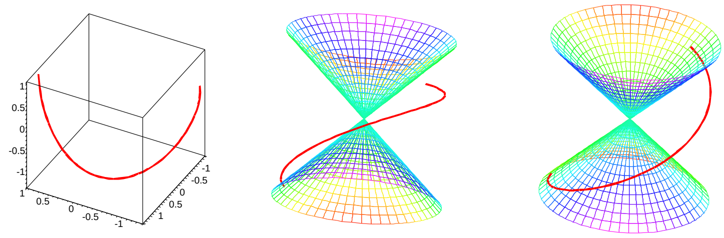


FIGURE 3. The graph of $k'(s)$.



(a) (b) (c)

FIGURE 4. Null curve $\gamma = \left(-\frac{k'}{k^3}, -\frac{k'}{k^3}\sin s + \frac{1}{k}\cos s, -\frac{k'}{k^3}\cos s - \frac{1}{k}\sin s\right)$.

4. Null curves on $\mathbb{S}^2_1$

4.1. As we saw in Proposition 3.6, some characterizations of spherical curves in $\mathbb{E}^3$ are known (see also [69], [70]). The following characterization is fundamental. (see *eg.* [57, p. 19, Theorem 4.1], [30, p. 161, Exercise 9]).

Proposition 4.1. *A curve $\gamma(s)$ parametrized by the arclength in $\mathbb{E}^3$ (which is not a straight line) is a spherical curve if and only if its curvature κ and the torsion τ satisfy*

- *$\tau = 0$ and κ is a nonzero constant, that is, γ is a circle or*
- *$\kappa' \neq 0$ and κ and τ satisfy*

$$\rho\tau + \left(\frac{\rho'}{\tau}\right)' = 0,$$

where $\rho := 1/\kappa$ is the curvature radius function.

In this section we prove the following result.

Theorem 4.1. *The only null curves lying on the pseudosphere $\mathbb{S}^2_1$ are null geodesics of $\mathbb{S}^2_1$.*

Denote by D and ∇ the Levi-Civita connections of $\mathbb{E}^3_1$ and $\mathbb{S}^2_1(\boldsymbol{m};r)$, respectively. Let $\boldsymbol{x}$ be the position vector field of $\mathbb{E}^3_1$. Then $-(\boldsymbol{x}-\boldsymbol{m})/r$ is a unit normal vector field of $\mathbb{S}^2_1(\boldsymbol{m};r)$ in $\mathbb{E}^3_1$.

The *Gauss formula* of $\mathbb{S}^2_1(\boldsymbol{m};r)$ in $\mathbb{E}^3_1$ is given by

$$D_XY = \nabla_XY - \langle X,Y\rangle(\boldsymbol{x}-\boldsymbol{m}),$$

where X and Y are arbitrary vector fields on $\mathbb{S}^2_1$.

Now we consider null curves on the pseudosphere $\mathbb{S}^2_1(\boldsymbol{m};r)$. First we recall the classification of null geodesics on $\mathbb{S}^2_1(\boldsymbol{m};r)$.

Let $\gamma(s)$ be a curve in $\mathbb{S}^2_1(\boldsymbol{m};r)$, then by the Gauss formula,

$$\frac{\mathrm{d}^2\gamma}{\mathrm{d}s^2} = \nabla_{\gamma'}\gamma' - \langle\gamma',\gamma'\rangle(\gamma-\boldsymbol{m}).$$

Now we assume that α is a null geodesic starts at $p \in \mathbb{S}^2_1(\boldsymbol{m};r)$ with initial velocity $q \in T_p\mathbb{S}^2_1(\boldsymbol{m};r)$. Note that since $q \in T_p\mathbb{S}^2_1(\boldsymbol{m};r)$, $\langle p-\boldsymbol{m},q\rangle = 0$. (This remark will be used in later discussions).

Then the Gauss formula implies

$$\frac{\mathrm{d}^2\gamma}{\mathrm{d}s^2} = 0, \ \alpha(0) = p, \ \frac{\mathrm{d}\gamma}{\mathrm{d}s} = q.$$

Hence $\gamma(s) = p + sq$, which is a null line contained in $\mathbb{S}^2_1(\boldsymbol{m}; r)$.

Remark 4.1. Since $\mathbb{S}^2_1(\boldsymbol{m}; r)$ is realized as a hyperboloid of one sheet in $\mathbb{E}^3_1$, it is a doubly ruled surface:

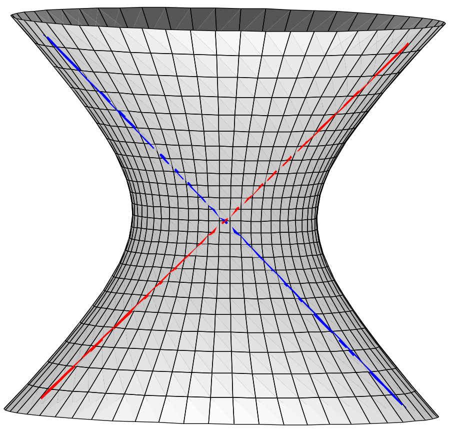
$$\begin{aligned} \varphi(u,v) &= \gamma(u) + v\ell^{\pm}(u), \\ \gamma(u) &= (m_1, m_2 + r\cos u, m_3 + r\sin u), \quad \boldsymbol{m} = (m_1, m_2, m_3), \\ \ell^{\pm}(u) &= (\pm r, -r\sin u, r\cos u). \end{aligned}$$

The base curve of these ruled patch is the circle $(\xi_2 - m_2)^2 + (\xi_3 - m_3)^2 = r^2$ in the spacelike plane $\xi_1 = m_1$. Such circles are spacelike geodesics of $\mathbb{S}^2_1(\boldsymbol{m}; r)$. The rulings are null lines in $\mathbb{E}^3_1$.

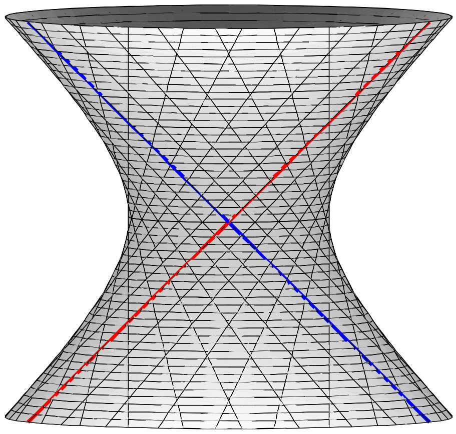
Remark 4.2. Here we would like to remark that at every point p of $\mathbb{S}^2_1$, there exists two null lines β and γ contained in $\mathbb{S}^2_1$ which make constant "angle". In fact, at every point $p \in \mathbb{S}^2_1$, there exist exactly two null directions in the tangent space $T_p\mathbb{S}^2_1$. We can take a pair of null vectors $\{u, v\}$ such that $\langle u, v\rangle = 1$. Then $\beta(s) = p + su$ and $\gamma(t) = p + tv$ are required null lines. In the left figure below, grid lines are timelike and spacelike geodesics. The null geodesics in red and blue have parametrizations

$$(v, -v/2, 1/2)\,, \quad (-v, -v/2, 1/2),$$

respectively.



(a)



(b)

FIGURE 5. Null lines on $\mathbb{S}^2_1$

This property characterizes $\mathbb{S}^2_1$ as follows:

Theorem 4.2. ([39]) *Let M be a timelike surface in $\mathbb{E}^3_1$. At every point p on M, if there exist two null lines through p contained in M then M is locally isometric to the pseudosphere or a timelike plane.*

4.2. Now we prove Theorem 4.1.

Lemma 4.1. (*cf.* [59]) *There are no nongeodesic parametrized null curves lying on the pseudosphere.*

Proof. Let $\gamma(s)$ be a null Frenet curve parametrized by the pseudoarc. Assume that γ lies in the pseudosphere $\mathbb{S}^2_1(\boldsymbol{m}; r)$. Without loss of generality, we may assume that $m = 0$ and $r = 1$. Then we have

$$\langle \gamma, \gamma \rangle = 1. \tag{4.1}$$

Differentiating (4.1), we have

$$\langle A, \gamma \rangle = 0. \tag{4.2}$$

Differentiating this equation, we get

$$\langle C, \gamma \rangle + \langle A, A \rangle = 0.$$

Hence we obtain

$$\langle C, \gamma \rangle = 0. \tag{4.3}$$

Differentiating (4.3),

$$\langle kA + B, \gamma \rangle = 0.$$

This together with (4.2), we have

$$\langle B, \gamma \rangle = 0.$$

Hence we conclude that $\gamma = 0$. This is a contradiction □

This Lemma together with Remark 4.2 implies Theorem 4.1.

Remark 4.3. In [59], Petrović-Torgašev and Šućurović claimed the nonexistence of null curves which lie in the pseudosphere in Minkowski 3-space. Their discussion consists of two parts. In the first part, they considered null lines contained in $\mathbb{S}^2_1(\boldsymbol{m}; r)$. And they claimed the nonexistence of null lines contained in $\mathbb{S}^2_1(\boldsymbol{m}; r)$. As we saw in Remark 4.2 at every point of $\mathbb{S}^2_1$, there exists two null lines through the point and contained in $\mathbb{S}^2_1$. This claim is not true. In the second part, they proved the nonexistence of non-geodesic null Frenet curves contained in the pseudosphere. The proof of this nonexistence is the same as the proof of Lemma 4.1.

Now we recall the first part of the proof of Theorem 4.1 in [59]. We use the notation of [59]: g denotes the induced metric of $\mathbb{S}^2_1(\boldsymbol{m}; r)$.

Let $\alpha(s) = p + sq$ be a null line contained in $\mathbb{S}^2_1(\boldsymbol{m}; r)$. Then we have

$$g(\alpha - \boldsymbol{m}, \alpha - \boldsymbol{m}) = r^2.$$

Differentiating this equation, we get

$$g(p + sq - \boldsymbol{m}, q) = 0.$$

Since q is null, we have

$$g(p, q) = g(\boldsymbol{m}, q).$$

From this relation, Petrović-Torgašev and Šućurović claimed that $p = m$ and hence $g(\alpha - \boldsymbol{m}, \alpha - \boldsymbol{m}) = 0$. However the relation $g(p, q) = g(m, q)$ does not imply $p = \boldsymbol{m}$.

