On $I_2$-Cauchy Double Sequences in Fuzzy Normed Spaces

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Abstract
In this paper, we investigate relationship between $I_2$-convergence and $I_2$-Cauchy double sequences in fuzzy normed spaces. After, we introduce the concepts of $I_2^*$-Cauchy double sequences and study relationships between $I_2$-Cauchy and $I_2^*$-Cauchy double sequences in fuzzy normed spaces.

Keywords: Double sequences, Fuzzy normed space, $I_2$-Cauchy, $I_2^*$-convergence.

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1. Introduction and background

Throughout the paper $\mathbb{N}$ and $\mathbb{R}$ denote the set of all positive integers and the set of all real numbers, respectively. The concept of convergence of a sequence of real numbers has been extended to statistical convergence independently by Fast [1] and Schoenfeld [2]. A lot of developments have been made in this area after the various studies of researchers [3, 4]. The idea of $I$-convergence was introduced by Kostyrko et al. [5] as a generalization of statistical convergence which is based on the structure of the ideal $I$ of subset of the set of natural numbers $\mathbb{N}$. Das et al. [6] introduced the concept of $I$-convergence of double sequences in a metric space and studied some properties of this convergence. A lot of developments have been made in this area after the works of [7, 8, 9, 10].

The concept of ordinary convergence of a sequence of fuzzy numbers was firstly introduced by Matloka [11] and proved some basic theorems for sequences of fuzzy numbers. Nanda [12] studied the sequences of fuzzy numbers and showed that the set of all convergent sequences of fuzzy numbers are a complete metric space. Şençimen and Pehlivan [13] introduced the notions of statistically convergent sequence and statistically Cauchy sequence in a fuzzy normed linear space. Hazarika [14] studied the concepts of $I$-convergence, $I^*$-convergence and $I$-Cauchy sequence in a fuzzy normed linear space. Dündar and Talo [15, 16] introduced the concepts of $I_2$-convergence and $I_2$-Cauchy sequence for double sequences of fuzzy numbers and studied some properties and relations of them. Hazarika and Kumar [17] introduced the notion of $I_2$-convergence and $I_2$-Cauchy double sequences in a fuzzy normed linear space. Dündar and Türkmen [18] studied some properties of $I_2$-convergence and $I_2^*$-convergence of double sequences in fuzzy normed spaces. A lot of developments have been made in this area after the various studies of researchers [19, 20, 21, 22].

Now, we recall the concept of ideal, convergence, statistical convergence, ideal convergence of sequence, double sequence and fuzzy normed and some basic definitions (see [1, 3, 4, 13, 15, 20, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34]).

Fuzzy sets are considered with respect to a nonempty base set $X$ of elements of interest. The essential idea is that each element $x \in X$ is assigned a membership grade $u(x)$ taking values in $[0, 1]$, with $u(x) = 0$ corresponding to nonmembership,
0 < u(x) < 1 to partial membership, and \( u(x) = 1 \) to full membership. According to Zadeh [35], a fuzzy subset of \( X \) is a nonempty subset \( \{(x, u(x)) : x \in X\} \) of \( X \times [0, 1] \) for some function \( u : X \to [0, 1] \). The function \( u \) itself is often used for the fuzzy set.

A fuzzy set \( u \) on \( \mathbb{R} \) is called a fuzzy number if it has the following properties:
1. \( u \) is normal, that is, there exists an \( x_0 \in \mathbb{R} \) such that \( u(x_0) = 1 \);
2. \( u \) is fuzzy convex, that is, for \( x, y \in \mathbb{R} \) and \( 0 \leq \lambda \leq 1 \), \( u(\lambda x + (1 - \lambda) y) \geq \min\{u(x), u(y)\} \);
3. \( u \) is upper semi-continuous;
4. \( \text{suppu} = \{x \in \mathbb{R} : u(x) > 0\} \), or denoted by \([u]_0\), is compact.

