INTERNATIONAL ENGINEERING, SCIENCE AND EDUCATION GROUP

**Middle East Journal of Science**
(2019) 5(1): 73-85
Published online June, 2019 (http://dergipark.org.tr/mejs)
doi: 10.23884/mejs.2019.5.1.08
e-ISSN 2618-6136
Received: December 10, 2018 Accepted: April 26, 2019
Submission Type: Research Article

# SOLVING INTEGRO-DIFFERENTIAL EQUATIONS USING EXPONENTIALLY FITTED COLLOCATION APPROXIMATE TECHNIQUE (EFCAT)

***FALADE K.I***
Department of Mathematics, Faculty Computing and Mathematical Sciences,
Kano University of Science and Technology, P.M.B 3244 Wudil Kano State Nigeria.
ORCID: 0000-0001-7572-5688
Corresponding author: faladekazeem2013@gmail.com

**Abstract:** *In this paper, we present and employ Exponentially Fitted Collocation Approximate Technique (EFCAT) to solve linear Volterra and Fredholm integro-differential equations.The collocated-perturbed Integro-differential equations were transformed into square matrix form which eventually solves using MAPLE 18 software. To demonstrate the applicability of the present method four examples is considered. It is observed that the present technique is in good agreement with the analytical solution and available methods in the literature.*

**Keywords***: Volterra and Fredholm integro-differential equations, exponentially fitted collocation approximate technique, analytical solution, Maple 18 software.*

## 1. Introduction

Integro-differential equations find special applicability within scientific and mathematical disciplines. It plays an important role in many branches of mathematical sciences and their applications in the theory of engineering, physics, mechanics, chemistry, astronomy, biology, economics, potential theory, and electrostatics. The theory and application of integrodifferential equations are important roles in engineering and applied sciences. The existence and uniqueness of the solutions of integrodifferential equations usually discussed in terms of their kernel have been established in [1]. The integro-differential equations are usually difficult to solve analytically thus, it requires suitable numerical techniques to obtain analytic-numeric solutions of Integro-differential equations. Therefore, several authors have proposed and applied different methods to obtain the solution of both linear and nonlinear IDEs such as, Adomian decomposition [2], Homotopy perturbation method [3], variation iteration method [4], Chebyshev polynomial collocation [5], The Taylor expansion approach [6], Bessel or Chebyshev polynomial approach are used to solve integro-differential equations in [7] and just mention a few.

Generally, integro-differential equations are difficult to solve, thus this present work is to apply a technique proposed in [8] to solve linear Volterra and Fredholm integro-differential equations, which promise to be a reliable, easy, fast and accurate numerical technique to obtain numerical solution of linear Volterra and Fredholm integro-differential equations. We obtain derivative of power series of function $y(x)$ and substitute into the linear integro differential equation. Slightly perturbation and collocation are carried out which eventually transform to square matrix form and MAPLE 18 software is used to obtain the unknown constants.

In this paper, we considered $pth$ order general integro-differential equation of the form:

$$\begin{cases} \dfrac{d^p y}{dx^p} + \alpha(x)\dfrac{d^{p-1}y}{dx^{p-1}} + \beta(x) + \dfrac{d^{p-2}y}{dx^{p-2}} + \cdots + y(x) \\ = g(x) + \psi(x)\displaystyle\int_a^b \gamma(t,x)\,y(t)dt \qquad x \in [a,b] \end{cases} \qquad (1)$$

subject to initial conditions

$$\begin{cases} y(a) = A \\ \dfrac{dy}{dx}(a) = B \\ \dfrac{d^2y}{dx^2}(a) = C \\ \quad . \\ \quad . \\ \quad . \\ \dfrac{d^{p-1}y}{dx^{p-1}}(a) = H \end{cases} \qquad (2)$$

Where $\alpha(x)$, $\beta(x)$, $g(x)$, $\psi(x)$ are continuous function on interval $[a,b]$, $\gamma(t,x)$ is a kernel and A,B,C,…..H are constants.

### 1.1 Defınıtıon of Chebyshev Polynomıals

The chebyshev polynomials of the first kind can be defined by the recurrence relation given by

$$T_0(x) = 1 \quad , \quad T_1(x) = 2x - 1$$

Thus, we have

$$T_{N+1}(x) = 2(2x-1)T_N(x) - T_{N-1}(x) \qquad N \geq 1 \quad (3)$$

**Table 1. The First ten (10) Chebyshev Polynomials**

| $T_N(x)$ | Chebyshev Polynomials |
|---|---|
| $T_0(x)$ | $1$ |
| $T_1(x)$ | $2x - 1$ |
| $T_2(x)$ | $8x^2 - 8x + 1$ |
| $T_3(x)$ | $32x^3 - 48x^2 + 18x - 1$ |
| $T_4(x)$ | $128x^4 - 258x^3 + 160x^2 - 32x + 1$ |
| $T_5(x)$ | $512x^5 - 1280x^4 + 1120x^3 - 400x^2 + 50x - 1$ |
| $T_6(x)$ | $2048x^6 - 6144x^5 + 6912x^4 - 3584x^3 + 640x^2 - 72x + 1$ |
| $T_7(x)$ | $8172x^7 - 28672x^6 + 39424x^5 - 26880x^4 + 9408x^3 - 1568x^2 + 98x - 1$ |
| $T_8(x)$ | $32768x^8 - 131072x^7 + 212992x^6 - 180224x^5 - 84480x^4 - 21504x^3 + 2688x^2 - 128x + 1$ |
| $T_9(x)$ | $131072x^9 - 589824x^8 + 1105920x^7 - 1118208x^6 + 658944x^5 - 228096x^4 + 44352x^3 - 4320x^2 + 162x - 1$ |
| $T_{10}(x)$ | $52488x^{10} - 2621440x^9 + 5570560x^8 - 6553600x^7 + 4659200x^6 - 2050048x^5 + 549120x^4 - 84480x^3 + 6600x^2 - 200x + 1$ |

