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The Graceful Coalescence of Alpha Cycles

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Abstract
The standard coalescence of two graphs is extended, allowing to identify two isomorphic subgraphs instead of a single vertex. It is proven here that any succesive coalescence of cycles of size n, where n is divisible by four, results in an α-graph, that is, the most restrictive kind of graceful graph, when the subgraphs identified are paths of sizes not exceeding $\frac{n}{2}$. Using the coalescence and another similar technique, it is proven that some subdivisions of the ladder $L_n = P_2 \times P_n$ also admit an α-labeling, extending and generalizing the existing results for this type of subdivided graphs.

Keywords: Coalescence, α-labeling, Graceful labeling, Ladder.
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1. Introduction

A *difference vertex labeling* of a graph G of size n is an injective mapping f from $V(G)$ into a set M of nonnegative integers, such that every edge uv of G has assigned a *weight* defined by $|f(u)-f(v)|$. All labelings considered in this work are difference vertex labelings. A labeling is called *graceful* when $M=\{0,1,\dots,n\}$ and the induced weights are $1,2,\dots,n$. If G admits such a labeling, it is called a *graceful graph*. Let G be a bipartite graph where $\{A,B\}$ is the natural bipartition of $V(G)$, we refer to A and B as *stable* sets of $V(G)$. A *bipartite labeling* of G is an injection $f: V(G) \to \{0,1,\dots,t\}$ for which there exists an integer λ, named the *boundary value* of f, such that $f(u) \leq \lambda < f(v)$ for every $(u,v) \in A \times B$, that induces n different weights. This is an extension of the definition given by Rosa and Širáň [1]. From the definition we conclude that the labels assigned by f on the vertices of A and B are in the interval $[0,\lambda]$ and $[\lambda+1,t]$, respectively. When $t=n$, the function f is an *α-labeling* and G is an *α-graph*.

Let $f: V(G) \to \{0,1,\dots,t\}$ be a labeling of a graph G of size $n \leq t$. The labeling $g: V(G) \to \{c,c+1,\dots,c+t\}$, defined for every $v \in V(G)$ and $c \in \mathbb{Z}$ as $g(v)=c+f(v)$, is the *shifting* of f in c units. Note that this labeling preserves the weights induced by f.

If f is bipartite with boundary value λ, the labeling $h: V(G) \to \{0,1,\dots,t+d-1\}$, defined for every $v \in V(G)$ and $d \in \mathbb{Z}^+$ as $h(v)=f(v)$ when $f(v) \leq \lambda$ and $h(v)=f(v)+d-1$ when $f(v) > \lambda$, is the *bipartite d-labeling* of G obtained from f. This labeling uses labels from $[1,\lambda] \cup [\lambda+d, t+d-1]$. In other terms, this labeling shifts the weights induced by f in $d-1$ units. Thus, if f is an α-labeling of G and d is a positive constant, then h is a d-graceful labeling. This concept was introduced, independently by Maheo and Thuillier [2] and Slater [3] in 1982.

In the following sections we study α-labelings of the coalescence of α-cycles. Suppose that G_1 and G_2 are two graphs such that H is an induced subgraph of both of them. The *H-coalescence*, or simply *coalescence*, of G_1 and G_2, denoted by $G_1 \cdot G_2$, is the graph obtained by identifying the copy of H in G_1 with the copy of H in G_2. Assuming that for $i=1,2$, the graph G_i has order n_i and size m_i, and H has order p and size q, then $G_1 \cdot G_2$ has order n_1+n_2-p and size m_1+m_2-q. In Figure 1.1 we show an example of this operation where G_1 and G_2 are isomorphic and $H \cong C_4$, which is represented in the picture with green edges.