Let \( L(\mathbb{R}) \) be set of all fuzzy numbers. If \( u \in L(\mathbb{R}) \) and \( u(t) = 0 \) for \( t < 0 \), then \( u \) is called a non-negative fuzzy number. We write \( L^+(\mathbb{R}) \) by the set of all non-negative fuzzy numbers. We can say that \( u \in L^+(\mathbb{R}) \) iff \( u_\alpha \geq 0 \) for each \( \alpha \in [0, 1] \). Clearly we have \( 0 \in L(\mathbb{R}) \). For \( u \in L(\mathbb{R}) \), the \( \alpha \)-level set of \( u \) is defined by

\[
[u]_\alpha = \left\{ x \in \mathbb{R} : u(x) \geq \alpha \right\}, \text{ if } \alpha \in (0, 1]
\]

A partial order \( \leq \) on \( L(\mathbb{R}) \) is defined by \( u \leq v \) if \( u_\alpha \leq v_\alpha \) and \( u_\alpha \leq v_\alpha \) for all \( \alpha \in [0, 1] \).

Arithmetic operations for \( t \in \mathbb{R}, +, \otimes, \circ \) and \( \oplus \) on \( L(\mathbb{R}) \) are defined by

\[
\begin{align*}
(u \circ v)(t) &= \sup_{\epsilon \in \mathbb{R}} \left\{ u(s) \wedge v(s - t) \right\}, \\
(u \otimes v)(t) &= \sup_{\epsilon \in \mathbb{R}} \{ u(s) \vee v(t + \epsilon) \}, \\
(u + v)(t) &= \sup_{\epsilon \in \mathbb{R}} \{ u(s) \vee v(t - \epsilon) \} \\
\text{and } (u \ominus v)(t) &= \sup_{\epsilon \in \mathbb{R}} \{ u(s) \wedge v(t + \epsilon) \}.
\end{align*}
\]

For \( k \in \mathbb{R}^+ \), \( ku \) is defined as \( ku(t) = u(t/k) \) and \( 0u(t) = 0, t \in \mathbb{R} \).

Some arithmetic operations for \( \alpha \)-level sets are defined as follows:

\[
\begin{align*}
[u \circ v]_\alpha &= [u_\alpha, u_\alpha^+] + [v_\alpha, v_\alpha^+] = [u_\alpha + v_\alpha, u_\alpha^+ + v_\alpha^+], \\
[u \otimes v]_\alpha &= [u_\alpha - v_\alpha, u_\alpha^+ - v_\alpha^+], \\
[u \ominus v]_\alpha &= [u_\alpha - v_\alpha, u_\alpha^+ - v_\alpha^+] \\
\text{and } [u \oplus v]_\alpha &= \left[ \frac{1}{\alpha_0 + \alpha_0}, \frac{1}{\alpha_0} \right] \text{, } u_\alpha > 0.
\end{align*}
\]

For \( u, v \in L(\mathbb{R}) \), the supremum metric on \( L(\mathbb{R}) \) defined as

\[
D(u, v) = \sup_{0 \leq \alpha \leq 1} \max \{|u_\alpha - v_\alpha|, |u_\alpha^+ - v_\alpha^+|\}.
\]

It is known that \( D \) is a metric on \( L(\mathbb{R}) \) and \( (L(\mathbb{R}), D) \) is a complete metric space.

A sequence \( x = (x_k) \) of fuzzy numbers is said to be convergent to the fuzzy number \( x_0 \), if for every \( \varepsilon > 0 \) there exists a positive integer \( k_0 \) such that \( \{x_k \mid x_k \in X \} \) consistent for \( k > k_0 \) and a sequence \( a = (a_k) \) of fuzzy numbers convergent to \( x_0 \) if and only if \( |x_k - x_0|_\alpha \leq \varepsilon \) and \( |x_k|_\alpha \leq |x_0|_\alpha \) for every \( \alpha \in (0, 1) \).

Let \( X \) be a vector space over \( \mathbb{R} \), \( \| \| : X \to L^+(\mathbb{R}) \) and the mappings \( L : R \) (respectively, left norm and right norm) : \([0, 1] \times [0, 1] \to [0, 1] \) be symmetric, nondecreasing in both arguments and satisfy \( L(0, 0) = 0 \) and \( R(1, 1) = 1 \).