## 2. Methodology

In this study, we consider the power series of the form

$$y(x) = \sum_{k=0}^{N} q_k x^k \qquad (4)$$

and exponentially fitted approximate solution proposed in [8]

$$y(x) \approx \sum_{k=0}^{N} q_k x^k + \tau_1 e^x \qquad (5)$$

Taking the first derivative of equation (4), we obtain

$$\frac{dy}{dx} = \sum_{k=1}^{N} k q_k x^{k-1} \qquad (6)$$

Suppose $p = 1$ and substitute equations (4) and (6) into equation (1), leads to

$$\sum_{k=1}^{N} k q_k x^{k-1} + \sum_{k=0}^{N} q_k x^k = g(x) + \psi(x)\left(\int_a^b \gamma(t,x) \sum_{k=0}^{N} q_k t^k dt\right) \qquad (7)$$

Expansion of (7), we obtain

$$(q_1 + 2q_2x + 3q_3x^2 + \cdots + Nq_Nx^{N-1}) + (q_0 + q_1x + q_2x^2 + q_3x^3 \ldots + q_Nx^N)$$
$$= g(x) + \psi(x)\left(\int_a^b \gamma(t,x)\,(q_0 + q_1t + q_2t^2 + q_3t^3 \ldots + q_Nt^N)dt\right) \qquad (8)$$

Collecting the likes terms, we have

$$\left(1 - \psi(x)\int_a^b \gamma(t,x)dt\right)q_0 + \left(1 + x - \psi(x)\int_a^b \gamma(t,x)tdt\right)q_1 +$$
$$\left(2x + x^2 - \psi(x)\int_a^b \gamma(t,x)t^2dt\right)q_2 + \left(3x^2 + x^3 - \psi(x)\int_a^b \gamma(t,x)t^3dt\right)q_3$$
$$+ \cdots + \left(Nx^{N-1} + x^N - \psi(x)\int_a^b \gamma(t,x)t^N dt\right)q_N = g(x) \qquad (9)$$

Slightly perturbe and collocate equation (9), we have

$$\left(1 - \psi(x)\int_a^b \gamma(t_i,x_i)dt\right)q_0 + \left(1 + x_i - \psi(x)\int_a^b \gamma(t_i,x_i)t_idt\right)q_1$$
$$+ \left(2x_i + x_i^2 - \psi(x)\int_a^b \gamma(t_i,x_i)x_i^2dt\right)q_2$$
$$+ \left(3x_i^2 + x_i^3 - \psi(x)\int_a^b \gamma(t_i,x_i)x_i^3dt\right)q_3$$
$$+ \left(Nx_i^{N-1} + x_i^N - \psi(x)\int_a^b \gamma(t_i,x_i)t_i^N dt\right)q_N - T_N(x_i)\tau_1 = g(x_i) \qquad (10)$$

Here $\tau_1$ are free tau parameter to be determined [8], $T_N(x_i)$ are the Chebyshev polynomials of degree N define in table 1 and

$$x_i = a + \frac{(b-a)i}{N+2}; \qquad i = 1,2,3 \dots . N+1$$

Hence, equation (10) gives rise to (N+1) algebraic linear system of equations in (N+2) unknown constants. One extra equation is obtained from the initial condition given as

$$y(a) = \sum_{k=0}^{N} q_k x^k + \tau_1 e^a \qquad (11)$$

Altogether, we obtained (N+2) algebraic linear equations in (N+2) unknown constants. Thus, we put the (N+2) algebraic equations in matrix form as

$$E_1 Q_1 = G_1(x) \qquad (12)$$

Where $E_1 =$ is the system of the equation of $N+2$ , $Q_1 = (q_0, q_1, q_2, q_3, \dots \dots \dots, q_N, \tau_1 )^T$

$$G_1(x) = (g(x_1), \dots \dots g(x_N), A )^T$$

MAPLE 18 software is used to obtain the unknown constants $q_0, q_1, q_2, q_3, \dots \dots \dots, q_N$ and $\tau_1$ and substitute into the exponentially fitted approximate solution (5).

Generally, we consider the order $p = 2,3,4,5$ for the equation (1) and repeat the procedures from equations (6) to (11), we obtain the follow:

When $p = 2$ we have

$$E_2 Q_2 = G_2(x) \qquad (13)$$

Where $E_2 =$ is the system of the equation of $N+3$ , $Q = (q_0, q_1, q_2, q_3, \dots \dots \dots, q_N, \tau_1, \tau_2 )^T$

$$G_2(x) = (g(x_1), g(x_2), \dots \dots g(x_N), A , B)^T$$

MAPLE 18 software is used to obtain the unknown constants $q_0, q_1, q_2, q_3, \dots \dots \dots, q_N$ and $\tau_1, \tau_2$ and substitute into the exponentially fitted approximate solution (5).

When $p = 3$ we have

$$E_3 Q_3 = G_3(x) \qquad (14)$$

Where $E_3 =$ is the system of the equation of $N+4$ , $Q_3 = (q_0, q_1, q_2, q_3, \dots \dots \dots, q_N, \tau_1, \tau_2 , \tau_3 )^T$

$$G_3(x) = (g(x_1), g(x_2), g(x_3), \dots \dots g(x_N), A , B, C)^T$$

MAPLE 18 software is used to obtain the unknown constants $q_0, q_1, q_2, q_3, \dots, q_N$ and $\tau_1, \tau_2, \tau_3$ *and* substitute into the exponentially fitted approximate solution (5).

When $p = 4$ we have

$$E_4 Q_4 = G_4(x) \qquad (15)$$

Where $E_4 =$ is the system of equation of $N+5$ , $Q_4 = (q_0, q_1, q_2, q_3, \dots \dots \dots, q_N, \tau_1, \tau_2 , \tau_3 , \tau_4)^T$

$$G_4(x) = (g(x_1), g(x_2), g(x_3), g(x_4) \dots \dots g(x_N), A , B, C, D)^T$$

MAPLE 18 software is used to obtain the unknown constants$q_0, q_1, q_2, q_3, \dots \dots \dots, q_N$ , $\tau_1$,$\tau_2$. $\tau_3, \tau_4$ and substitute into the exponentially fitted approximate solution (5).