The relation $g(p, q) = g(\boldsymbol{m}, q)$ means that $g(p - \boldsymbol{m}, q) = 0$, namely q is a tangent vector of $\mathbb{S}^2_1(\boldsymbol{m}; r)$ at p.

Note that obviously null lines starting at m (corresponding to the case $p = \boldsymbol{m}$) are not contained in $\mathbb{S}^2_1$.

5. Screen bundles

In this section, we would like to discuss how our formulation relates to the formulation due to the book [19].

Let $\gamma : I \subset \mathbb{R} \to \mathbb{E}^3_1$ be a parametrized null curve. Denote by $T\gamma$ the *tangent bundle* of γ:

$$T\gamma := \bigcup_{t \in I} \mathbb{R}\dot{\gamma(t)}, \quad \dot{\gamma} = \frac{\mathrm{d}\gamma}{\mathrm{d}t}.$$

The tangent bundle $T\gamma$ is a line bundle over I.

Define a vector bundle $T^{\perp}\gamma$ of rank 2 over I by

$$T^{\perp}\gamma := \bigcup T^{\perp}_p\gamma, \quad T^{\perp}_p\gamma = \{v_p \in T_p\mathbb{E}^3_1 \mid \langle v_p, \dot{\gamma}\rangle = 0\}.$$

The tangent bundle $T\gamma$ is a vector subbundle of $T^{\perp}\gamma$. Moreover both $T\gamma$ and $T^{\perp}\gamma$ are vector subbundles of the pull-back bundle:

$$\gamma^* T\mathbb{E}^3_1 = \bigcup_t T_{\gamma(t)}\mathbb{E}^3_1.$$

A complementary vector subbundle $\mathcal{S} = \mathcal{S}(T^{\perp}\gamma)$;

$$T^{\perp}\gamma = T\gamma \oplus \mathcal{S}(T^{\perp}\gamma).$$

is called a *screen vector bundle* of γ. The screen bundle is **not** uniquely determined. In fact, let ξ be a local section of the screen bundle. Then for any function $\lambda(s)$, $\xi(s) + \lambda(s)\gamma'(s)$ spans another screen bundle. It is easy to see that every screen bundle is spacelike (and hence, nondegenerate). Thus we have the following orthogonal decomposition:

$$\gamma^* T\mathbb{E}^3_1 = \mathcal{S}(T^{\perp}\gamma) \oplus \mathcal{S}(T^{\perp}\gamma)^{\perp}.$$

Now we fix a screen bundle of γ. Then we have:

Lemma 5.1. (Bejancu) *Let $\gamma(t)$ be a parametrized null curve in $\mathbb{E}^3_1$ with screen vector bundle $\mathcal{S}$. Then there exists a unique line bundle $\mathcal{N}$ such that there exists a unique section N of $\mathcal{N}$ satisfying*

$$\langle \frac{\mathrm{d}\gamma}{\mathrm{d}t}, N\rangle = 1, \quad \langle N, N\rangle = \langle N, X\rangle = 0$$

for any section X of $\mathcal{S}$.

The line bundle $\mathcal{N} = \mathcal{N}_{\mathcal{S}}$ is called the *null transversal bundle* of γ with respect to the screen bundle $\mathcal{S}$.

The null transversal bundle depends on the choice of screen bundle. Take a unit section W of $\mathcal{S}(T^{\perp}\gamma)$ so that $\det(\gamma', N, W) = 1$. Then (L, N, W), $L = \gamma'$ is a null frame field of γ. Thus (L, N, W) satisfies the Frenet equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}(L, N, W) = (L, N, W) \begin{pmatrix} k_1 & 0 & -k_3 \\ 0 & -k_1 & -k_2 \\ k_2 & k_3 & 0 \end{pmatrix}.$$

In general this frame field does not satisfy $k_1 = 0$.

Let $\gamma(s)$ be a nongeodesic null curve. Then there exists a null frame (A, B, C) with $k_1 = 0$ according to the construction given in Proposition 2.1.

The principal normal vector field $C(s)$ is defined by the formula:

$$\gamma''(s) = k_2(s)C(s).$$

The principal normal vector field defines a screen bundle $\mathcal{S}$;

$$\mathcal{S}(T^{\perp}\gamma) = \bigcup_s \mathcal{S}(T_s^{\perp}\gamma), \quad \mathcal{S}(T_s^{\perp}\gamma) = \mathbb{R}C(s).$$

Let $B(s)$ be the binormal vector field. Then the null transversal bundle is spanned by $B(s)$;

$$\mathcal{N} = \bigcup \mathcal{N}_s, \quad \mathcal{N}_s = \mathbb{R}B(s).$$

As we saw before, for any function λ,

$$\mathcal{S}_\lambda(T^{\perp}\gamma) = \bigcup_s \mathcal{S}_\lambda(T_s^{\perp}\gamma), \quad \mathcal{S}_\lambda(T_s^{\perp}\gamma) = \mathbb{R}(C(s) + \lambda(s)A(s))$$

gives another screen bundle.

Part II Applications

6. Applications: Ruled surfaces

6.1. Ruled patch. First of all, we recall basic ingredients about ruled surfaces from O'Neill's textbook [55, pp. 140–143, 231–234] and Spivak's textbook [65, pp. 146–150, 182–183].

Let $\mathbb{R}^3$ be Cartesian 3-space. A *ruled surface* in $\mathbb{R}^3$ is an immersed surface swept out by a straight line ℓ moving along a curve γ. The various position of the generating line ℓ are called the *ruling* (or *ruler*) of the surface. Such a surface thus has a parametrisation of the following *ruled form*:

$$\varphi(u, v) = \gamma(u) + vP(u) \quad (\text{or} \quad \gamma(v) + uP(v)).$$

The curve γ is called the *base curve* or *generating curve*. The curve $P(u)$ is called the *director curve*. Alternatively P is regarded as a *vector field* along γ It is necessary to restrict v to some interval to guarantee the immersion property of φ.

In Euclidean 3-space, the following result is well-known:

Theorem 6.1. *Flat surfaces in $\mathbb{E}^3$ are generalized cones, cylinders or tangent developables.*

Corollary 6.1. (Pogorelov-Hartman-Nirenberg-Massey) *Complete flat surfaces are cylinders.*

Ruled surfaces which are not generalized cones or cylinders or tangent developables are sometimes called *scrolls*.

Now we study ruled surfaces in Minkowski 3-space.

Let $\gamma : \mathcal{I} \to \mathbb{E}_1^3$ be a parametrized curve and P be a vector field along γ. Consider the mapping $\varphi(u, v) := \gamma(u) + vP(u) : \mathcal{I} \times \mathbb{R}^* \to \mathbb{E}_1^3$. Then the pulled-back symmetric tensor $\mathrm{I} = \langle \mathrm{d}\varphi, \mathrm{d}\varphi \rangle$ is given by [1]:

$$\mathrm{I} = \begin{pmatrix} \langle \dot{\gamma}, \dot{\gamma} \rangle + 2v\langle \dot{\gamma}, \dot{P} \rangle + v^2\langle \dot{P}, \dot{P} \rangle & \langle \dot{\gamma}, P \rangle + v\langle \dot{P}, P \rangle \\ \langle \dot{\gamma}, P \rangle + v\langle \dot{P}, P \rangle & \langle P, P \rangle \end{pmatrix}.$$

Here we used the dot "$\cdot$" for $\mathrm{d}/\mathrm{d}u$. According to the causal character of γ and P, there are four possibilities for *nondegenerate* ruled surfaces:

(I) $\dot{\gamma}$ and P are non-null and $\dot{\gamma}$ and P are linearly independent;
(II) $\dot{\gamma}$ is null and P is non-null with $\langle \dot{\gamma}, P \rangle \neq 0$;

(Ⅲ) $\dot{\gamma}$ is non-null and P is null with $\langle \dot{\gamma}, P\rangle \neq 0$;
(Ⅳ) $\dot{\gamma}$ and P are null with $\langle \dot{\gamma}, P\rangle \neq 0$;

The following result is due to L. J. Alías, A. Ferrández, P. Lucas and M. A. Meroño.

Lemma 6.1. (*cf.* [1]) *After an appropriate change of the base curve γ, cases* (Ⅱ) *and* (Ⅲ) *can be reduced locally to* (I) *and* (Ⅳ)*, respectively.*

Proof. Let $\varphi(u,v) = \gamma(u) + vP(u)$ be a ruled of type (Ⅱ). Then by a suitable reparametrization of γ and rescaling of P, we get

$$\langle P, P\rangle = \nu = \pm 1, \quad \langle \dot{\gamma}, P\rangle = 1$$

so that

$$\det \mathrm{I} = \nu\{2v\langle \dot{\gamma}, \dot{P}\rangle + v^2\langle \dot{P}, \dot{P}\rangle\} - 1 < 0.$$

Now we look for a curve $\alpha(u) = \gamma(u)+v(u)P(u)$ in the ruled surface with $\langle \dot{\alpha}, \dot{\alpha}\rangle = \nu$ and such that $\dot{\alpha}$ and X are linearly independent. The condition $\langle \dot{\alpha}, \dot{\alpha}\rangle = \nu$ is equivalent to

$$\dot{v}^2 + 2\nu\dot{v} + \det \mathrm{I} = 0. \tag{6.1}$$

This equation has positive discriminant from $\det \mathrm{I} < 0$. Thus we can locally integrate this to obtain $v = v(u)$. Besides, $\dot{\alpha}$ and X are linearly independent because $\langle \dot{\alpha}, \dot{\alpha}\rangle = \langle P, P\rangle = \nu$ and $\langle \dot{\alpha}, P\rangle = 1 + \nu\ \dot{v} \neq \pm\nu$. This shows that $\varphi(u,v)$ is reparametrized as in case (I) taking α as the base curve.