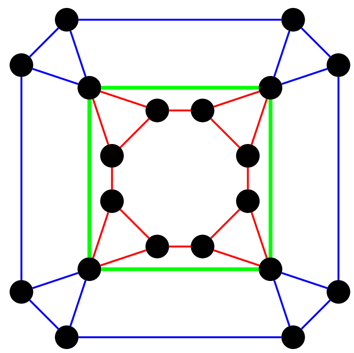

Figure 1.1. The C_4-coalescence of two isomorphic graphs of order 12 and size 20

In [4], Barrientos proved that if $H \cong P_1$ and G_1 and G_2 are α-graphs, there is a coalescence (also named one-point union or vertex amalgamation) of them is an α-graph. In [5], Barrientos and Minion showed that if $H \cong P_2$ and G_1 and G_2 are α-graphs, the coalescence (or edge amalgamation) of them is an α-graph if the edge of minimum weight in G_1 is identified with the edge of maximum weight of G_2. In this article, we extend the idea of the edge amalgamation, presented in [5], considering H to be a path. All graphs considered here are finite with no loops or multiple edges. We use the notation and terminology used in [6] and [7].

2. Preliminary results

In his seminal paper, Rosa [8] showed that when $n \equiv 0 (\text{mod } 4)$, there exists an α-labeling of the cycle C_n. We present here two labelings of C_n that are going to be used in the proof of the main result of the next section.

Suppose that $V(C_n) = \{v_1, v_2, \ldots, v_n\}$ is the vertex set of C_n and its edge set is $\{v_i v_{i+1} : 1 \leq i \leq n\}$ where the addition is taken modulo n. The labelings f and g given below are two well-known α-labelings of C_n. The interested reader can easily verify this statement.

$$f(v_i) = \begin{cases} \frac{i-1}{2} & \text{if } i \text{ is odd and } 1 \leq i \leq \frac{n}{2}-1, \\ \frac{i+1}{2} & \text{if } i \text{ is odd and } \frac{n}{2}+1 \leq i \leq n-1, \\ n+1-\frac{i}{2} & \text{if } i \text{ is even.} \end{cases}$$

$$g(v_i) = \begin{cases} \frac{i-1}{2} & \text{if } i \text{ is even and } 2 \leq i \leq \frac{n}{2}-1, \\ \frac{i}{2} & \text{if } i \text{ is even and } \frac{n}{2}+2 \leq i \leq n, \\ n-\frac{i-1}{2} & \text{if } i \text{ is odd.} \end{cases}$$

In Figure 2.1 we show two examples for each of these labelings. The graphs on the first row are labeled using the function f, while g is used to label the graphs on the second row. The arrow inside the cycle shows the orientation of the vertices within each graph.

3. Graceful coalescence

Let $C_{n_1}, C_{n_2}, \ldots, C_{n_k}$ be a collection of cycles, where the vertex set is $V(C_{n_j}) = \{v_{1,j}, v_{2,j}, \ldots, v_{n_j,j}\}$ and the edge set is $E(C_{n_j}) = \{v_{1,j}v_{2,j}, v_{2,j}v_{3,j}, \ldots, v_{n_k-1,j}v_{n_k,j}, v_{n_k,j}v_{1,j}\}$ for each $j \in \{1,2,\ldots,k\}$. For every $j \in \{1,2,\ldots,k-1\}$, select a positive integer t_j such that $2t_j \leq \min\{n_j, n_{j+1}\}$. A graph G is a *coalescence* of the cycles $C_{n_1}, C_{n_2}, \ldots, C_{n_k}$ when the vertices $v_{1,j+1}, v_{2,j+1}, \ldots, v_{t_j,j+1}$ of $C_{n_{j+1}}$ (together with the induced edges) are identified with the vertices $v_{n_j-t_j+1,j}, v_{n_j-t_j+2,j}, \ldots v_{n_j,j}$ of C_{n_j}, respectively. Note that G is a graph of order

$$\sum_{j=1}^{k} n_j - \sum_{j=1}^{k-1} t_j = n_k + \sum_{j=1}^{k-1} (n_j - t_j)$$