When $p = 5$ we have

$$E_5 Q_5 = G_5(x) \qquad (16)$$

Where $E_5 =$ is the system of equation of $\mathrm{N}+6$ , $Q_4 = \left(q_0, q_1, q_2, q_3, \dots \dots \dots, q_N, \tau_1, \tau_2, \tau_3, \tau_4, \tau_{5,}\right)^T$

$$G_5(\mathrm{x}) = (g(x_1), g(x_2), g(x_3), g(x_4), g(x_5) \dots \dots g(x_N), A, B, C, D, E)^T$$

MAPLE 18 software is used to obtain the unknown constants$q_0, q_1, q_2, q_3, \dots \dots \dots, q_N$ , $\tau_1$, $\tau_2$. $\tau_3, \tau_4, \tau_5$ and substitute into the exponentially fitted approximate solution (5).

## 3. Numerical Application

In this section, we implement exponentially fitted collocation approximate technique on first, second, third and fifth orders Volterra and Fredholm Integro-Differential Equations. Four examples are considered to illustrate the accuracy and efficiency of the proposed method.

Example 1. Consider the First-order Volterra integro-differential equation [9]

$$\frac{dy}{dx} + y(x) = (x^2 + 2x + 1)e^{-x} + 5x^2 + 8 - \int_0^x t\, y(t)dt \qquad x \in [0,1] \qquad (17)$$

Subject to initial condition $y(0) = 10$ (18)

Analytical solution $y(x) = 10 - xe^{-x}$ (19)

EFCAT technique

Comparing equation (17) with equation (10) and $\psi(x) = -1$

Choosing computational length N=10, we obtain linear system of equation.

$$\begin{cases} \left(1 + \frac{1}{2}x_i^3\right)q_0 + \left(1 + x_i + \frac{1}{3}x_i^3\right)q_1 + \left(2x_i + x_i^3 + \frac{1}{4}x_i^4\right)q_2 + \\ \left(3x_i^3 + x_i^3 + \frac{1}{5}x_i^5\right)q_3 + \left(4x_i^3 + x_i^4 + \frac{1}{6}x_i^6\right)q_4 + \left(5x_i^4 + x_i^5 + \frac{1}{7}x_i^7\right)q_5 + \\ \left(6x_i^5 + x_i^6 + \frac{1}{8}x_i^8\right)q_6 + \left(7x_i^6 + x_i^7 + \frac{1}{9}x_i^9\right)q_7 + \left(8x_i^7 + x_i^8 + \frac{1}{10}x_i^{10}\right)q_8 + \\ \left(9x_i^8 + x_i^9 + \frac{1}{11}x_i^{11}\right)q_9 + \left(10x_i^9 + x_i^{10} + \frac{1}{12}x_i^{12}\right)q_{10} \\ -\left\{\begin{matrix} 52488x_i^{10} - 2621440x_i^9 + 5570560x_i^8 - 6553600x_i^7 + 4659200x_i^6 \\ -2050048x_i^5 + 549120x_i^4 - 84480x_i^3 + 6600x_i^2 - 200x_i + 1 \end{matrix}\right\}\tau_1 = \\ (x_i^3 + 2x_i + 1)e^{-x_i} + 5x_i^3 + 8 \end{cases} \qquad (20)$$

Collocate equation (20) as follows:

$$x_i = a + \frac{(b-a)i}{N+2}; \quad i = 1,2,3 \dots \dots .. N+1 \text{ Where } a = 0 \ , \ b = 1 \ , \ N = 10$$

$$x_1 = \frac{1}{12}, x_2 = \frac{2}{12}, x_3 = \frac{3}{12}, x_4 = \frac{4}{12}, x_5 = \frac{5}{12}, x_6 = \frac{6}{12},$$

$$x_7 = \frac{7}{12}, x_8 = \frac{8}{12}, x_9 = \frac{9}{12}, x_{10} = \frac{10}{12}, x_{11} = \frac{11}{12}$$

Consider initial condition (18) and using MAPLE 18 software to obtain twelve (12) unknown constants of equations (20), we obtain the following constants

$$\begin{cases} q_0 = 10.00000000000, q_1 = -0.99999569981 \\ q_2 = 0.999929888700, q_3 = -0.49945051670 \\ q_4 = 0.164233879700, q_5 = -0.03520284380 \\ q_6 = -0.00217697048, q_7 = 0.008870692442 \\ q_8 = -0.00536139901, q_9 = 0.001307283762 \\ q_{10} = -0.00003367108, \tau_1 = -0.00000007663 \end{cases}$$

Substitute the above values into approximation solution (5).The approximate solution of First-order Volterra integro-differential equation (17) can be written as

$$y(x) \approx \begin{cases} 10.000000000 - 0.99999569981x + 0.999929888700x^2 \\ -0.49945051670x^3 + 0.164233879700x^4 - 0.03520284380x^5 \\ -0.00217697048x^6 + 0.008870692442x^7 - 0.00536139901x^8 \\ 0.001307283762x^9 - 0.00003367108x^{10} - 0.00000007663e^x \end{cases} \tag{21}$$

Example 2. Consider the Second-order Volterra integro-differential equation [10]

$$\frac{d^2y}{dx^2} = 1 + xe^x - \int_0^x e^{x-t}\, y(t)dt \qquad x \in [0,1] \tag{22}$$

Subject to initial conditions

$$\begin{cases} y(0) = 0 \\ \frac{dy}{dx}(0) = 1 \end{cases} \tag{23}$$

Analytical solution $\mathrm{y}(x) = \mathrm{e}^x - 1$ (24)