The quadruple \((X, \| \|, L, R)\) is called fuzzy normed linear space (briefly \( FNS \)) and \( \| \| \) a fuzzy norm if the following axioms are satisfied

\[
\begin{align*}
1. & \quad \|x\| = 0 \text{ iff } x = 0, \\
2. & \quad \|x\| = \|r \circ \|x\|| \text{ for } x \in X, r \in \mathbb{R}, \\
3. & \begin{align*}
(a) & \quad \|x + y\|_S (s + t) \geq L(\|x\|, \|y\|) \text{, whenever } s \leq \|x\|_S, t \leq \|x\|_S \text{ and } s + t \leq \|x + y\|_S, \\
(b) & \quad \|x + y\|_S (s + t) \leq R(\|x\|, \|y\|) \text{, whenever } s \geq \|x\|_S, t \geq \|y\|_S \text{ and } s + t \geq \|x + y\|_S.
\end{align*}
\]

Let \( \text{(X, \| \|, C)} \) be an ordinary normed linear space. Then, a fuzzy norm \( \| \| \) on \( X \) can be obtained by

\[
\|x\|_C = \begin{cases} 
0, & \text{if } 0 \leq t \leq a \|x\|_C \text{ or } t \geq b \|x\|_C, \\
\frac{t}{(t-a)\|x\|_C} - \frac{a}{t-a}, & \text{if } a \|x\|_C \leq t \leq \|x\|_C, \\
\frac{b}{(b-t)\|x\|_C} \frac{t}{b-t}, & \text{if } \|x\|_C \leq t \leq b \|x\|_C.
\end{cases}
\]

where \( \|x\|_C \) is the ordinary norm of \( x \neq 0 \), \( 0 < a < 1 \) and \( 1 < b < \infty \). For \( x = 0 \), define \( \|x\| = \tau \). Hence, \( \text{(X, \| \|)} \) is a fuzzy normed linear space.

Let us consider the topological structure of an \( FNS \) \((X, \| \|)\). For any \( \varepsilon > 0, \alpha \in [0, 1] \) and \( x \in X \), the \((\varepsilon, \alpha) \) - neighborhood of \( x \) is the set \( N(x, \varepsilon, \alpha) = \{y \in X : \|x - y\|_\alpha < \varepsilon \}. \)
Let $(X, \| \cdot \|)$ be an FNS. A sequence $(x_n)_{n=1}^\infty$ in $X$ is convergent to $x \in X$ with respect to the fuzzy norm on $X$ and we denote by $x_n \xrightarrow{FN} x$, provided that $(D) - \lim_{n \to \infty} \| x_n - x \| = \widetilde{0}$; i.e., for every $\varepsilon > 0$ there is an $N(\varepsilon) \in \mathbb{N}$ such that $D \left( \| x_n - x \|, 0 \right) < \varepsilon$ for all $n \geq N(\varepsilon)$. This means that for every $\varepsilon > 0$ there is an $N(\varepsilon) \in \mathbb{N}$ such that for all $n \geq N(\varepsilon)$, \[ \sup_{x \in \mathbb{[0,1[}} \| x_n - x \|_\alpha = \| x_n - x \|_0 < \varepsilon. \]

Let $(X, \| \cdot \|)$ be an FNS. Then a double sequence $(x_{jk})$ is said to be convergent to $x \in X$ with respect to the fuzzy norm on $X$ if for every $\varepsilon > 0$ there exist a number $N = N(\varepsilon)$ such that $D \left( \| x_{jk} - x \|, 0 \right) < \varepsilon$, for all $j, k \geq N$.

In this case, we write $x_{jk} \xrightarrow{FN} x$. This means that, for every $\varepsilon > 0$ there exist a number $N = N(\varepsilon)$ such that \[ \sup_{\alpha \in [0,1]} \| x_{jk} - x \|_\alpha = \| x_{jk} - x \|_0 < \varepsilon, \] for all $j, k \geq N$. In terms of neighborhoods, we have $x_{jk} \xrightarrow{FN} x$ provided that for any $\varepsilon > 0$, there exists a number $N = N(\varepsilon)$ such that $x_{jk} \in \mathcal{N}(\varepsilon, 0)$, whenever $j, k \geq N$.