Next let φ be a ruled surface of type (Ⅲ). Then by a suitable reparametrisation of γ and the rescaling of P, we have

$$\langle \dot{\gamma}, \dot{\gamma}\rangle = \nu, \quad \langle \dot{\gamma}, P\rangle = 1.$$

We can find a null curve $\alpha(u)$ of the form $\alpha(u) = \gamma(u) + v(u)P(u)$ in the ruled surface with $\langle \dot{\alpha}, P\rangle \neq 0$. In fact, since the condition $\langle \dot{\alpha}, \dot{\alpha}\rangle = 0$ is equivalent to

$$-2\dot{v} = \nu + 2v\langle \dot{\alpha}, \dot{P}\rangle + v^2\langle \dot{P}, \dot{P}\rangle. \tag{6.2}$$

This equation can be locally integrated to obtain $v = v(u)$. Moreover $\langle \dot{\alpha}, P\rangle = \langle \dot{\gamma}, P\rangle \neq 0$. Thus we can take α as a base curve of φ. □

In the rest of this section we shall investigate the case (Ⅳ). The main purpose of the rest of this section is to give the "standard parametrization" for ruled surfaces of type (Ⅳ).

6.2. B-scrolls.

Graves [29] introduced the notion of B-scroll.

Definition 6.1. Let $\gamma(u)$ be a null Frenet curve parametrized by the distinguished parameter u and framed by (A, B, C). Then the timelike ruled surface

$$\varphi(u,v) := \gamma(u) + vB(u)$$

is called the *B-scroll* of γ.

The first derivatives of the B-scroll are given by

$$\varphi_u = A(u) - vk_3(u)C(u), \quad \varphi_v = B(u).$$

Thus the metric I of the B-scroll is

$$\mathrm{I} = \begin{pmatrix} k_3(u)^2v^2 & 1 \\ 1 & 0 \end{pmatrix}.$$

Take a unit normal vector field $\boldsymbol{n}$ by $\boldsymbol{n} = -k_3(u)vB(u) + C(u)$. Then the shape operator of φ derived from $\boldsymbol{n}$ is

$$S = \begin{pmatrix} k_3(u) & 0 \\ k_2(u) + vk_3'(u)^2 & k_3(u) \end{pmatrix}.$$

Thus the Gaussian curvature K and the mean curvature H of the B-scroll are given by

$$K = k_3(u)^2, \; H = k_3(u).$$

In particular, the B-scroll is totally umbilical if and only if γ is a geodesic and k_3 is a constant.

Thus if γ is not a geodesic, then S has real repeated principal curvatures with 1-dimensional eigenspace spanned by B. (Hence this surface is not totally umbilic).

B-scrolls give examples of timelike surfaces which have no Euclidean or spacelike counterparts. In fact, B-scrolls of nongeodesic null Frenet curves are not totally umbilic but have real repeated principal curvatures everywhere.

It is easy to see that B-scroll has vanishing mean curvature if and only if it is flat. Moreover the base curve of flat B-scroll is a generalized null cubic.

Remark 6.1. B-scrolls can be totally umbilical.

Let M be a totally umbilical timelike surface which is not totally geodesic. It is known that such a surface is congruent to an open portion of a pseudosphere $\mathbb{S}^2_1(r)$ centered at the origin.

Now we show that $\mathbb{S}^2_1(r)$ is locally expressed as a B-scroll. As we saw before, a B-scroll is totally umbilical if and only if the generating curve is a null geodesic and has constant torsion k_3. Take a null geodesic $\gamma(s)$ defined by

$$\gamma(s) = (0, 0, 1) + s(1, 1, 0)/\sqrt{2}.$$

Then γ admits a null Frenet frame:

$$A(s) = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0), \; B(s) = (-\frac{s^2}{2\sqrt{2}r^2} - \frac{1}{\sqrt{2}} - \frac{s^2}{2\sqrt{2}r^2} + \frac{1}{\sqrt{2}}, \frac{s}{r}),$$

$$C(s) = (-\frac{s}{r\sqrt{2}}, -\frac{s}{r\sqrt{2}}, 1).$$

The torsion of γ is $k_3 = 1/r$. The B-scroll of γ is $\mathbb{S}^2_1(r)$.

Remark 6.2. Let $\gamma(u)$ be a null Frenet curve. Then $\psi(u, v) = \varphi(u) + vA(u)$ is a lightlike surface and called the *lightlike developable* of γ [37].

B-scrolls have been appeared in many contexts of differential geometry. Moreover, the notion of B-scroll can be generalized to 3-dimensional Lorentzian space forms. See [15] and [26].

Here we would like to exhibit some examples.

6.3. Bonnet surfaces and HIMC surfaces.

Definition 6.2. Let $\varphi : M \to \mathbb{E}^3_1$ be a timelike surface. Then (M, φ) is said to be a *timelike Bonnet surface* if there exists a (local) isometric deformation family $\{\varphi_\lambda\}$ preserving the mean curvature through φ.

Theorem 6.2. ([26]) *Every B-scroll is a timelike Bonnet surface.*

Next a timelike surface M in Minkowski 3-space is said to be a *timelike surface with harmonic inverse mean curvature* (timelike HIMC surface, in short) if its reciprocal mean curvature function $1/H$ is a Lorentz harmonic function [27].

Proposition 6.1. ([27]) *For every non-geodesic null Frenet curve, its B-scroll is a timelike HIMC surface.*

6.4. Finite type immersions. Let $\varphi : M \to \mathbb{E}^3_1$ be an immersed semi-Riemannian surface. Denote by Δ the Laplace-Beltrami operator with respect to the induced metric I. Then (M, φ) is said to be of *finite type* if φ can be written as a *finite* sum of eigenfunctions of Δ:

$$\varphi = \varphi_0 + \varphi_1 + \cdots + \varphi_k,$$

where φ_0 is a constant vector and $\Delta\varphi_i = \lambda_i \varphi_i$, for $i = 1, \cdots, k$. If in particular all eigenvalues $\{\lambda_i\}$ are mutually different, then (M, φ) is said to be of k-type.

F. Dillen, I. Van de Woestyne, L. Verstraelen and J. Walrave classified ruled surfaces of finite type in $\mathbb{E}^3_1$.

Theorem 6.3. ([17]) *A timelike ruled surfaces M in $\mathbb{E}^3_1$ is of finite type if and only if*

(1) *M is minimal,* $(H = 0)$*;*
(2) *M is a portion of a timelike circular cylinder $\mathbb{E}^1_1 \times \mathbb{S}^1$;*
(3) *pseudocircular cylinder $\mathbb{S}^1_1 \times \mathbb{E}^1$ or*
(4) *M is an isoparametric surface with null rulings.*

Timelike minimal surfaces and various cylinders are 1-type timelike surfaces, that is, $\varphi = \varphi_0 + \varphi_i$ with $\lambda_i \neq 0$.

Timelike isoparametric surfaces are of *null* 2*-type*, that is, φ has the decomposition:

$$\varphi = \varphi_0 + \varphi_i + \varphi_j$$

with $\lambda_i = 0,\ \lambda_j \neq 0$.

Note that timelike isoparametric surfaces can be reparametrized as B-scrolls of null Frenet curves with constant torsion.

6.5. Gauss maps. Let $\varphi : M \to \mathbb{E}^3_1$ be a timelike surface. Then its unit normal vector field can be regarded as a smooth map into the unit pseudosphere $\mathbb{S}^2_1$. The resulting map $\psi : M \to \mathbb{S}^2_1$ is called the *Gauss map* of φ.

Like Euclidean geometry, the following Ruh-Vilms type theorem is obtained:

Proposition 6.2. ([49]) *Let $\varphi : M \to \mathbb{E}^3_1$ be a timelike surface with Gauss map $\psi : M \to \mathbb{S}^2_1$. Then ψ is a (Lorentz) harmonic map if and only if the mean curvature of φ is constant.*

Based on the Ruh-Vilms property, a loop group theoretic construction of timelike constant mean curvature surfaces in $\mathbb{E}^3_1$ is established in [18]. See also [38].

Let ψ be a smooth map from a semi-Riemannian 2-manifold M into the unit pseudo 2-sphere $\mathbb{S}^2_1$ or hyperbolic 2-space $\mathbb{H}^2$. We embbed $\mathbb{S}^2_1$ and $\mathbb{H}^2$ in $\mathbb{E}^3_1$ as hyperboloids. Then ψ is harmonic if and only if

$$\Delta\psi = \rho\ \psi$$

for some function ρ. Mappings which satisfy these equations are called *pointwise* 1*-type map* by some authors (See *eg.*, [41] *etc*).

Kim and Yoon classified nondegenerate ruled surfaces of constant mean curvature in $\mathbb{E}^3_1$.

Proposition 6.3. ([41]) *The only nondegenerate cylindrical ruled surfaces of constant mean curvature whose base curves are non-null are open portions of*

(1) *spacelike or timelike planes,*
(2) *hyperbolic cylinder* $\mathbb{H}^1 \times \mathbb{E}^1$ *(spacelike surface),*
(3) *timelike circular cylinder* $\mathbb{E}^1_1 \times \mathbb{S}^1$ *(timelike surface), or*
(4) *pseudocircular cylinder* $\mathbb{S}^1_1 \times \mathbb{E}^1$ *(timelike surface).*

Proposition 6.4. ([41]) *Nondegenerate noncylindrical ruled surfaces of constant mean curvature with non-null base curves have mean curvature* 0*.*

Note that ruled spacelike maximal surfaces and ruled timelike minimal surfaces are classified by O. Kobayashi [43], L. McNertney [48] and Van de Wostijne [67].

Proposition 6.5. ([41]) *B-scrolls over null curves with constant* k_3 *are the only ruled surfaces of constant mean curvature in* $\mathbb{E}^3_1$ *with null rulings.*

Choi, Ki, Suh [14], Alías, Ferrández, Lucas and Merõno [1] studied Gauss maps of timelike ruled surfaces in Minkowski 3-space.