EFCAT technique

Comparing equation (22) with equation (13) and choosing computational length N=8, we have

$$\begin{cases} (e^{x_i} - 1)q_0 + \\ (e^{x_i} - x_i - 1)q_1 + \\ (2e^{x_i} - x_i^2 - 2x_i)q_2 + \\ (6e^{x_i} - x_i^3 - 3x_i^2 - 6)q_3 + \\ (24e^{x_i} - x_i^4 - 4x_i^3 - 24x_i - 24)q_4 + \\ (120e^{x_i} - x_i^5 - 5x_i^4 - 60x_i^2 - 120x_i - 120)q_5 + \\ (720e^{x_i} - x_i^6 - 6x_i^5 - 120x_i^3 - 360x_i^2 - 720x_i - 720)q_6 + \\ (5040e^{x_i} - x_i^7 - 7x_i^6 - 210x_i^4 - 840x_i^3 - 2520x_i^2 - 5040x_i - 5040)q_7 + \\ (40320e^{x_i} - x_i^8 - 8x_i^7 - 336x_i^5 - 1680\mathrm{x_i}^4 - 6720x_i^3 - 20160x_i^2 - 40320x_i - 5040)q_8 \\ -\begin{Bmatrix} 32768x_i^8 - 131072x_i^7 + 212992x_i^6 - 180224x_i^5 - \\ 84480x_i^4 - 21504x_i^3 + 2688x_i^2 - 128x_i + 1 \end{Bmatrix}\tau_1 \\ -\begin{Bmatrix} 8172x_i^7 - 28672x_i^6 + 39424x_i^5 - 26880x_i^4 + \\ 9408x_i^3 - 1568x_i^2 + 98x_i - 1 \end{Bmatrix}\tau_2 = 1 + x_i e^{x_i} \end{cases} \tag{25}$$

Collocate equation (25) as follows:

$$x_i = a + \frac{(b-a)i}{N+2};\ \ i = 1,2,3 \ldots\ldots.N+1 \ \text{ where } \ a = 0\ ,\ b = 1\ ,\ N = 8$$

$$x_1 = \frac{1}{10}, x_2 = \frac{2}{10}, x_3 = \frac{3}{10}, x_4 = \frac{4}{10}, x_5 = \frac{5}{10}, x_6 = \frac{6}{10}, x_7 = \frac{7}{10}, x_8 = \frac{8}{10}, x_9 = \frac{9}{10}$$

Consider initial condition (23) and using MAPLE 18 software to obtain eleven (11) unknown constants in equations (25), thus we obtain

$$\begin{cases} q_0 = 0.000000043736, q_1 = 1.000000044 \\ q_2 = 0.4999999721, \quad q_3 = 0.1666663308 \\ q_4 = 0.04167069796, \quad q_5 = 0.008317273866 \\ q_6 = 0.001420233805, \; q_7 = \; 0.000166139345 \\ q_8 = 0.0000411775232, \tau_1 = -0.00000000163 \\ \tau_2 = -0.0000000373624797 \end{cases}$$

Substitute the above values into approximation solution (5). The approximate solution of Second-order Volterra integro-differential equation (22) can be written as

$$y(x) \approx \begin{cases} 0.000000043736 + 1.000000044x + 0.4999999721x^2 + \\ 0.1666663308x^3 + 0.04167069796x^4 + 0.008317273866x^5 + \\ 0.001420233805x^6 + 0.000166139345x^7 + 0.00004117752315x^8 \\ -0.0000000373624797e^x \end{cases} \quad (26)$$

Example 3. Consider the Third-order Volterra integro-differential equation [11]

$$\frac{d^3y}{dx^3} - x\frac{d^2y}{dx^2} = \frac{4}{7}x^9 - \frac{8}{5}x^7 - x^6 + 6x^2 - 6 + 4\int_0^x x^2t^3\, y(t)dt \qquad x \in [0,1] \qquad (27)$$

Subject to initial conditions

$$\begin{cases} y(0) = 1 \\ \frac{dy}{dx}(0) = 2 \\ \frac{d^2y}{dx^2}(0) = 0 \end{cases} \qquad (28)$$

Analytical solution $y(x) = -x^3 + 2x + 1$ (29)

**EFCAT technique**

Comparing equation (27) with equation (14) and $\psi(x) = 4$

Choosing computational length N=8, we have linear system of equation

$$\begin{cases} (-x_i{}^6)q_0 + \\ \left(-\frac{4}{5}x_i{}^7\right)q_1 + \left(-2x_i - \frac{2}{3}x_{i_i}{}^8\right)q_2 + \\ \left(-6x_i + 6 - \frac{4}{9}x_i{}^9\right)q_3 + \left(-12x_i{}^3 + 24x_i{}^3 - \frac{1}{2}x_i{}^{10}\right)q_4 + \\ \left(-20x_i{}^4 + 60x_i{}^2 - \frac{4}{9}x_i{}^{11}\right)q_5 + \left(-30x_i{}^4 + 120x_i{}^3 - \frac{2}{5}x_i{}^{12}\right)q_6 + \\ \left(-42x_i{}^6 + 210x_i{}^4 - \frac{4}{11}x_i{}^{13}\right)q_7 + \left(-56x_i{}^7 + 336x_i{}^5 - \frac{1}{3}x_i{}^{14}\right)q_8 + \\ -\begin{Bmatrix} 32768x_i{}^8 - 131072x_i{}^7 + 212992x_i{}^6 - 180224x_i{}^5 \\ -84480x_i{}^4 - 21504x_i{}^3 + 2688x_i{}^2 - 128x_i + 1 \end{Bmatrix}\tau_1 \\ -\begin{Bmatrix} 8172x_i{}^7 - 28672x_i{}^6 + 39424x_i{}^5 - 26880x_i{}^4 \\ +9408x_i{}^3 - 1568x_i{}^2 + 98x_i - 1 \end{Bmatrix}\tau_2 \\ -\begin{Bmatrix} 2048x_i{}^6 - 6144x_i{}^5 + 6912x_i{}^4 - 3584x_i{}^3 + \\ 640x_i{}^2 - 72x_i + 1 \end{Bmatrix}\tau_3 \\ = \left(\frac{4}{9}x_i{}^9 - \frac{8}{5}x_i{}^7 - x_i{}^6 + 6x_i{}^2 - 6\right) \end{cases} \qquad (30)$$

Collocate equation (30) as follows:

$x_i = a + \frac{(b-a)i}{N+2}$; $i = 1,2,3 \ldots\ldots..N+1$ Where $a = 0$ , $b = 1$ , $N = 8$