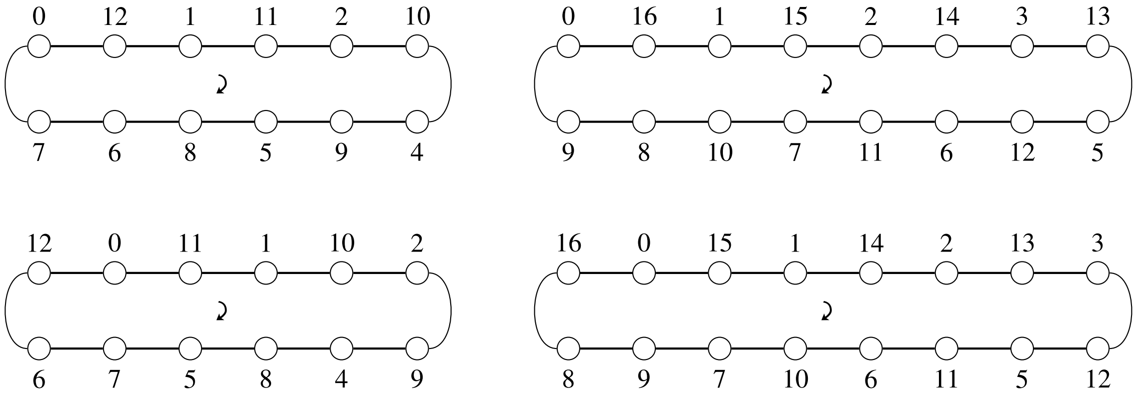

Figure 2.1. Two α-labelings of C_{12}, and C_{16}

and size

$$\sum_{j=1}^{k} n_j - \sum_{j=1}^{k-1} (t_j - 1) = n_k + \sum_{j=1}^{k-1} (n_j - t_j + 1)$$

In Figure 3.1 we show an example of this construction where $n_1 = 8$, $n_2 = 12$, $n_3 = 12$, $n_4 = 8$, and $n_5 = 4$, and $t_1 = 3$, $t_2 = 4$, $t_3 = 4$, and $t_4 = 2$. The numbers inside each cycle correspond to their respective vertices within that cycle.

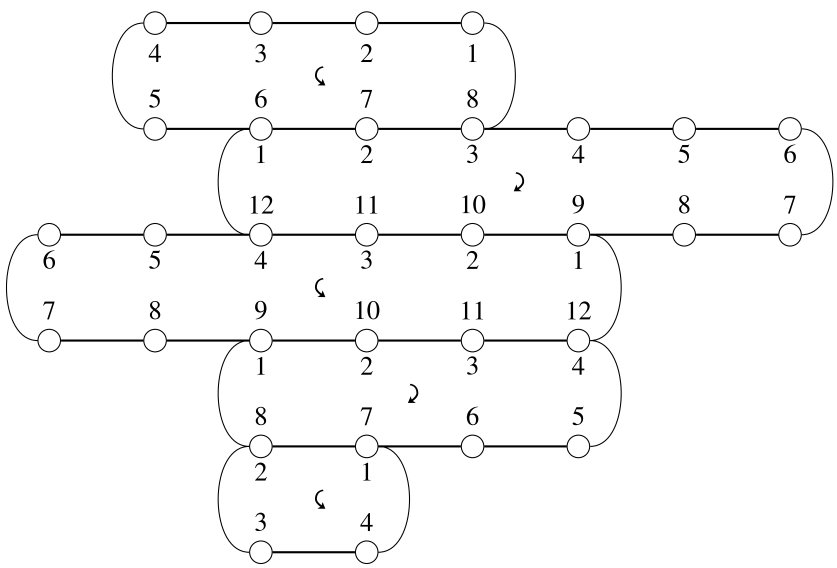

Figure 3.1. A coalescence of C_8, C_{12}, C_{12}, C_8 and C_4

We claim that when each $n_j \equiv 0(\text{mod } 4)$, the coalescence G of the cycles $C_{n_1}, C_{n_2}, \ldots, C_{n_k}$, determined by $t_1, t_2, \ldots, t_{k-1}$, is an α-graph. Within the proof of this theorem we use the labelings f and g of C_n given in Section 2.

Theorem 3.1. *Let G be the coalescence of the cycles $C_{n_1}, C_{n_2}, \ldots, C_{n_k}$ determined by the integers $t_1, t_2, \ldots, t_{k-1}$, where $2t_j \leq min\{n_j, n_{j+1}\}$. If for every $j \in \{1, 2, \ldots, k\}$, $n_j \equiv 0(mod\ 4)\}$, then G is an α-graph.*

Proof. Let G be the coalescence of the cycles $C_{n_1}, C_{n_2}, \ldots, C_{n_k}$ determined by the integers $t_1, t_2, \ldots, t_{k-1}$, where, for every $j \in \{1, 2, \ldots, k\}$, $2t_j \leq min\{n_j, n_{j+1}\}$ and $n_j \equiv 0(\text{mod } 4)\}$. Thus, every C_{n_j} admits an α-labeling.