Theorem 6.4. ([1],[13],[14])

(1) *The only timelike ruled surfaces with non-null rulings in* $\mathbb{E}^3_1$ *for which the Gauss map satisfies* $\Delta\psi = \Lambda\psi$*, where* Δ *is the Laplace operator on the surface and* Λ *some fixed endomorphism of* $\mathbb{E}^3_1$*, are open portions of a timelike plane, timelike circular cylinder* $\mathbb{E}^1_1 \times \mathbb{S}^1$*, pseudocircular cylinder* $\mathbb{S}^1_1 \times \mathbb{E}^1$*.*
(2) *B-scrolls over null curves with constant torsion are the only ruled surfaces in* $\mathbb{E}^3_1$ *with null rulings whose Gauss map* ψ *satisfies the condition* $\Delta\psi = \Lambda\psi$*.*

In all the cases, the Gauss maps are (Lorentz) harmonic maps into $\mathbb{S}^2_1$.

Note that Ferrández and Lucas [25] generalized the result (2) in the preceding Theorem to that for B-scrolls in 3-dimensional Lorentzian space forms [15].

As we mentioned before, B-scrolls of constant (mean) curvature are null 2-type. Null 2-type timelike surfaces in 3-dimensional Lorentzian space forms are classified by Kim and Kim [40]. In case of $\mathbb{E}^3_1$, they proved:

Proposition 6.6. *The only null* 2*-type timelike surfaces are constant mean curvature timelike cylinders or B-scrolls of constant mean curvature.*

6.6. Affine differential geometry. B-scrolls have been appeared in affine differential geometry.

First, we recall the following result:

Lemma 6.2. ([16]) *Up to conjugation, the following three cases cover all possible non-trivial one-parameter subgroups of the isometry group* $\mathrm{SO}^+_1(3) \ltimes \mathbb{R}^3$*:*

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos t & \sin t \\ 0 & -\sin t & \cos t \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} + h\begin{pmatrix} t \\ 0 \\ 0 \end{pmatrix}; \tag{6.3}$$

$$\begin{pmatrix} \cosh t & \sinh t & 0 \\ \sinh t & \cosh t & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} + h\begin{pmatrix} 0 \\ 0 \\ t \end{pmatrix}; \tag{6.4}$$

$$\begin{pmatrix} 1+t^2/2 & -t^2/2 & t \\ t^2/2 & 1-t^2/2 & t \\ t & -t & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} + h\begin{pmatrix} t^3/3+t \\ t^3/3-t \\ t^2 \end{pmatrix}. \tag{6.5}$$

The one-parameter subgroups in the list are called *helicoidal motion* with pitch h. The third type helicoidal motion is also called a *cubic screw motion* (kubische Schraubung).

A nondegenerate surface in $\mathbb{E}^3_1$ is said to be a *helicoidal surface* if there exists a non-trivial one-parameter subgroup $\{g_t\}$ of $\mathrm{SO}^+_1(3) \ltimes \mathbb{R}^3$ such that the surface is invariant under the action of $\{g_t\}$.

Let us consider the orbit of the line $\mathbb{R}\mathbf{e}_3$ under the helicoidal motion (6.5). The orbit is parametrized as

$$\varphi(s,v) = \big(h(s^3+s), h(s^3-s), hs^2\big) + v\,(s,s,1)\,.$$

Note that this surface has vanishing mean curvature.

This helicoidal surface is a timelike ruled surface with null base curve $\gamma(s) = (h(s^3+s), h(s^3-s), hs^2)$ with director curve $P(s) = (s,s,1)$. (Since, $\langle \gamma', P\rangle = -2h \neq 0$. One can reparametrize φ as a B-scroll). If we choose $h = -1/2$, then φ is a B-scroll of γ. This example is an indefinite real form of the so-called *Lie's minimal surface.*

It is pointed out in [16], this timelike ruled surface is equiaffinly congruent (but not isometric) to the Cayley's ruled surface of third degree which has been investigated in affine differential geometry. For more detailed historical discussions about Cayley's minimal surface, we refer to [16].

F. Dillen and W. Kühnel classified ruled Weingarten surfaces in Minkowski 3-space. The following is a slight modification of [16, Theorem 2].

Theorem 6.5. *Let $\varphi(s,v) = \gamma(s) + vP(s)$ be a nondegenerate Weingarten ruled surface in $\mathbb{E}^3_1$. Then*

(1) *Any non-flat ruled Weingarten surface with nonnull rulings is an open portion of a helicoidal ruled surface,*
(2) *Any ruled surface with null rulings is a Weingarten surface satisfying $H^2 = K$.*

6.7. Distribution parameter. A. Fujioka and the first named author have proved the following fundamental result:

Theorem 6.6. ([26]) *Let M be a timelike surface in $\mathbb{E}^3_1$. Assume that M has real repeated principal curvatures with corresponding 1-dimensional eigenspace everywhere. Then M is a B-scroll of a null Frenet curve.*

This implies that every null scrolls can be reparametrized as a B-scroll of a null Frenet curve. For instance, as we have seen before (Remark 6.1), the pseudosphere $\mathbb{S}^2_1$ can be represented as B-scroll.

Hereafter we study geometric meaning of these reparametrizations in detail.

Let M be a ruled surface in $\mathbb{E}^3_1$ with null rulings. Assume that M is non-cylindrical. Take any regular curve α on the surface which intersects the rulings transversally, and let X be a vector field along α such that $X(u)$ is pointing in the direction of the ruling through $\alpha(u)$. Then M is parametrized as

$$\varphi(u,v) = \alpha(u) + vX(u),$$

where $\langle X, X\rangle = 0$. Since M is non-cylindrical, $\dot{X} \neq 0$. Moreover, since M is timelike,

$$\langle \dot{X}, \dot{X}\rangle > 0, \quad \langle \dot{\alpha}, X\rangle \neq 0.$$

The following result is due to [17]:

Lemma 6.3. *The function* $\rho = \langle \dot{X}, \dot{X} \rangle / \langle \dot{\alpha}, X \rangle^2$ *is independent of the choice of* α*, the parametrisation of* α *or the scaling of* X*. The shape operator of* M *with respect to the coordinates* (u, v) *has the form:*

$$\begin{pmatrix} \sqrt{\rho} & 0 \\ \text{something} & \sqrt{\rho} \end{pmatrix}.$$

We call the function ρ the *distribution parameter* of the ruled surface M with null rulings. In particular, if M is a B-scroll of a null Frenet curve, $\rho = k_3^2 = K$.

Remark 6.3. (Extension B-scroll) In [58], Nasar and Fathi introduced the notion of "extension B-scroll". Let $\alpha(s)$ be a null Frenet curve and P be a null vector field along γ. Nasar and Fathi call the ruled surface $\phi(s, v) = \alpha(s) + vL(s)$ an *extension B-scroll.*

However, one can see that every extension B-scroll can be reparametrized as a B-scroll.

7. Applications: Timelike minimal surfaces

7.1. Lorentz conformal structure. Let M be an oriented 2-manifold and h_1 and h_2 Lorentzian metrics on M. Then h_1 and h_2 are said to be *conformally equivalent* if there exists a smooth positive function μ on M such that $h_2 = \mu\, h_1$. An equivalence class $\mathcal{C}$ of a Lorentzian metric on M is called a *Lorentz conformal structure.* An orderd pair $(M, \mathcal{C})$ consisiting of an oriented surface and a Lorentz conformal structure compatible to the given orientation is called a *Lorentz surface.*

A local coordinate system (u, v) is said to be *null* if

$$h\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial u}\right) = h\left(\frac{\partial}{\partial v}, \frac{\partial}{\partial v}\right) = 0$$

for any $h \in \mathcal{C}$.

Proposition 7.1. (*cf.* [68, p. 13]) *Let* $(M, \mathcal{C})$ *be a Lorentz surface. Then there exists, in some neighborhood of any point, a null coordinate system* (u, v)*.*

Remark 7.1. Let $(M, \mathcal{C})$ be a Lorentz surface. Take a Lorentzian metric $h \in \mathcal{C}$. Denote by $\Box_h$ the Laplace-Beltrami operator of h (frequently called *d'Alembert operator*). With respect to a null coordinate system (u, v) such that $h = e^{\omega}\, \mathrm{d}u\mathrm{d}v$, $\Box_h$ is given by

$$\Box_h = 4e^{-\omega}\partial_u\partial_v.$$

A smooth function $f \in C^\infty(M)$ is said to be a (*Lorentz*) *harmonic function* if $\Box_h f = 0$. The explicit formula of $\Box$ implies that the *harmonicity of functions* is invariant under conformal transformations of $(M, \mathcal{C})$.

The exterior differential operator d is decomposed with respect to the conformal structure:

$$\mathrm{d} = \mathrm{d}' + \mathrm{d}'', \quad \mathrm{d}' := \frac{\partial}{\partial u}\mathrm{d}u, \quad \mathrm{d}'' := \frac{\partial}{\partial v}\mathrm{d}v.$$

It is easy to see that d' and d'' are independent of the choice of null coordinate system.

Definition 7.1. Let $f : M \to \mathbb{R}$ be a smooth function. Then f is said to be a (real) *Lorentz holomorphic function* if $\mathrm{d}''f = 0$. Similarly, f is said to be a (real) *Lorentz anti holomorphic function* if $\mathrm{d}'f = 0$.

It is easy to see that $f \in C^2(M)$ is Lorentz harmonic if and only if $\mathrm{d}''\mathrm{d}'f = 0$. In particular, (Lorentz) $\pm$-holomorphic functions are harmonic.

Proposition 7.2. (d'Alembert formula) *Let M be a Lorentz surface and F a Lorentz harmonic function. Then there exist a Lorentz holomorphic function X and a Lorentz anti holomorphic function Y such that $F = X + Y$.*

For general theory of Lorentz surfaces, we refer to [38], [68].

7.2. Classical representation formula. Let $M = (M, \mathcal{C})$ be a Lorenzt surface. An immersion $\varphi : M \to N_1^3$ to a Lorentz 3-manifold (N_1^3, h) is said to be a *timelike surface* if φ is conformal, *i.e.*, $\varphi^* h \in \mathcal{C}$. A timelike surface is said to be *minimal* (or *extremal*) if its mean curvature vanishes.