$x_1 = \frac{1}{10}, x_2 = \frac{2}{10}, x_3 = \frac{3}{10}, x_4 = \frac{4}{10}, x_5 = \frac{5}{10}, x_6 = \frac{6}{10}, x_7 = \frac{7}{10}, x_8 = \frac{8}{10}, x_9 = \frac{9}{10}$

Consider initial condition (28) and using MAPLE 18 software to obtain twelve (12) unknown constants of equations (30), thus we obtain the following constants

$$\begin{cases} q_0 = 1.0000000000000, \quad q_1 = 2.00000000 \\ q_2 = 0.000000000190070, q_3 = -1.0000000000 \\ q_4 = 0.000000000084219129, \quad q_5 = -0.00000000007143969 \\ q_6 = 0.00000000151048311, \quad q_7 = -0.00000000130707181 \\ q_8 = 0.000000000415825808, \tau_1 = -0.000000000030283257 \\ \tau_2 = -0.0000000000515148426, \tau_3 = -0.000000000019007089 \end{cases}$$

Substitute the above values into approximate solution (5).The approximate solution of Third-order Volterra integro-differential equation (27) can be written as

$$y(x) \approx \begin{cases} 1.00000000 + 2.00000000\,x + 0.000000000190070x^2 \\ -1.0000000000x^3 + 0.000000000084219129x^4 \\ -0.00000000007143969x^5 + 0.00000000151048311x^6 \\ +\,0.00000000151048311x^6 - 0.00000000130707181x^7 \\ +0.000000000415825808x^8 - 0.000000000019007089e^x \end{cases} \qquad (31)$$

Example 4. Consider the Fifth -order Fredholm integro-differential equation

$$\frac{d^5y}{dx^5} - x^2\frac{d^3y}{dx^3} - \frac{dy}{dx} - xy(x) = x^2\cos(x) - x\sin(x) + 4\int_{-1}^{1} y(t)dt \qquad x \in [-1,1] \quad (32)$$

subject to initial conditions

$$\begin{cases} y(0) = 0 \\ \frac{dy}{dx}(0) = 1 \\ \frac{d^2y}{dx^2}(0) = 0 \\ \frac{d^3y}{dx^3}(0) = -1 \\ \frac{d^4y}{dx^4}(0) = 0 \end{cases} \qquad (33)$$

Analytical solution $y(x) = \sin(x)$ . (34)

**EFCAT technique**

Comparing equation (32) with equation (16) and when $\psi(x) = 4$

Choosing computational length N=8, we obtain a linear system of equations

$$\begin{cases}(-x_i+8)q_0+(-x_i^2-1)q_1+\left(-x_i^3-2x_i-\frac{8}{3}\right)q_2+(-x_i^4-9x_i^2)q_3+\\ \left(-x_i^5-28x_i^2-\frac{8}{5}\right)q_4+(-x_i^6-65x_i^4)q_5+\left(-x_i^7-126x_i^5-\frac{5032}{7}\right)q_6\\ (-x_i^8-217x_i^6+5040x_i)q_7+\left(-x_i^9-344x_i^7+20160x_i^2-\frac{8}{7}\right)q_8\\ -\begin{Bmatrix}32768x_i^8-131072x_i^7+212992x_i^6-180224x_i^5-\\ 84480x_i^4-21504x_i^3+2688x_i^2-128x_i+1\end{Bmatrix}\tau_1\\ -\begin{Bmatrix}8172x_i^7-224x_i^5-26880x_i^4+\\ 9408x_i^3-1568x_i^2+98x_i-1\end{Bmatrix}\tau_2\\ -\begin{Bmatrix}2048x_i^6-6144x_i^5+6912x_i^4-3584x_i^3\\ +640x_i^2-72x_i+1\end{Bmatrix}\tau_3\\ -\begin{Bmatrix}512x_i^5-128x_i^4+1120x_i^3-400x_i^2\\ +50x_i-1\end{Bmatrix}\tau_4\\ -\begin{Bmatrix}128x_i^4-258x_i^3+160x_i^2\\ -32x_i+1\end{Bmatrix}\tau_5=x_i^2\cos(x_i)-x_i\sin(x_i)\end{cases} \quad (35)$$

Collocate equation (35) as follows:

$$x_i=a+\frac{(b-a)i}{N+2};\ \ i=1,2,3\ldots\ldots N+1 \ \text{ where } \ a=0\ ,\ b=1\ ,\ N=8$$

$$x_1=\frac{1}{10}, x_2=\frac{2}{10}, x_3=\frac{3}{10}, x_4=\frac{4}{10}, x_5=\frac{5}{10}, x_6=\frac{6}{10}, x_7=\frac{7}{10}, x_8=\frac{8}{10}, x_9=\frac{9}{10}$$

Consider initial condition (33) and using MAPLE 18 software to obtain fourteen (14) unknown constants of equations (35), thus we obtain the following constants

$$\begin{cases} q_0=0.001882931290,\quad q_1=1.001882931\\ q_2=0.0009414656452,\quad q_3=-0.1663528448\\ q_4=0.00007845547043,\quad q_5=0.003352490348\\ q_6=0.001374368978,\quad q_7=-0.0000000715278865\\ q_8=-0.0000310258822, \tau_1=0.0000000328467432\\ \tau_2=0.000001712327853, \tau_3=0.000002396932743\\ \tau_4=-0.0004291360005, \tau_5=-0.00188293129\end{cases}$$

Substitute the above values into approximation solution (5). The approximate solution of Fifth-order Fredholm integro-differential equation (31) can be written as

$$y(x)\approx\begin{cases}0.001882931290+1.001882931x+0.0009414656452x^2-\\ 0.1663528448x^3+0.00007845547043x^4+0.003352490348x^5+\\ 0.001374368978x^6-0.0000000715278865x^7-\\ 0.0000310258822x^8-0.00188293129\,e^x\end{cases} \quad (36)$$