Let $X \neq \emptyset$. A class $\mathcal{I}$ of subsets of $X$ is said to be an ideal in $X$ provided:

(i) $\emptyset \in \mathcal{I}$, (ii) $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$, (iii) $A \in \mathcal{I}, B \subset A$ implies $B \in \mathcal{I}$.

$\mathcal{I}$ is called a nontrivial ideal if $\mathcal{I} \neq \mathcal{I}$. A nontrivial ideal $\mathcal{I}$ in $X$ is called admissible if $(\{ x \} \in \mathcal{I}$ for each $x \in X$.

A nontrivial ideal $\mathcal{I}$ of $\mathbb{N} \times \mathbb{N}$ is called strongly admissible if (i) $\times \in \mathcal{I}$ and $\times \times \{ i \} \in \mathcal{I}$ for each $i \in \mathbb{N}$. It is evident that a strongly admissible ideal is also admissible.

Let $\mathcal{I} = \{ A \subset \mathbb{N} \times \mathbb{N} : (\exists m(\mathcal{A}), (i, j) \geq m(\mathcal{A}) \Rightarrow (i, j) \not\in \mathcal{A}) \}$. Then $\mathcal{I}$ is a nontrivial strongly admissible ideal and clearly an ideal $\mathcal{I}$ is strongly admissible if and only if $\mathcal{I} \setminus \mathcal{I} \subset \mathcal{I}$.

Let $X \neq \emptyset$. A non empty class $\mathcal{F}$ of subsets of $X$ is said to be a filter in $X$ provided:

(i) $\emptyset \notin \mathcal{F}$, (ii) $A, B \in \mathcal{F}$ implies $A \cap B \in \mathcal{F}$, (iii) $A \in \mathcal{F}, A \subset B$ implies $B \in \mathcal{F}$.

$\mathcal{F}$ is a nontrivial filter in $X, X \neq \emptyset$, then the class $\mathcal{F}(\mathcal{I}) = \{ M \subset X : (\exists A \in \mathcal{I})(M = X \setminus A) \}$ is a filter on $X$, called the filter associated with $\mathcal{F}$.

Let $(X, \rho)$ be a linear metric space and $\mathcal{F} \subset 2^{\mathbb{N} \times \mathbb{N}}$ be a strongly admissible ideal. A double sequence $(x_{mn})$ in $X$ is said to be $\mathcal{F}_2$-convergent to $L \in X$, if for any $\varepsilon > 0$ we have $A(\varepsilon) = \{ (m, n) \in \mathbb{N} \times \mathbb{N} : \rho(x_{mn}, L) \geq \varepsilon \} \in \mathcal{F}_2$ and we write $x_{mn} \xrightarrow{\mathcal{F}_2} L$.

The element $L$ is called the $\mathcal{F}$-limit of $(x_{mn})$ in $X$.

Let $(X, \| \cdot \|)$ be a fuzzy normed space. A double sequence $x = (x_{mn})_{(m,n)\in\mathbb{N} \times \mathbb{N}}$ in $X$ is said to be $\mathcal{F}_2$-convergent to $L_1 \in X$ with respect to fuzzy norm on $X$ if for each $\varepsilon > 0$, the set $A(\varepsilon) = \{ (m, n) \in \mathbb{N} \times \mathbb{N} : \| x_{mn} - L_1 \|_0 \geq \varepsilon \} \in \mathcal{F}_2$. In this case, we write $x_{mn} \xrightarrow{\mathcal{F}_2} L_1$. The element $L_1$ is called the $\mathcal{F}_2$-limit of $(x_{mn})$ in $X$. In terms of neighborhoods, we have $x_{mn} \xrightarrow{\mathcal{F}_2} L_1$ provided that for each $\varepsilon > 0$, $\{ (m, n) \in \mathbb{N} \times \mathbb{N} : x_{mn} \notin \mathcal{N}(\varepsilon, 0) \} \in \mathcal{F}_2$. A useful interpretation of the above definition is the following:

$$ x_{mn} \xrightarrow{\mathcal{F}_2} L_1 \Leftrightarrow \lim_{m,n \to \infty} \| x_{mn} - L_1 \|_0^\alpha = 0. $$