Now let $\varphi : M \to \mathbb{E}_1^3$ be a timelike surface in Minkowski 3-space. Then it is easy to see that M is minimal if and only if φ is a vector-valued Lorentz harmonic function. Hence the immersion φ can be written locally:

$$\varphi(u, v) = X(u) + Y(v)$$

as a sum of two curves $X(u)$ and $Y(v)$ by Proposition 7.2. By computing the first fundamental form of the right hand side, we obtain

$$\langle X_u, X_u \rangle = \langle Y_v, Y_v \rangle = 0, \tag{7.1}$$

and X_u and Y_v are linearly independent. Hence $X(u)$ and $Y(v)$ are *null curves* in $\mathbb{E}_1^3$.

Here we arrive at the classical representation formula:

Proposition 7.3. ([48, Theorem 3.5]) *Let $\varphi : M \to \mathbb{E}_1^3$ be a timelike minimal surface. Then φ is expressed locally as a sum of null curves*:

$$\varphi(u, v) = X(u) + Y(v).$$

The velocity vector fields of the null curves $X(u)$ and $Y(v)$ are linearly independent.

Conversely, let $X(u)$ and $Y(v)$ be null curves defined on open intervals I_u and I_v, respectively. Assume that the velocity vector fields X_u and Y_v are linearly independent. Then $\varphi(u, v) = X(u) + Y(v)$ is a timelike minimal immersion of $(I_u \times I_v, \mathrm{d}u\mathrm{d}v)$ into $\mathbb{E}_1^3$ with metric $\mathrm{I} = 2\langle X_u, Y_v \rangle \mathrm{d}u\mathrm{d}v$.

Let $\varphi(u, v) = X(u) + Y(v)$ be a timelike minimal surface. Then $\hat{\varphi}(u, v) := X(u) - Y(v)$ is also a timelike minimal surface. The timelike minimal surface $\hat{\varphi}$ is called the *conjugate timelike minimal surface* of φ.

Example 7.1 (Timelike catenoid and Timelike helicoid with spacelike axis)**.** Let us consider the following timelike minimal surface:

$$\varphi(u, v) = X(u) + Y(v),$$

where

$$X(u) = (\sinh u, \cosh u, u), \quad Y(v) = (-\sinh v, -\cosh v, -v).$$

Then

$$\varphi = \left(2 \sinh\left(\frac{u - v}{2}\right) \cosh\left(\frac{u + 2}{2}\right), 2 \sinh\left(\frac{u - v}{2}\right) \sinh\left(\frac{u + 2}{2}\right), u - v\right),$$

$$\hat{\varphi} = \left(2 \cosh\left(\frac{u - v}{2}\right) \sinh\left(\frac{u + 2}{2}\right), 2 \cosh\left(\frac{u - v}{2}\right) \cosh\left(\frac{u + 2}{2}\right), u + v\right).$$

The surface φ is a timelike minimal surface of revolution with timelike profile curve and spacelike axis, which is called the *Lorentz catenoid* (or *timelike catenoid*) with *spacelike* axis. The conjugate timelike minimal surface $\hat{\varphi}$ is a timelike ruled minimal surface which is called the *timelike helicoid* with *spacelike* axis.

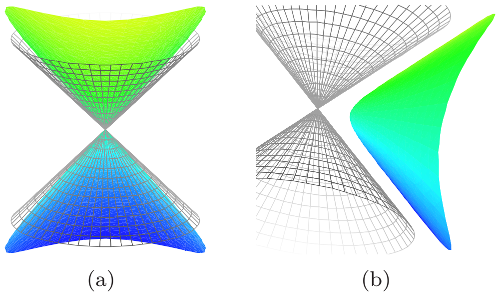


FIGURE 6. Timelike catenoid φ (a) and Timelike helicoid $\hat{\varphi}$ (b) with spacelike axis

Example 7.2 (Timelike catenoid and Timelike helicoid with timelike axis)**.** Next, let us consider the following timelike minimal surfaces:

$$\varphi(u,v) = X(u) + Y(v),$$

where

$$X(u) = (u, \sin u, -\cos u), \quad Y(v) = (-v, -\sin v, \cos v).$$

Then

$$\varphi(u,v) = \left(u-v, 2\sin\left(\frac{u-v}{2}\right)\cos\left(\frac{u+v}{2}\right), 2\sin\left(\frac{u-v}{2}\right)\cos\left(\frac{u+v}{2}\right)\right),$$

$$\hat{\varphi}(u,v) = \left(u+v, 2\cos\left(\frac{u-v}{2}\right)\sin\left(\frac{u+v}{2}\right), -2\cos\left(\frac{u-v}{2}\right)\cos\left(\frac{u+v}{2}\right)\right).$$

The surface φ is a timelike minimal surface of revolution with timelike profile curve and timelike axis, which is called the *Lorentz catenoid* (or *timelike catenoid*) with *timelike axis*. The conjugate timelike minimal surface $\hat{\varphi}$ is a timelike ruled minimal surface which is called the *timelike helicoid* with *timelike axis*.

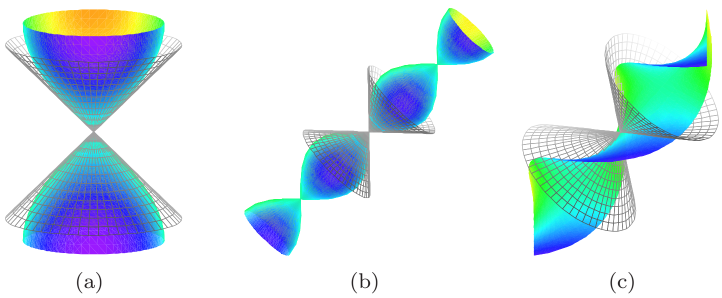


FIGURE 7. Timelike catenoid φ (a), (b) and Timelike helicoid $\hat{\varphi}$ (c) with timelike axis

7.3. The Weierstrass formula. Let $\varphi : M \to \mathbb{E}^3_1$ be a timelike surface parametrized by null coordinates (u, v). Hereafter we assume that M is a simply connected region of the Minkowski plane $\mathbb{E}^2_1 = (\mathbb{R}^2, \mathrm{d}u\mathrm{d}v)$ containing the origin $(0, 0)$.

Let $\boldsymbol{n}$ be the unit normal vector field to M, *i.e.*, $\boldsymbol{n}$ is a vector field along φ which satisfies

$$\langle \varphi_u, \boldsymbol{n}\rangle = \langle \varphi_v, \boldsymbol{n}\rangle = 0, \quad \langle \boldsymbol{n}, \boldsymbol{n}\rangle = 1.$$

then one can check that $Q\mathrm{d}u^2 + R\mathrm{d}v^2$ is globally defined on the Lorentz surface M, where $Q = \langle \varphi_{uu}, \boldsymbol{n}\rangle$ and $R = \langle \varphi_{vv}, \boldsymbol{n}\rangle$. The differential $Q\mathrm{d}u^2 + R\mathrm{d}v^2$ is called the *Hopf differential* of M.

The Codazzi equations of (M, φ) imply that the mean curvature H is constant if and only if $Q_v = R_u = 0$ (See [36], [18], [38]).

Now let $\varphi : M \to \mathbb{E}^3_1$ be a timelike minimal surface parametrized by null coordinates (u, v).

Define two vector valued functions $\xi = (\xi_1, \xi_2, \xi_3)$ and $\eta = (\eta_1, \eta_2, \eta_3)$ by

$$\xi(u) := \varphi_u, \quad \eta(v) := \varphi_v,$$

in other words,

$$X(u) = \int_0^u \xi(u)\ \mathrm{d}u, \quad Y(v) = \int_0^v \eta(v)\ \mathrm{d}v$$

for the timelike minimal surface $\varphi(u, v) = X(u) + Y(v)$. By definition, ξ and η satisfy

$$-\xi_1^2 + \xi_2^2 + \xi_3^2 = -\eta_1^2 + \eta_2^2 + \eta_3^2 = 0.$$

Define the functions $f(u)$,$q(u)$, $g(v)$ and $r(v)$ by

$$-\xi_1 + \xi_2 = f, \quad \xi_3 = qf,$$

$$\eta_1 + \eta_2 = g, \quad \eta_3 = rg.$$

Then we have

$$\xi(u) = (-\frac{1}{2}(1 + q(u)^2), \frac{1}{2}(1 - q(u)^2), q(u))f(u),$$

$$\eta(v) = (\frac{1}{2}(1 + r(v)^2), \frac{1}{2}(1 - r(v)^2), r(v))g(v).$$

Thus the original timelike minimal surface is represented by

$$\varphi(u, v) = \int_0^u \left(-\frac{1}{2}(1 + q(u)^2), \frac{1}{2}(1 - q(u)^2), q(u)\right) f(u)\mathrm{d}u + \int_0^v \left(\frac{1}{2}(1 + r(v)^2), \frac{1}{2}(1 - r(v)^2), r(v)\right) g(v)\mathrm{d}v \tag{7.2}$$

up to translations. This is the *Weierstrass formula* obtained by Magid. See [46, Theorem 4.3 and p. 456, Notes 2]. The first fundamental form of φ is given by

$$\mathrm{I} = \{1 + q(u)r(v)\}^2 f(u)g(v)\ \mathrm{d}u\mathrm{d}v.$$

The unit normal vector field $\boldsymbol{n}$ of φ is given by

$$\boldsymbol{n} = \frac{1}{1 + qr}(q - r, q + r, -1 + qr).$$

Under the stereographic projection ([38, Example 3.18]):

$$\wp_+ : \mathbb{S}^2_1 \setminus \{z = -1\} \to \mathbb{E}^2_1 \setminus \mathbb{H}^1_0,$$

the unit normal $\boldsymbol{n}$ is projected to

$$\frac{1}{2qr}(q-r, q+r) \in \mathbb{E}^2_1.$$

The $\mathbb{E}^2_1$-valued function (q, r) is frequently referred as the *projected Gauss map* of (M, φ).