## 3. Numerical results

**Table 2: Analytical and Numerical results Example 1**

| $x$ | Analytical solution | EFCAT solution | TimothyA. Anake [9] | Absolute Error $E_{EFCAT}$ | Absolute Error $E_{[9]}$ |
|---|---|---|---|---|---|
| 0.0 | 10.00000000 | 10.00000000 | 10.00000000 | 0.000000000 | 0.000000000 |
| 0.1 | 9.909516258 | 9.909516343 | 9.909518155 | 0.000000085 | 0.000000000 |

| | | | | | |
|---|---|---|---|---|---|
| 0.2 | 9.836253849 | 9.836253909 | 9.836255661 | 0.000000060 | 0.000001812 |
| 0.3 | 9.777754534 | 0.777754572 | 9.777756147 | 0.000000038 | 0.000001613 |
| 0.4 | 9.731871982 | 9.731871988 | 9.731873400 | 0.000000006 | 0.000001418 |
| 0.5 | 9.696734670 | 9.696734654 | 9.696735769 | 0.000000016 | 0.000001099 |
| 0.6 | 9.670713018 | 9.670712980 | 9.670713761 | 0.000000038 | 0.000000743 |
| 0.7 | 9.652390287 | 9.652390219 | 9.652390809 | 0.000000068 | 0.000000522 |
| 0.8 | 9.640536829 | 9.640536736 | 9.640537243 | 0.000000093 | 0.000000414 |
| 0.9 | 9.634087306 | 9.634087186 | 9.634087439 | 0.000000120 | 0.000000133 |
| 1.0 | 9.632120559 | 9.632120518 | 9.632120150 | 0.000000041 | 0.000000405 |

**Table 3: Analytical and Numerical results Example 2**

| $x$ | Analytical solution | EFCAT solution | Akinboro et al [10] | Absolute Error $E_{EFCAT}$ | Absolute Error $E_{[10]}$ |
|---|---|---|---|---|---|
| 0.0 | 0.000000000 | 0.0000000000 | 0.0000000000 | 0.0000000000 | 0.0000000000 |
| 0.1 | 0.105170918 | 0.1051709175 | 0.1051709278 | 0.0000000005 | 0.0000000098 |
| 0.2 | 0.221402758 | 0.2214027563 | 0.2214027799 | 0.0000000017 | 0.0000000236 |
| 0.3 | 0.349858808 | 0.3498588044 | 0.3498587427 | 0.0000000036 | 0.0000000617 |
| 0.4 | 0.491824698 | 0.4918246921 | 0.4918247080 | 0.0000000056 | 0.0000000100 |
| 0.5 | 0.648721271 | 0.6487212613 | 0.6487212399 | 0.0000000097 | 0.0000000311 |
| 0.6 | 0.822118800 | 0.8221187864 | 0.8221188288 | 0.0000000012 | 0.0000000288 |
| 0.7 | 1.013752707 | 1.013752688 | 1.0137527320 | 0.0000000019 | 0.0000000250 |
| 0.8 | 1.225540928 | 1.225540903 | 1.2255410370 | 0.0000000025 | 0.0000001090 |
| 0.9 | 1.459603111 | 1.459603080 | 1.4596032700 | 0.0000000031 | 0.0000001590 |
| 1.0 | 1.718281828 | 1.718281795 | 1.7182822950 | 0.0000000033 | 0.0000004670 |

**Table 4: Analytical and Numerical results Example 3**

| $x$ | Analytical solution | EFCAT solution | Jalil and Ali [11] | Absolute Error $E_{EFCAT}$ | Absolute Error $E_{[11]}$ |
|---|---|---|---|---|---|
| 0.0 | 1.000000000 | 1.000000000 | 1.000000000 | 0.0000000000 | 0.0000000000 |
| 0.1 | 1.199000000 | 1.199000000 | 1.199000000 | 0.0000000000 | 0.0000000000 |
| 0.2 | 1.392000000 | 1.392000000 | 1.392000000 | 0.0000000000 | 0.0000000000 |
| 0.3 | 1.573000000 | 1.573000000 | 1.573000000 | 0.0000000000 | 0.0000000000 |
| 0.4 | 1.736000000 | 1.736000000 | 1.736000000 | 0.0000000000 | 0.0000000000 |
| 0.5 | 1.875000000 | 1.875000000 | 1.875000000 | 0.0000000000 | 0.0000000000 |
| 0.6 | 1.984000000 | 1.984000000 | 1.984000000 | 0.0000000000 | 0.0000000000 |
| 0.7 | 2.057000000 | 2.057000000 | 2.057000000 | 0.0000000000 | 0.0000000000 |
| 0.8 | 2.088000000 | 2.088000000 | 2.088000000 | 0.0000000000 | 0.0000000000 |
| 0.9 | 2.071000000 | 2.071000000 | 2.071000000 | 0.0000000000 | 0.0000000000 |
| 1.0 | 2.000000000 | 2.000000000 | 2.000000000 | 0.0000000000 | 0.0000000000 |

**Table 5: Analytical and Numerical results Example 4**

| $x$ | Analytical solution | EFCAT solution | Absolute Error $E_{EFCAT}$ |
|---|---|---|---|
| -1.0 | -0.8414709840 | -0.8353290012 | 0.0061419836 |
| -0.8 | -0.7173560909 | -0.7154055909 | 0.0019505000 |
| -0.6 | -0.5646424734 | -0.5641959781 | 0.0004464953 |

| | | | |
|---|---|---|---|
| -0.4 | -0.3894183423 | -0.3893619029 | 0.0000564394 |
| -0.2 | -0.1986693308 | -0.1986676467 | 0.0000016841 |
| 0.0 | 0.0000000000 | 0.0000000000 | 0.0000000000 |
| 0.2 | 0.1986693308 | 0.1986678220 | 0.0000015088 |
| 0.4 | 0.3894183423 | 0.3893730997 | 0.0000452426 |
| 0.6 | 0.5646424734 | 0.5643229354 | 0.0003195380 |
| 0.8 | 0.7173560909 | 0.7161143586 | 0.0012417323 |
| 1.0 | 0.8414709848 | 0.8380103627 | 0.0034606221 |