Note that $\mathcal{F}_2 - \lim_{m,n \to \infty} \| x_{mn} - L_1 \|_0^\alpha = 0$ implies that

$$ \mathcal{F}_2 - \lim_{m,n \to \infty} \| x_{mn} - L_1 \|_0^\alpha = \mathcal{F}_2 - \lim_{m \to \infty} \| x_{mn} - L_1 \|_0^\alpha = 0, $$

for each $\alpha \in [0,1]$, since $0 \leq \| x_{mn} - L_1 \|_0^\alpha \leq \| x_{mn} - L_1 \|_0 \leq \| x_{mn} - L_1 \|_0^\alpha$ holds for every $m, n \in \mathbb{N}$ and for each $\alpha \in [0,1]$.

Let $(X, \| \cdot \|)$ be a fuzzy normed space. A double sequence $x = (x_{mn})$ in $X$ is said to be $\mathcal{F}_2$-Cauchy (or $F, \mathcal{F}_2$-Cauchy) double sequence with respect to the fuzzy norm on $X$ if, for each $\varepsilon > 0$, there exists integers $p = p(\varepsilon)$ and $q = q(\varepsilon)$ such that the set $\{ (m, n) \in \mathbb{N} \times \mathbb{N} : \| x_{mn} - x_{pq} \|_0 \geq \varepsilon \}$ belongs to $\mathcal{F}_2$.

We say that an admissible ideal $\mathcal{I} \subset 2^{\mathbb{N} \times \mathbb{N}}$ satisfies the property (AP2), if for every countable family of mutually disjoint sets $\{ A_1, A_2, \ldots \}$ belonging to $\mathcal{I}$, there exists a countable family of sets $\{ B_1, B_2, \ldots \}$ such that $A_j \cap B_j \subset 2^{\mathbb{N} \times \mathbb{N}}$, i.e., $A_j \cap B_j$ is included in the finite union of rows and columns in $\mathbb{N} \times \mathbb{N}$ for each $j \in \mathbb{N}$ and $B = \bigcup_{j=1}^\infty B_j \in \mathcal{I}$ (hence $B_j \in \mathcal{I}$ for each $j \in \mathbb{N}$).

**Lemma 1.1.** ([27], Theorem 3.3) Let $\{ P_i \}_{i=1}^\infty$ be a countable collection of subsets of $\mathbb{N} \times \mathbb{N}$ such that $P_i \in \mathcal{F}(\mathcal{F}_2)$ for each $i$, where $\mathcal{F}(\mathcal{F}_2)$ is a filter associated with a strongly admissible ideal $\mathcal{I}$ with the property (AP2). Then there exists a set $P \subset \mathbb{N} \times \mathbb{N}$ such that $P \in \mathcal{F}(\mathcal{F}_2)$ and the set $P \setminus P_i$ is finite for all $i$. 
Lemma 1.2. ([17], Theorem 3.5) Let \((X, ||.||))\) be fuzzy normed space and \(I\) be a admissible ideal. Then, every \(I\)-convergent sequence is \(I\)-Cauchy sequence.

2. Main results

In this section, we investigate relationship between \(I\)-convergence and \(I\)-Cauchy double sequences in fuzzy normed spaces. After, we introduce the concepts of \(I\)-Cauchy double sequences and study relationships between \(I\)-Cauchy and \(I^*\)-Cauchy double sequences in fuzzy normed spaces.

Theorem 2.1. Let \((X, ||.||))\) be a fuzzy normed space. Then, a double sequence \((x_{mn})\) is \(I\)-convergent if and only if it is \(I\)-Cauchy double sequence.

Proof. Hazarika and Kumar proved that every \(I\)-convergent sequence is \(I\)-Cauchy sequence in Lemma 1.2.