The Hopf differential of φ is given by

$$Q(u) = \frac{\mathrm{d}q}{\mathrm{d}u}(u)f(u), \quad R(v) = \frac{\mathrm{d}r}{\mathrm{d}v}(v)g(v).$$

Locally one can reparametrize φ so that $f(u) = g(v) = 1$. Then under this parametrization, the null curves

$$X(u) = \int_0^u \left(-\frac{1}{2}(1+q(u)^2), \frac{1}{2}(1-q(u)^2), q(u)\right) \mathrm{d}u,$$

$$Y(v) = \int_0^v \left(\frac{1}{2}(1+r(v)^2), \frac{1}{2}(1-r(v)^2), r(v)\right) \mathrm{d}v$$

have null Frenet frame fields

$$A^X(u) = \left(-\frac{1}{2}(1+q(u)^2), \frac{1}{2}(1-q(u)^2), q(u)\right),$$

$$B^X(u) = (1,1,0), \quad C^X(u) = (-q(u), -q(u), 1),$$

$$A^Y(v) = \left(\frac{1}{2}(1+r(v)^2), \frac{1}{2}(1-r(v)^2), r(v)\right),$$

$$B^Y(v) = (-1,1,0), \quad C^Y(v) = (r(v), -r(v), 1),$$

respectively. The curvature functions of X and Y are

$$k^X_2(u) = q_u = Q, \quad k^Y_2(v) = r_v = R, \ k^X_3(u) = k^Y_3(v) = 0.$$

Example 7.3 (Timelike Enneper surfaces)**.** Take $q = \varepsilon u = \pm u$, $r = v$ in

$$\varphi^{(\varepsilon)}_u = (-\frac{1}{2}(1+q^2), \frac{1}{2}(1-q^2), q), \quad \varphi^{(\varepsilon)}_v = (\frac{1}{2}(1+r^2), \frac{1}{2}(1-r^2), r),$$

Then we obtain the following immersion:

$$\varphi^{(\varepsilon)}(u, v) = X(u) + Y(v),$$

where

$$X(u) = \frac{1}{2}(-(u+\frac{u^3}{3}), u - \frac{u^3}{3}, \varepsilon u^2),$$

$$Y(v) = \frac{1}{2}(v + \frac{v^3}{3}, v - \frac{v^3}{3}, v^2).$$

These formulas show that the parameters u and v are pseudoarc paremeters of $X(u)$ and $Y(v)$, respectively. And hence $X(u)$ and $Y(v)$ are null cubics.

The timelike surface $\varphi^{(1)}$ may be regarded as "Lorentzian cousin" of Enneper's minimal surface. The Hopf differential and the Gaussian curvature of $\varphi^{(\varepsilon)}$ are

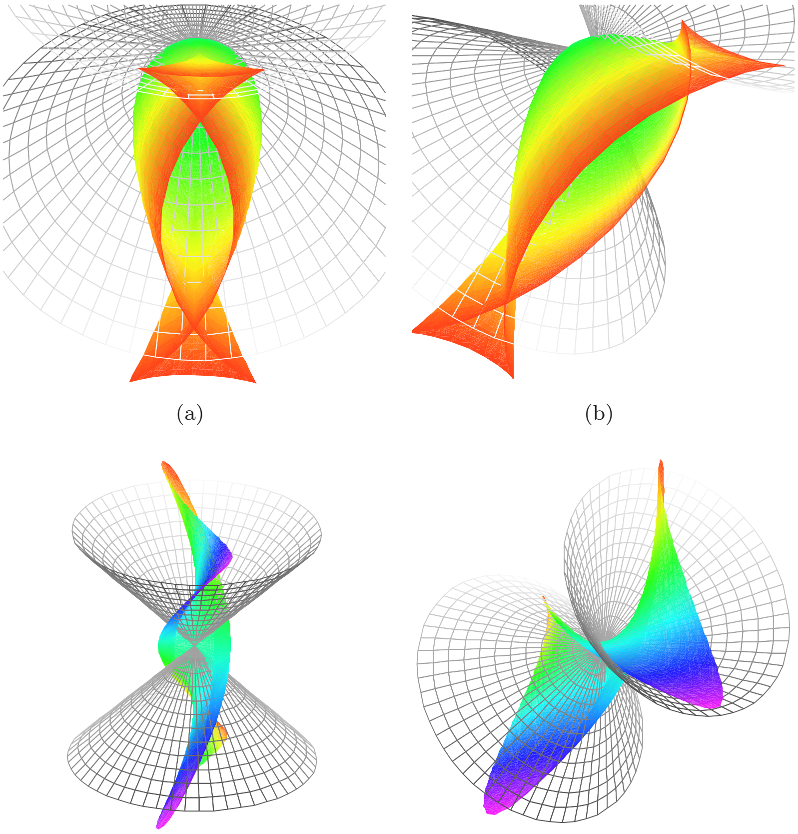
$$Q = \varepsilon, \ R = 1, \ K = -4\varepsilon(1+\varepsilon uv)^{-4}. \tag{7.3}$$

The metric of $\varphi^{(\varepsilon)}$ is

$$\mathrm{I} = (1+\varepsilon uv)^2 \mathrm{d}u\mathrm{d}v. \tag{7.4}$$

The surface $\varphi^{(1)}$ has real distinct principal curvatures. On the contrary, $\varphi^{(-1)}$ has no Euclidean counterpart, since it has imaginary principal curvatures. Both the surfaces are *foliated by null cubics* discussed in Example 2.2.

The Enneper's minimal surface in Euclidean 3-space does not have such a property.



(a) (b) (c) (d)

FIGURE 8. Timelike Enneper surface (isothermic type) (a), (b) and Timelike Enneper surface (anti-isothermic type) (c), (d)

Remark 7.2 (Isothermic and anti-isothermic coordinate systems). Let $\varphi : M \to N^3_1$ be a timelike surface in a Lorentz 3-manifold.

(1) A local null coordinate system (u, v) is said to be *isothermic* if the metric and Hopf differential of the surface have the form:

$$\mathrm{I} = e^{\omega}\mathrm{d}u\mathrm{d}v, \quad Q(u,v) = \frac{1}{2}\mathfrak{q}(u,v)\varrho(u), \quad R(u,v) = \frac{1}{2}\mathfrak{q}(u,v)\sigma(v),$$

where ϱ and σ are positive Lorentz holomorphic function and positive Lorentz anti-holomorphic function, respectively.

(2) A local null coordinate system (u, v) is said to be *anti-isothermic* if the metric and Hopf differential of the surface have the form:

$$\mathrm{I} = e^{\omega}\mathrm{d}u\mathrm{d}v, \quad Q(u,v) = \frac{1}{2}\mathfrak{q}(u,v)\varrho(u), \quad R(u,v) = -\frac{1}{2}\mathfrak{q}(u,v)\sigma(v),$$

where ϱ and σ are positive Lorentz holomorphic function and positive Lorentz anti-holomorphic function, respectively. The anti-isothermic property implies that the timelike surface has non-real (imaginary) principal curvatures.

(3) The formulas (7.3)–(7.4) imply that the timelike Enneper surface $\varphi^{(1)}$ is parametrized by isothermic coordinate system (u, v). On the other hand, $\varphi^{(-1)}$ is parametrized by anti-isothermic coordinate system (u, v). For more details on isothermic and anti-isothermic coordinate system, we refer to [26]–[27].

Example 7.4 (Flat B-scoll)**.** Next we consider a timelike minimal surface with with $q \neq 0$ and $r = 0$. By the classical formula, we obtain

$$\varphi(u,v) = X(u) + Y(v),$$

$$X(u) = \int_0^u \left(-\frac{1}{2}(1+q(u)^2), \frac{1}{2}(1-q(u)^2), q(u)\right)\mathrm{d}u, \quad Y(v) = v\left(\frac{1}{2}, \frac{1}{2}, 0\right)$$

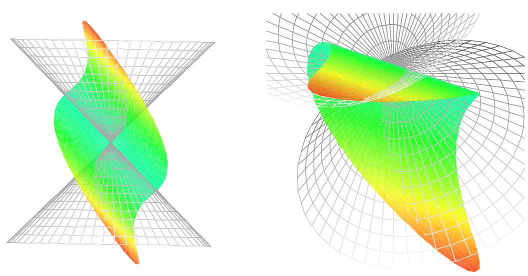
This explicit representation shows that φ is a ruled surface. The null Frenet frame field (A^X, B^X, C^X) along the null curve $X(u)$ is given by

$$A^X(u) = X_u(u) = \left(-\frac{1}{2}(1+q(u)^2), \frac{1}{2}(1-q(u)^2), q(u)\right),$$

$$B^X(u) = (1, 1, 0), \quad C^X(u) = (-q, -q, 1).$$

Hence we notice the $Y(v) = vB^X(u)/2$. Thus φ is the B-scroll of $X(u)$. Note that the unit normal vector field of φ is $N = C$. The second fundamental form of φ is described as $Q = q_u(u)$, $R = 0$, $K = H = 0$. Hence φ has principal curvature 0 with multiplicity 2. However φ is not totally geodesic since $q_u \neq 0$ by our assumption $q \neq 0$. This timelike surface has no Euclidean counterpart. Reparametrize (u, v) so that $Q = 1$. Without loss of generality, we may assume that $q(u) = u$. Then u is the pseudoarc parameter of $X(u)$. Then φ is parametrized as

$$\varphi(u,v) = \frac{1}{2}\left(-\frac{u^3}{3} - u + v, -\frac{u^3}{3} + u + v, u^2\right).$$



(a) (b)

FIGURE 9. Flat B-scroll

Hence φ is a cylinder over a parabora $2\xi_3 = (-\xi_1+\xi_2)^2$. P. Mira and J. A. Pastor called a minimal B-scroll a *parabolic null cylinder* [50].

Example 7.5 (Timelike catenoid and Timelike helicoid with spacelike axis)**.** The timelike catenoid and timelike helicoid with spacelike axis in Example 7.1 can be obtained by the Weierstrass formula (7.2) with the data $q(u) = -e^u$, $f(u) = -e^{-u}$, $r(v) = e^{-v}$, $g(v) = \mp e^v$, respectively. The ordered pair $(q, r) = (-e^u, e^{-v})$ is the Gauss map projected in $\mathbb{E}^2_1(u, v)$. The timelike catenoid and timelike helicoid share the same Gauss map analogously to the Euclidean case.