To start, we label the vertices of C_{n_1} using the labeling f. For each $j \in \{2, 3, \ldots, k\}$, the selection of the initial labeling used on the vertices of C_{n_j} depends on the labeling used on $C_{n_{j-1}}$, according to the following criteria:

- If t_{j-1} is even, both $C_{n_{j-1}}$ and C_{n_j} have the same type of labeling.
- If t_{j-1} is odd, $C_{n_{j-1}}$ and C_{n_j} have different types of labelings.

Now that every cycle C_{n_j} has been α-labeled, we proceed to modify these initial labelings to obtain the desired α-labeling of G.

Recall that for every $j \in \{1,2,\ldots,k\}$, the size of the coalescence of the cycles $C_{n_1}, C_{n_2}, \ldots, C_{n_k}$ determined by the integers $t_1, t_2, \ldots, t_{k-1}$, is

$$\sum_{i=j}^{k} n_i - \sum_{i=j}^{k-1}(t_i - 1) = n_k + \sum_{i=j}^{k-1}(n_i - t_i + 1),$$

where the term $\sum_{i=j}^{k-1}(t_i - 1)$ is the number of edges shared by C_{n_j} and $C_{n_{j+1}}$.

The α-labeling of C_{n_j} is transformed into a d_j-graceful labeling (the intermediate labeling), where

$$d_j = \left(1 + n_k + \sum_{i=j}^{k-1}(n_i - t_i + 1)\right) - n_j.$$

In this way, the weights on the edges of C_{n_j} form the interval

$$I_j = \left[n_k + n_k + \sum_{i=j}^{k-1}(n_i - t_i + 1) - (n_j - 1), n_k + \sum_{i=j}^{k-1}(n_i - t_i + 1)\right].$$

Since $\min\{I_j : 1 \le j \le k\} = 1$ and $\max\{I_j : 1 \le j \le k\} = n_k + \sum_{i=1}^{k-1}(n_i - t_i + 1)$, that is, the size of G, we get

$$\bigcup_{j=1}^{k} I_j = [1, n_k + \sum_{i=1}^{k-1}(n_i - t_i + 1)].$$

Now, we need to shift these labelings to perform the coalescence of the labeled cycles. The labels assigned to the vertices of C_{n_1} constitute the final labeling of this cycle. For every $j \in \{2,3,\ldots,k\}$, the final labeling of C_{n_j} is obtained recursively in the following manner:

Assume that the labeling of $C_{n_{j-1}}$ is its final labeling. Let L_{j-1} be the set of the labels assigned to the vertices shared by $C_{n_{j-1}}$ and C_{n_j}. The final labeling of C_{n_j} is obtained by adding the constant $\min L_{j-1}$ to every label of C_{n_j}. Thus, the only overlapping of vertex labels between $C_{n_{j-1}}$ and C_{n_j} occurs on the vertices used to produce the coalescence.

Once this process has been completed, we have a bipartite labeling of G where the induced weights are $1, 2, \ldots, n_k + \sum_{j=1}^{k-1}(n_j - t_j + 1)$, with no label repeated.