Assume that \((x_{mn})\) is \(I\)-Cauchy double sequence. We prove that \((x_{mn})\) is \(I\)-convergent. To this effect, let \((\varepsilon_{pq})\) be a strictly decreasing sequence of numbers converging to zero. Since \((x_{mn})\) is \(I\)-Cauchy double sequence, there exist two strictly increasing sequences \((k_p)\) and \((l_q)\) of positive integers such that the set

\[
A(\varepsilon_{pq}) = \{ (m,n) \in \mathbb{N} \times \mathbb{N} : \|x_{mn} - x_{k_pl_q}\|_0^+ \geq \varepsilon_{pq} \}
\]

belongs to \(I\), \((p,q) \in \mathbb{N}\). This implies that

\[
\emptyset \neq \{ (m,n) \in \mathbb{N} \times \mathbb{N} : \|x_{mn} - x_{k_pl_q}\|_0^+ < \varepsilon_{pq} \}
\]

belongs to \(I\) \((\mathcal{F} \cup I)\), \((p,q) \in \mathbb{N}\). Let \(p,q,s,t\) be four positive integers such that \(p \neq q\) and \(s \neq t\). By (2.1), both the sets

\[
D(\varepsilon_{pq}) = \{ (m,n) \in \mathbb{N} \times \mathbb{N} : \|x_{mn} - x_{k_pl_q}\|_0^+ < \varepsilon_{pq} \}
\]

and

\[
C(\varepsilon_{st}) = \{ (m,n) \in \mathbb{N} \times \mathbb{N} : \|x_{mn} - x_{k_pl_q}\|_0^+ < \varepsilon_{st} \}
\]

are non empty sets in \(I\) \((\mathcal{F} \cup I)\). Since \(\mathcal{F} \cup I\) is a filter on \(\mathbb{N} \times \mathbb{N}\), therefore \(\emptyset \neq D(\varepsilon_{pq}) \cap C(\varepsilon_{st}) \in \mathcal{F} \cup I\). Thus, for each pair \((p,q)\) and \((s,t)\) of positive integers with \(p \neq q\) and \(s \neq t\), we can select a pair \((m_{p,q}(s,t)), n_{p,q}(s,t)) \in \mathbb{N} \times \mathbb{N}\) such that

\[
\|x_{m_{pq}n_{pq}} - x_{k_pl_q}\|_0^+ < \varepsilon_{pq} \text{ and } \|x_{m_{pq}n_{pq}} - x_{k_pl_q}\|_0^+ < \varepsilon_{st}.
\]

It follows that

\[
\|x_{k_pl_q} - x_{k_pl_q}\|_0^+ \leq \|x_{m_{pq}n_{pq}} - x_{k_pl_q}\|_0^+ + \|x_{m_{pq}n_{pq}} - x_{k_pl_q}\|_0^+ < \varepsilon_{pq} + \varepsilon_{st} \rightarrow 0, \text{ as } p,q,s,t \rightarrow \infty.
\]

This implies that \((x_{k_pl_q})\) \((p,q) \in \mathbb{N}\) is a Cauchy double sequence in fuzzy normed space, therefore it satisfies the Cauchy convergence criterion. Thus, the sequence \((x_{k_pl_q})\) converges to a finite limit \(L_1\) that is,

\[
\lim_{p,q \rightarrow \infty} x_{k_pl_q} = L_1.
\]

Also, we have \(\varepsilon_{pq} \rightarrow 0\) as \(p,q \rightarrow \infty\), so for each \(\varepsilon > 0\) we can choose the positive integers \(p_0,q_0\) such that \(p \geq p_0\) and \(q \geq q_0\).

\[
\varepsilon_{p_0q_0} < \frac{\varepsilon}{2} \text{ and } \|x_{k_pl_q} - L_1\|_0^+ < \frac{\varepsilon}{2}.
\]

(2.2)

Now, we define the set

\[
A(\varepsilon) = \{ (m,n) \in \mathbb{N} \times \mathbb{N} : \|x_{mn} - L_1\|_0^+ \geq \varepsilon \}.
\]