Example 7.6 (Timelike catenoid and Timelike helicoid with timelike axis)**.** The timelike catenoid and timelike helicoid with timelike axis in Example 7.2 can be obtained by the Weierstrass formula (7.2) with the data $q(u) = \frac{\sin u}{-1+\cos u}$, $f(u) = -1 + \cos u$, $r(v) = \frac{\sin v}{1+\cos v}$, $g(v) = \mp(1 + \cos v)$, respectively. The ordered pair $(q, r) = \left(\frac{\sin u}{-1+\cos u}, \frac{\sin v}{1+\cos v}\right)$ is the Gauss map projected in $\mathbb{E}^2_1(u, v)$. The timelike catenoid and timelike helicoid share the same Gauss map analogously to the Euclidean case.

8. Applications: Timelike CMC-1 surfaces

In this section, we study a representation formula for timelike surfaces of constant mean curvature 1 (abbreviated as CMC-1) in anti-de Sitter 3-space $\mathbb{H}^3_1$ [34], [45]. It is interesting that timelike CMC-1 surfaces in $\mathbb{H}^3_1$ can be constructed by a pair of Lorentz holomorphic and Lorentz anti-holomorphic null curves in $\mathrm{PSL}(2;\mathbb{R}) = \mathrm{SL}(2;\mathbb{R})/\{\pm\mathrm{id}\}$. That is, there is a Weierstrass formula for timelike CMC-1 surfaces in $\mathbb{H}^3_1$. This is not a coincidence due to Lawson-Guichard correspondence between timelike minimal surfaces in $\mathbb{E}^3_1$ and timelike CMC-1 surfaces (See [26], [27], [45].)

The *anti-de Sitter* 3*-space* $\mathbb{H}^3_1$ is realized as a hyperquadric:

$$\mathbb{H}^3_1 = \{(\xi_0, \xi_1, \xi_2, \xi_3) \in \mathbb{E}^4_2 \mid -\xi_0^2 - \xi_1^2 + \xi_2^2 + \xi_3^2 = -1\}$$

in the semi-Euclidean 4-space $\mathbb{E}^4_2$ with index 2. The semi-Euclidean 4-space $\mathbb{E}^4_2$ is identified with the space $\mathrm{M}(2;\mathbb{R})$ of all 2 by 2 real matrices via the correspondence;

$$(\xi_0, \xi_1, \xi_2, \xi_3) \longleftrightarrow \begin{pmatrix} \xi_0 - \xi_3 & -\xi_1 + \xi_2 \\ \xi_1 + \xi_2 & \xi_0 + \xi_3 \end{pmatrix}. \tag{8.1}$$

The scalar product $\langle \cdot, \cdot \rangle$ of $\mathbb{E}^4_2$ corresponds to the scalar product

$$\langle X, Y \rangle = \frac{1}{2}\{\mathrm{tr}(XY) - \mathrm{tr}(X)\mathrm{tr}(Y)\} \tag{8.2}$$

on $\mathrm{M}(2;\mathbb{R})$. Under the identification $\mathbb{E}^4_2 = \mathrm{M}(2;\mathbb{R})$, the anti de Sitter 3-space $\mathbb{H}^3_1$ is identified with the real special linear group $\mathrm{SL}(2;\mathbb{R})$. The metric on $\mathbb{H}^3_1$ induced by (8.2) is bi-invariant. In fact, the product group $\mathrm{SL}(2;\mathbb{R})\times\mathrm{SL}(2;\mathbb{R})$ acts isometrically and transitively on $\mathbb{H}^3_1$ via the action:

$$(a, b) \cdot X = aXb^t.$$

The isotropy subgroup at the identity is

$$K = \{(a, (a^t)^{-1}) \mid a \in \mathrm{SL}(2;\mathbb{R})\}.$$

This isotropy group is isomorphic to $\mathrm{SL}(2;\mathbb{R})$. Hence $\mathbb{H}^3_1$ is represented as $\mathbb{H}^3_1 = \mathrm{SL}(2;\mathbb{R}) \times \mathrm{SL}(2;\mathbb{R})/\mathrm{SL}(2;\mathbb{R})$ as a Lorentzian symmeteric space. The natural projection $\pi : \mathrm{SL}(2;\mathbb{R})\times\mathrm{SL}(2;\mathbb{R}) \to \mathbb{H}^3_1$ is given by $\pi(a,b) = ab^t$. For more information on the $\mathrm{SL}(2;\mathbb{R})$-model of $\mathbb{H}^3_1$, we refer to [15], [36], [38], [45].

The following result is fundamental in this section.

Proposition 8.1. *Let $\gamma : I \to \mathbb{H}^3_1$ be a curve. then the following conditions are mutually equivalent*:

- *γ is a null curve, i.e., γ' is a null vector field along γ,*
- *$\det(\gamma^{-1}\gamma') = 0$.*

This characterization motivates us the following definition.

Definition 8.1. Let M be a 2-manifold and $F : M \longrightarrow \mathrm{SL}(2;\mathbb{R})$ a map. Then F is said to be *null* if $\det(F^{-1}\mathrm{d}F) = 0$.

Theorem 8.1 (Weierstrass-Bryant representation [34],[45])**.** *Let M be a Lorentz surface and $F = (F_1, F_2) : M \longrightarrow \mathrm{SL}(2;\mathbb{R}) \times \mathrm{SL}(2;\mathbb{R})$ an immersion such that*

(1) *F_1 is Lorentz holomorphic, i.e., $(F_1)_v = 0$ and F_2 is Lorentz anti-holomorphic, i.e., $(F_2)_u = 0$,*
(2) *F_1 and F_2 are null curves, i.e., $\det(F_1^{-1}\mathrm{d}F_1) = \det(F_2^{-1}\mathrm{d}F_2) = 0$.*

Then

$$\psi := F_1 F_2^t \tag{8.3}$$

is a smooth conformal timelike immersion into $\mathbb{H}^3_1$ with CMC-1. *Conversely, let M be an oriented and simply-connected Lorentz surface with globally defined null coordinates. If $\psi : M \longrightarrow \mathbb{H}^3_1$ is a smooth conformal immersion with* CMC-1*, then there exists an immersion $F = (F_1, F_2) : M \longrightarrow \mathrm{SL}(2;\mathbb{R}) \times \mathrm{SL}(2;\mathbb{R})$ such that F_1, F_2 satisfy the conditions (1), (2), and $\psi = F_1F_2^t$.*

Let $\psi : M \longrightarrow \mathbb{H}^3_1$ be a timelike CMC-1 surface. Then, by Theorem 8.1, there exists a smooth immersion $F = (F_1, F_2) : M \longrightarrow \mathrm{SL}(2;\mathbb{R})\times\mathrm{SL}(2;\mathbb{R})$ satisfying (1), (2) and $\psi = F_1F_2^t$.

Locally in an open set $U \subset M$,

$$F_1^{-1}\mathrm{d}F_1 = \begin{pmatrix} p_1q_1 & -p_1^2 \\ q_1^2 & -p_1q_1 \end{pmatrix} e^{\frac{\omega}{2}}\mathrm{d}u, \quad F_2^{-1}\mathrm{d}F_2 = \begin{pmatrix} p_2q_2 & -p_2^2 \\ q_2^2 & -p_2q_2 \end{pmatrix} e^{\frac{\omega}{2}}\mathrm{d}v,$$

where $\begin{pmatrix} p_1 & -q_2 \\ q_1 & p_2 \end{pmatrix} \in \mathrm{SL}(2;\mathbb{R})$ and $\omega : M \longrightarrow \mathbb{R}$.

Let $q := \frac{p_1}{q_1}$, $f(u) := q_1^2$, $r := \frac{p_2}{q_2}$, and $g(v) := q_2^2$. Then the Weierstrass formula (7.2) defines a timelike minimal surface $\varphi : U \longrightarrow \mathbb{E}^3_1$. Note that the ordered pair (q, r) coincides with the *stereographically* projected Gauss map of the timelike minimal surface φ. The induced metric I_ψ of (U, ψ) is related to the induced metric I_φ of (U, φ) by

$$\mathrm{I}_\psi = e^\omega(1 + qr)^2 f(u)g(v)\mathrm{d}u\mathrm{d}v = e^\omega\ \mathrm{I}_\varphi.$$

Thus the induced metric I_ψ of the timelike CMC-1 surface ψ is conformal to I_φ.

Conversely, assume that a timelike minimal surface $\varphi : M \longrightarrow \mathbb{E}^3_1$ is given by the Weierstrass formula (7.2) with data $(q, f(u))$ and $(r, g(v))$. Consider the following system of differential equations:

$$\frac{\mathrm{d}F_1}{\mathrm{d}u} = F_1 \begin{pmatrix} q & -q^2 \\ 1 & -q \end{pmatrix} e^{\frac{\omega}{2}} f(u), \quad \frac{\mathrm{d}F_2}{\mathrm{d}v} = F_2 \begin{pmatrix} r & -r^2 \\ 1 & -r \end{pmatrix} e^{\frac{\omega}{2}} g(v). \tag{8.4}$$

The linear system (8.4) is called the *Brayant-Umehara-Yamada* type representation formula. These equations in (8.4) satisfy the integrability condition, so there exists a solution $(F_1, F_2) : M \longrightarrow \mathrm{SL}(2;\mathbb{R}) \times \mathrm{SL}(2;\mathbb{R})$ satisfying the conditions (1) and (2). By Theorem 8.1, $\psi := F_1 F_2^t : M \longrightarrow \mathbb{H}^3_1$ defines a timelike CMC-1 surface in $\mathbb{H}^3_1$.