Therefore, G is an α-graph. □

In Figure 3.2 we show the final α-labeling of the coalescence of the cycles C_{16}, C_{12}, C_8, C_{12}, C_8, determined by the integers $t_1 = 5$, $t_2 = 3$, $t_3 = 4$, $t_4 = 3$. The starting α-labelings of the cycles are: (0, 16, 1, 15, 2, 14, 3, 13, 5, 12, 6, 11, 7, 10, 8, 9), (12, 0, 11, 1, 10, 2, 9, 4, 8, 5, 7, 6), (0, 8, 1, 7, 3, 6, 4, 5), (0, 12, 1, 11, 2, 10, 4, 9, 5, 8, 6, 7), (8, 0, 7, 1, 6, 3, 5, 4). The intermediate d-graceful labelings are: (0, 45, 1, 44, 2, 43, 3, 42, 5, 41, 6, **40, 7, 39, 8, 38**), (**33, 0, 32, 1, 31**, 2, 30, 4, 29, **5, 28, 6**), (**0, 23, 1**, 22, **3, 21, 4, 20**), (**0, 18, 1, 17**, 2, 16, 4, 15, 5, **14, 6, 13**), (**8, 0, 7**, 1, 6, 3, 5, 4). The shifting constants are 7, 12, 15, and 21, respectively. The highlighted numbers correspond to the vertices that are going to be identified to produce the graph G.

Suppose that we want to form all nonisomorphic coalescence graphs constructed with k copies of C_n, where $n \equiv 0 (\mathrm{mod}\ 4)$. How many of these graphs exist? Since two consecutive copies of C_n shared at most $\frac{n-2}{2}$ edges, any graph obtained by the coalescence of these cycles can be described by a sequence (or string) of numbers from $\{1, 2, \ldots, \frac{n-2}{2}\}$. Thus, counting nonismorphic coalescence graphs is equivalent to count nonoriented strings with $k-1$ beads of $\frac{n-2}{2}$ or fewer colors. This number is known and can be found in OEIS A277504 [9]. In the following table we show the first values for $n \in \{4, 8, 12, 16, 20\}$ and $k \in \{1, 2, \ldots, 11\}$.

In Figure 3.3 we show the α-labelings of the six graphs obtained using three copies of C_8.

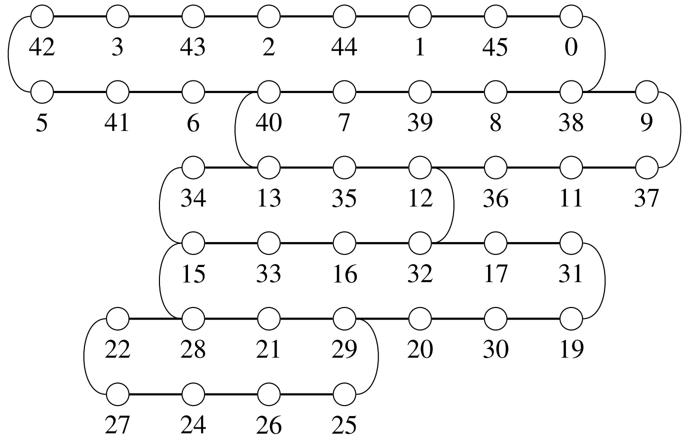

Figure 3.2. α-labeling of a coalescence of cycles

$k\backslash n$	4	8	12	16	20
1	1	1	1	1	1
2	1	3	5	7	9
3	1	6	15	28	45
4	1	18	75	196	405
5	1	45	325	1225	3321
6	1	135	1625	8575	29889
7	1	378	7875	58996	266085
8	1	1134	39375	412972	2394765
9	1	3321	195625	2883601	21526641
10	1	9963	978125	20185207	193739769
11	1	29643	4884375	141246028	1743421725

Table 1. Number of nonisomorphic coalescence graphs formed with k copies of C_8

4. Graceful subdivision of ladders

In this section we present two graceful labelings of subdivisions of ladders; the first result is a corollary of Theorem 3.1, the second one is a new construction. The *ladder* L_n is the result of the Cartesian product of the paths P_2 and P_n. The edges of P_2 within $L_n = P_2 \times P_n$ are called the *steps* of L_n. This type of graph can be seen as the coalescence of $n-1$ copies of C_4, therefore, L_n is an α-graph. The fact that L_n is graceful was proven first by Acharya and Gill [10].

In a graph G, a *subdivision* of an edge uv is the operation of replacing uv with a path u, w, v through a new vertex w. If the edhe uv is replaced with the path $u, w_1, w_2, \ldots, w_t, v$, we say that uv has been *subdivided an even* (resp. *odd*) *number of times* when t is even (resp. odd). Kathiresan [11] has shown that graphs obtained from ladders by subdividing each step exactly once are graceful.