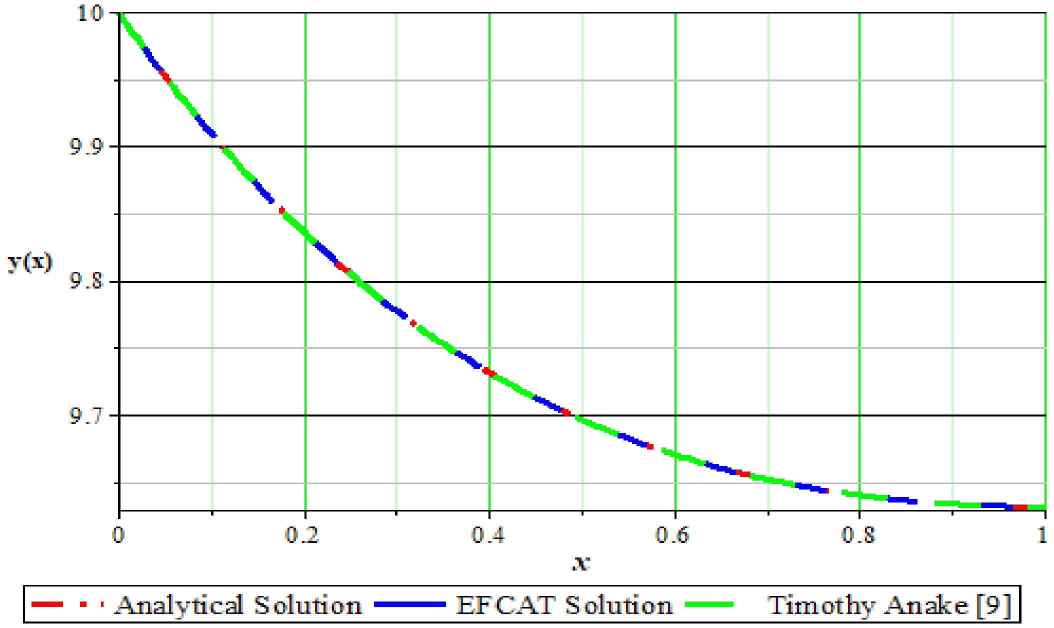


**Figure 1. First-order Volterra Integro-differential equation Example 1**

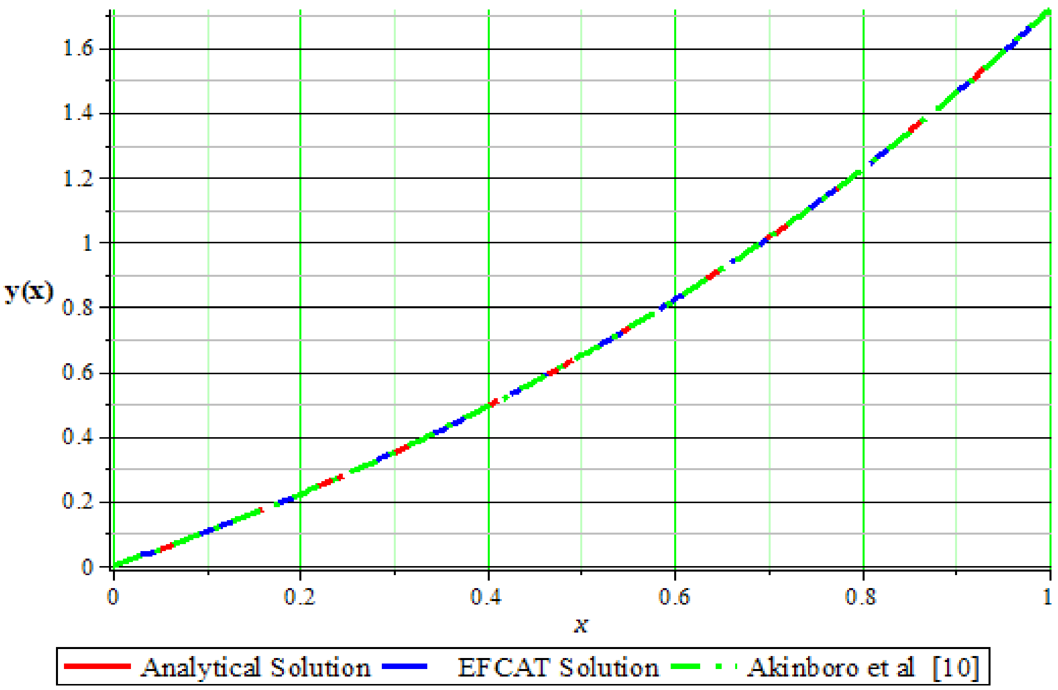


**Figure 2. Second-order Volterra Integro-differential equation Example 2**

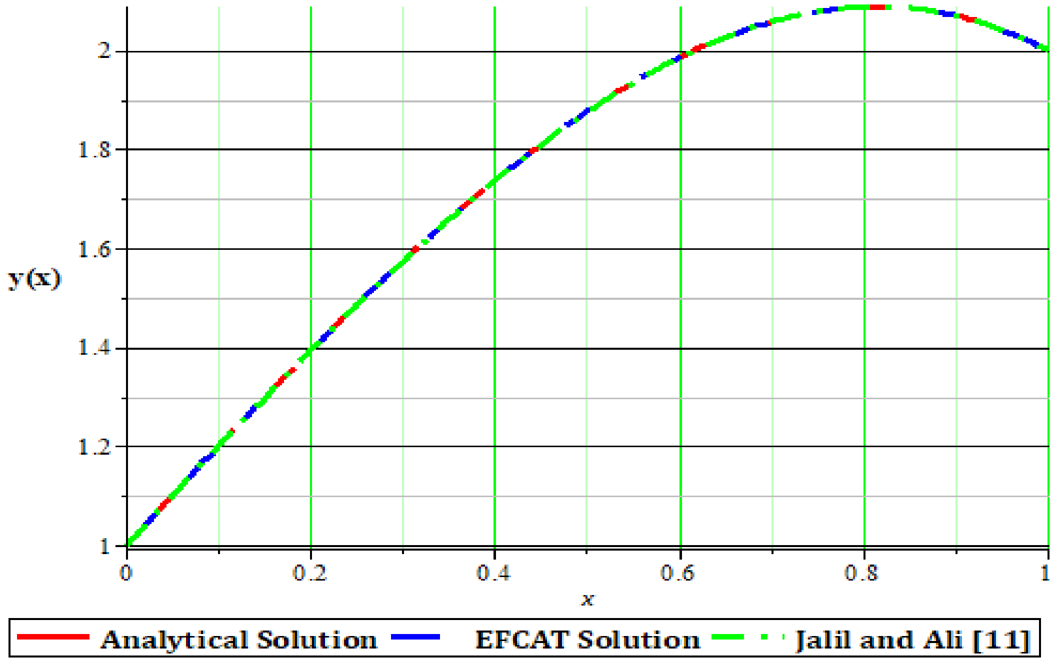


**Figure 3. Third-order Volterra Integro-differential equation Example 3**

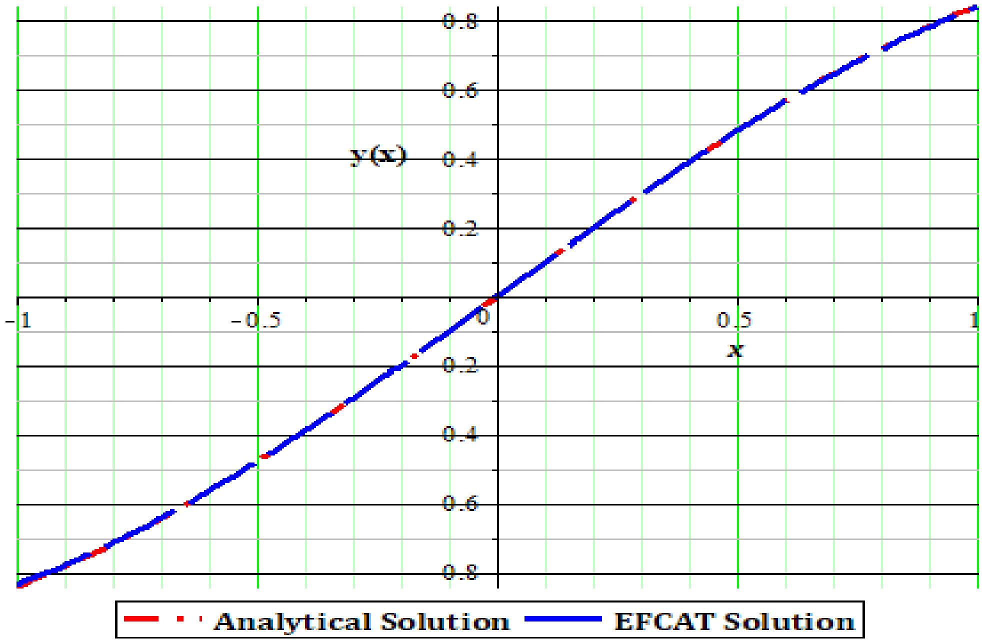


**Figure 4. Fifth-order Fredholm Integro-differential equation Example 4**

## 5. Conclusion

The proposed technique has been successfully applied to find numerical solution of the 1st,2nd, 3rd, and 5th–orders Volterra and Fredholm integro-differential equations. In order to investigate the accuracy of the method, four examples were considered and the numerical solutions were compared to the analytical solution and some existing methods available in the literature. The comparison of absolute errors (table 2, table 3, table 4 and table 5) of the results obtained by the present method with analytical solution and existing method give close or same results obtained by [9],[10],[11]. Moreover, the method is efficient and reliable for the numerical solutions of integro-differential equations of Volterra and Fredholm type as illustrated in all examples consider. We conclude that the proposed technique is a promising tool to solve higher order integro-differential equations and many applied problems in mathematical physics.

**Acknowledgment**

I wish to thank my supervisor Professor O.A Taiwo of the Department of Mathematics, University of Ilorin, Ilorin, Nigeria, for his supervision throughout my research work.

**References**

[1] P. Linz, *Analytic and Numerical Methods for Volterra Equations*, SIAM, Philadelphia, Pa, USA, Pp 132-139, 1985.

[2] G. Adomian, *Solving Frontier problems of physics: The decomposition method*, Kluwer Pp 123-134, 1994.