Hence, we see that timelike CMC-1 surfaces in $\mathbb{H}^3_1$ are closely related to timelike minimal surfaces in $\mathbb{E}^3_1$. More specifically, there is a one-to-one correspondence between timelike CMC-1 surfaces in $\mathbb{H}^3_1$ and timelike minimal surfaces in $\mathbb{E}^3_1$. Note that the Gauss-Codazzi equations are preserved by this correspondence, *i.e.*, those correspondents in $\mathbb{H}^3_1$ and in $\mathbb{E}^3_1$ satisfy the same Gauss-Codazzi equations. This is a special case of the so-called *Lawson-Guichard correspondence* between timelike CMC surfaces in semi-Riemannian space forms $\mathbb{E}^3_1$, $\mathbb{S}^3_1$, and $\mathbb{H}^3_1$. Here, $\mathbb{S}^3_1$ denotes the *de Sitter* 3*-space* which can be realized as the hyperquadric in Minkowski 4-space $\mathbb{E}^4_1$:

$$\mathbb{S}^3_1 := \{(\xi_0, \xi_1, \xi_2, \xi_3) \in \mathbb{E}^4_1 \mid -\xi_0^2 + \xi_1^2 + \xi_2^2 + \xi_3^2 = 1\}.$$

Let $\wp_+ : \mathbb{H}^3_1 \setminus \{\xi_0 = -1\} \longrightarrow \mathbb{E}^3_1 \setminus \mathbb{S}^2_1$ be the stereographic projection from $-\boldsymbol{e}_0 = (-1, 0, 0, 0)$:

$$\wp_+(\xi_0, \xi_1, \xi_2, \xi_3) = \left(\frac{\xi_1}{1+\xi_0}, \frac{\xi_2}{1+\xi_0}, \frac{\xi_3}{1+\xi_0}\right). \tag{8.5}$$

Let $\wp_- : \mathbb{H}^3_1 \setminus \{\xi_0 = 1\} \longrightarrow \mathbb{E}^3_1 \setminus \mathbb{S}^2_1$ be the stereographic projection from $e_0 = (1, 0, 0, 0)$:

$$\wp_-(x_0, x_1, x_2, x_3) = \left(\frac{x_1}{1-x_0}, \frac{x_2}{1-x_0}, \frac{x_3}{1-x_0}\right). \tag{8.6}$$

Cut $\mathbb{H}^3_1$ into two halves by the hyperplane $\xi_0 = 0$. Denote by $(\mathbb{H}^3_1)_+$ (resp. $(\mathbb{H}^3_1)_-$) the half containing $\boldsymbol{e}_0 = (1, 0, 0, 0)$ (resp. $-\boldsymbol{e}_0 = (-1, 0, 0, 0)$). Then $\wp_+ : (\mathbb{H}^3_1)_+ \longrightarrow \mathrm{Int}\ \mathbb{S}^2_1$ and $\wp_- : (\mathbb{H}^3_1)_- \longrightarrow \mathrm{Int}\ \mathbb{S}^2_1$.

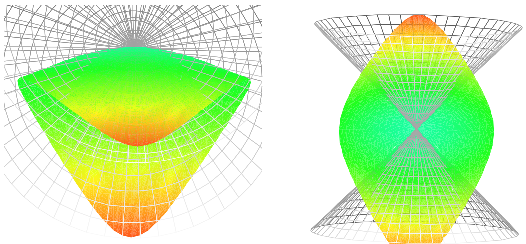
Example 8.1 (Timelike Enneper Cousin in $\mathbb{H}^3_1$ of isothermic type)**.** Let $(q(u), r(v)) = (u, v)$. Then using the Bryant-Umehara-Yamada type representation formula (8.4), we set up the following initial value problem:

$$F_1^{-1}\mathrm{d}F_1 = \begin{pmatrix} u & -u^2 \\ 1 & -u \end{pmatrix} \mathrm{d}u,\ F_2^{-1}\mathrm{d}F_2 = \begin{pmatrix} v & -v^2 \\ 1 & -v \end{pmatrix} \mathrm{d}v$$

with the initial condition $F_1(0) = F_2(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. This initial value problem has a unique solution

$$F_1(u) = \begin{pmatrix} \cosh u & \sinh u - u\cosh u \\ \sinh u & \cosh u - u \sinh u \end{pmatrix},$$
$$F_2(v) = \begin{pmatrix} \cosh v & \sinh v - v\cosh v \\ \sinh v & \cosh v - v \sinh v \end{pmatrix}$$

which are Lorentz holomorphic and Lorentz anti-holomorphic null curves in $\mathrm{SL}(2;\mathbb{R})$. The Weierstrass-Bryant representation formula (8.3) then yields a timelike CMC-1 surfaces in $\mathbb{H}^3_1$. The resulting surface is a correspondent of isothermic type timelike Enneper surface in $\mathbb{E}^3_1$ under the Lawson-Guichard correspondence. For this reason, the resulting surface is called *isothermic type timelike Enneper cousin* in $\mathbb{H}^3_1$.



(a) (b)

FIGURE 10. Isothermic type timelike Enneper cousin projected into Int $\mathbb{S}^2_1$ via $\wp_+$ with lightcone in $\mathbb{E}^3_1$

Example 8.2 (Timelike Enneper Cousin in $\mathbb{H}^3_1$ of anti-isothermic type)**.** Let $(q(u), r(v)) = (-u, v)$. Then using the Bryant-Umehara-Yamada type representation (8.4), we set up the following initial value problem:

$$F_1^{-1}\mathrm{d}F_1 = \begin{pmatrix} -u & -u^2 \\ 1 & u \end{pmatrix} \mathrm{d}u, \; F_2^{-1}\mathrm{d}F_2 = \begin{pmatrix} v & -v^2 \\ 1 & -v \end{pmatrix} \mathrm{d}v$$

with the initial condition $F_1(0) = F_2(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. This initial value problem has a unique solution

$$F_1(u) = \begin{pmatrix} \cos u & -\sin u + u\cos u \\ \sin u & \cos u + u \sin u \end{pmatrix},$$
$$F_2(v) = \begin{pmatrix} \cosh v & \sinh v - v\cosh v \\ \sinh v & \cosh v - v \sinh v \end{pmatrix}$$

which are Lorentz holomorphic and Lorentz anti-holomorphic null curves into $\mathrm{SL}(2;\mathbb{R})$. The Weierstrass-Bryant representation formula (8.3) then yields a timelike CMC-1 surface in $\mathbb{H}^3_1$. The resulting surface is a correspondent of anti-isothermic type timelike Enneper surface in $\mathbb{E}^3_1$ under the Lawson-Guichard correspondence. For this reason, the resulting surface is called *anti-isothermic type timelike Enneper cousin* in $\mathbb{H}^3_1$. Figure 11 shows the anti-isothermic type timelike Enneper cousin in $\mathbb{H}^3_1$ projected via $\wp_+$ into the interior of the boundary $\mathbb{S}^2_1$.

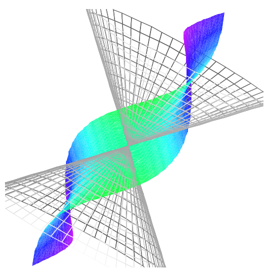


FIGURE 11. Anti-isothermic type timelike Enneper cousin projected into Int $\mathbb{S}^2_1$ via $\wp_+$ with lightcone in $\mathbb{E}^3_1$

Example 8.3 (Timelike Catenoid Cousins in $\mathbb{H}^3_1$)**.** Let $q(u) = -e^u$, $f(u) = -e^{-u}$, $r(v) = e^{-v}$ and $g(v) = -e^v$. We solve the initial value problem:

$$F_1^{-1}\mathrm{d}F_1 = \begin{pmatrix} 1 & e^u \\ -e^{-u} & -1 \end{pmatrix}\mathrm{d}u, \quad F_2^{-1}\mathrm{d}F_2 = \begin{pmatrix} -1 & e^{-v} \\ -e^{v} & 1 \end{pmatrix}\mathrm{d}v$$

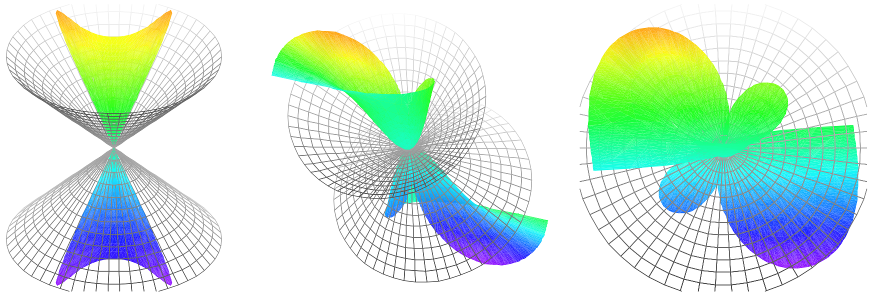
with $F_1(0) = F_2(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

The resulting surface $F_1F_2^t$ (Figure 12(a)) is a timelike catenoid cousin $\mathbb{H}^3_1$, which is corresponded to timelike catenoid with spacelike axis in $\mathbb{E}^3_1$ via the Lawson-Guichard correspondence.

Now, let us take $q(u) = \frac{\sin u}{-1+\cos u}$, $f(u) = -1 + \cos u$, $r(v) = \frac{\sin v}{1+\cos v}$ and $g(v) = -(1+\cos v)$. Then we solve the initial value problem:

$$F_1^{-1}\mathrm{d}F_1 = \begin{pmatrix} \sin u & -\frac{\sin^2 u}{-1+\cos u} \\ -1+\cos u & -\sin u \end{pmatrix}\mathrm{d}u,$$
$$F_2^{-1}\mathrm{d}F_2 = \begin{pmatrix} -\sin v & \frac{\sin^2 v}{1+\cos v} \\ -1-\cos v & \sin v \end{pmatrix}\mathrm{d}v$$

with $F_1(\pi/2) = F_2(\pi/2) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. The resulting surface $F_1F_2^t$ (Figures 12(b), 12(c)) is a timelike catenoid cousin $\mathbb{H}^3_1$, which is corresponded to timelike catenoid with timelike axis in $\mathbb{E}^3_1$ (Example 7.1, Example 7.5) via the Lawson-Guichard correspondence.



(a) (b) (c)

FIGURE 12. Timelike Enneper cousins projected into Int $\mathbb{S}^2_1$ via $\wp_+$ with lightcone in $\mathbb{E}^3_1$

For more details on timelike CMC-1 surfaces in $\mathbb{H}^3_1$, we refer to [45].

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