If every step of L_n is subdivided an even number of times, then two consecutive subdivided steps, together with the two edges connecting them, form a cycle of size divisible by 4. Using Theorem 3.1, we can prove that this type of subdivided ladder is a graceful graph; in fact, it is an α-graph.

Corollary 4.1. *If every step of the ladder $L_n = P_2 \times P_n$ is subdivided an even number of times, the resulting graph is an α-graph.*

In Figure 4.1 we show, together with the original labeling of L_5, two examples of this subdivided ladder.

Unfortunately, the argument used in this corollary does not work when the edges are subdivided an odd number of times. So it is an open problem to find an α-labeling or a graceful labeling of these subdivided ladders.

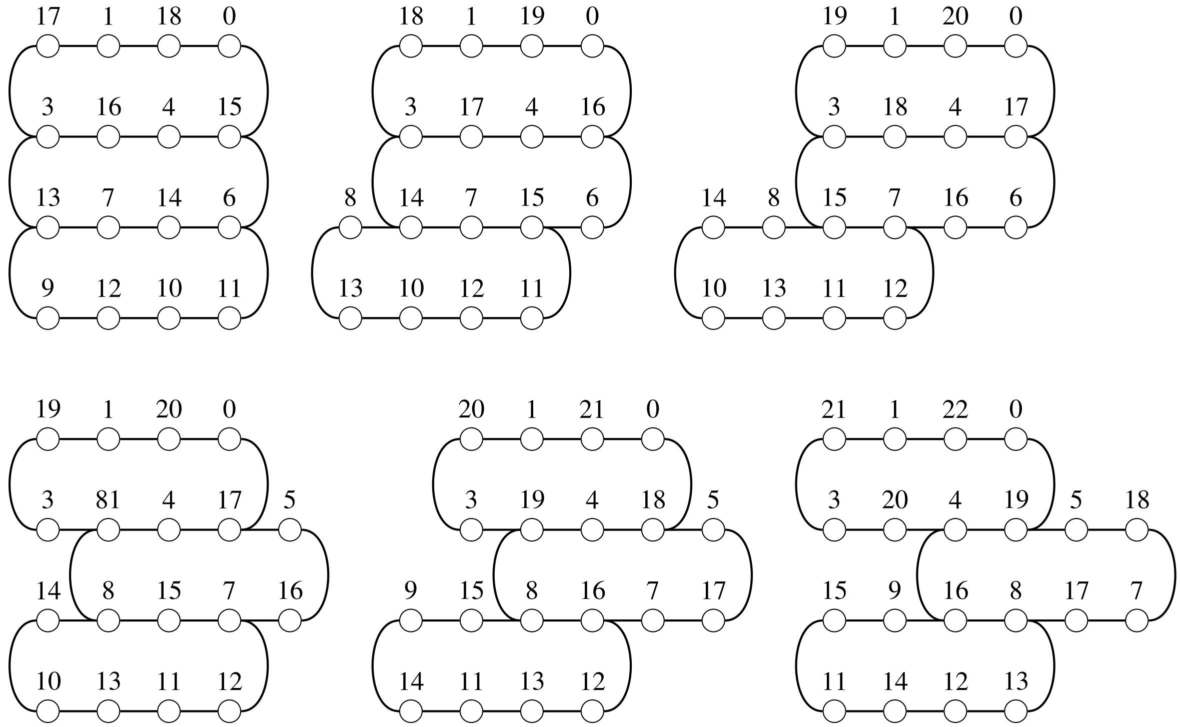

Figure 3.3. α-labelings of all the coalescences of three copies of C_8

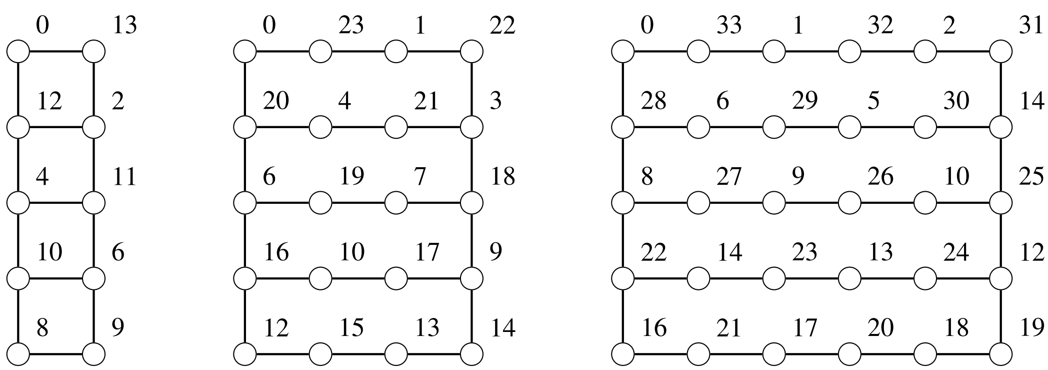

Figure 4.1. α-labeling of the ladder L_5 and some of its even subdivisions

Now we turn our attention to the graph obtained by subdividing every edge of L_n exactly once. We claim that this graph admits an α-labeling. Even when the resulting graph can be seen as the coalescence of $n-1$ copies of C_8, we use here a different construction based on the facts that C_8 and P_5 are α-graphs. The basic labelings of these graphs are given below:

- For C_8, the consecutive labels are: 5, 0, 8, 1, 7, 3, 6, 4.
- For P_5, the consecutive labels are: 2, 1, 3, 0, 4.

Suppose that G_n denotes the graph obtained by subdividing once all the edges of L_n. When $n=2$, $G_2 \cong C_8$; we use on G_2 the α-labeling given above. To obtain an α-labeling of G_3 we transform the α-labeling of G_2 by shifting its weights in such a way that the new largest label is 14 = 8 + 6, that is, the size of G_2 plus 6. The labeling of P_5 is shifted λ_2+2 units, where λ_2 is the boundary value of the α-labeling of G_2. The vertices λ_2-1 and λ_2 in G_2 are connected to the vertices to the vertices λ_2+4 and λ_2+6 in P_5; thus, the new edges have weights 5 and 6, respectively. The resulting graph is G_3 with an α-labeling. We continue this process until G_n has been labeled. In this way we have proved the following theorem.

Theorem 4.2. *The graph G, obtained by subdividing every edge of the ladder L_n exactly once, is an α-graph.*

In Figure 4.2 we show the first four cases of this construction.

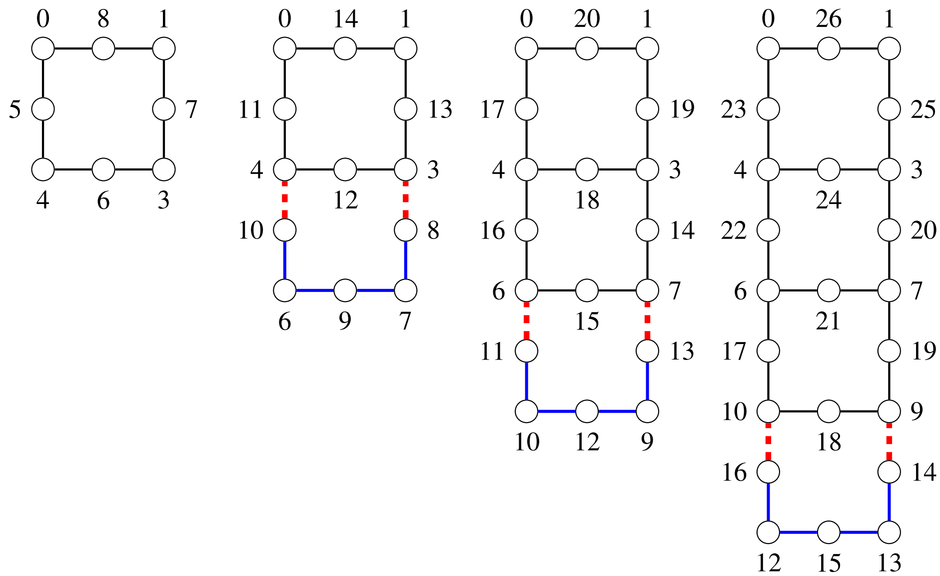

Figure 4.2. α-labelings of subdivided ladders

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