[3] J.H. He, “Homotopy perturbation technique”, *Computation Meth. Appl. Mech. Eng*., 178, 257-262 1999.

[4] J.H. He, “Approximate analytical solution for seepage flow with fractional derivatives in porous media”, *Computer. Meth. Appl. Mech. Engn*., 167, 57-68, 1998.

[5] O.A. Taiwo, K.I. Falade and A. K. Bello “Application of Collocation Methods for the numerical solution of Integro – Differential Equations by Chebyshev Polynomials”, *Intern. Journal of Biological and Physical Sciences*, 16, 12 – 20, 2011.

[6] R. Kanwall, K. Liu. “A Taylor expansion approach for solving integral equations”. *International Journal of Mathematical Education in Science and Technology*, 20, 411–414, 1989.

[7] S. Yuzbasi, N. Sahin, M. Sezer. “Bessel polynomial solutions of high-order linear Volterra integro-differential equations”. *Computers & Mathematics with Applications*, 62, 1940–1956, 2011.

[8] Falade K.I, “Numerical Solution of Higher Order Singular Initial Value Problems (SIVP) by Exponentially Fitted Collocation Approximate” *Method American International Journal of Research in Science, Technology, Engineering & Mathematics*, 21(1), 48-55, 2017.

[9] Olumuyiwa A. Agbolade and Timothy A. Anake “Solutions of First-Order Volterra Type Linear Integrodifferential Equations by Collocation Method” *Hindawi Journal of Applied Mathematics*, Article ID 1510267, page 3, 2017.

[10] S. Alao, F.S. Akinboro, F.O. Akinpelu, R.A. Oderinu Numerical “Solution of Integro-Differential Equation Using Adomian Decomposition and Variational Iteration Methods.” IOSR *Journal of Mathematics (IOSR)*, 10(4),20, 2014.

[11] Jalil Rashidinia and Ali Tahmasebi “Approximate solution of linear integro-differential equations by using modified Taylor expansion method” *World Journal of Modelling and Simulation*, 9(4), 300, 2013.

.
.