We prove that \(A(\varepsilon) \subset A(\varepsilon_{p_0q_0})\). Let \((m,n) \in A(\varepsilon)\), then by second half of (2.2) we have

\[
\varepsilon \leq \|x_{mn} - L_1\|_0^+ \leq \|x_{mn} - x_{k_pl_q}\|_0^+ + \|x_{k_pl_q} - L_1\|_0^+ \leq \|x_{mn} - x_{k_pl_q}\|_0^+ + \frac{\varepsilon}{2}.
\]
This implies that
\[ \frac{\varepsilon}{2} \leq \left\| x_{mn} - x_{kp} \right\|_0^+ \]
and therefore by first half of (2.2) we have
\[ \varepsilon_{kp} \leq \left\| x_{mn} - x_{kp} \right\|_0^+ . \]
This implies that \((m, n) \in A(\varepsilon_{kp})\) and therefore \(A(\varepsilon)\) is contained in \(A(\varepsilon_{kp})\). Since \(A(\varepsilon_{kp})\) belongs to \(\mathcal{F}_2\) therefore, \(A(\varepsilon)\) belongs to \(\mathcal{F}_2\). This proves that \((x_{mn})\) is \(F, \mathcal{F}_2\)-convergent to \(L_1\).

**Definition 2.2.** Let \((X, \|\|)\) be a fuzzy normed space. A double sequence \(x = \{x_{mn}\}\) in \(X\) is said to be \(\mathcal{F}_2\)-Cauchy (or \(F, \mathcal{F}_2\)-Cauchy) double sequence with respect to fuzzy norm on \(X\) if, there exists a set \(M \in \mathcal{F}(\mathcal{F}_2)\) (i.e., \(H = \mathbb{N} \times \mathbb{N} \setminus M \in \mathcal{F}_2\)) and \(k_0 = k_0(\varepsilon)\) such that for every \(\varepsilon > 0\) and for \((m, n), (s, t) \in M\), \(\|x_{mn} - x_{st}\|_0^+ < \varepsilon\), whenever \(m, n, s, t > k_0\). In this case, we write
\[ \lim_{m,n,s,t \to \infty} x_{mn} - x_{st} = 0. \]

**Theorem 2.3.** Let \(\mathcal{F}_2\) be an admissible ideal of \(\mathbb{N} \times \mathbb{N}\). If a double sequence \((x_{mn})\) in \(X\) is an \(F, \mathcal{F}_2\)-Cauchy sequence, then it is \(F, \mathcal{F}_2\)-Cauchy sequence.

**Proof.** Suppose that \((x_{mn})\) is an \(F, \mathcal{F}_2\)-Cauchy sequence. Then, there exists a set \(M \in \mathcal{F}(\mathcal{F}_2)\) (i.e., \(H = \mathbb{N} \times \mathbb{N} \setminus M \in \mathcal{F}_2\)) and \(k_0 = k_0(\varepsilon)\) such that for every \(\varepsilon > 0\) and for \((m, n), (s, t) \in M\), \(\|x_{mn} - x_{st}\|_0^+ < \varepsilon\), whenever \(m, n, s, t > k_0\). Then,
\[ A(\varepsilon) = \{ (m, n) \in \mathbb{N} \times \mathbb{N} : \|x_{mn} - x_{st}\|_0^+ \geq \varepsilon \} \subset H \cup [M \cap (\mathbb{N} \times \{1, \ldots, k_0\}) \times (\mathbb{N} \times \{1, \ldots, k_0\})]. \]
Since \(\mathcal{F}_2\) be an admissible ideal, then
\[ H \cup [M \cap (\mathbb{N} \times \{1, \ldots, k_0\}) \times (\mathbb{N} \times \{1, \ldots, k_0\})] \in \mathcal{F}_2. \]
Therefore, we have \(A(\varepsilon) \in \mathcal{F}_2\). This shows that \((x_{mn})\) is \(F, \mathcal{F}_2\)-Cauchy sequence in \(X\).

**Theorem 2.4.** Let \(\mathcal{F}_2\) be an admissible ideal of \(\mathbb{N} \times \mathbb{N}\) with the property (AP2) and \((x_{mn})\) be a double sequence in \(X\). Then, the concepts \(\mathcal{F}_2\)-Cauchy double sequence with respect to fuzzy norm on \(X\) and \(\mathcal{F}_2\)-Cauchy double sequence with respect to fuzzy norm on \(X\) coincide.

**Proof.** If a double sequence is \(F, \mathcal{F}_2\)-Cauchy, then it is \(F, \mathcal{F}_2\)-Cauchy by Theorem 2.3, where \(\mathcal{F}_2\) need not have the property (AP2). Now, it is sufficient to prove that a double sequence \((x_{mn})\) in \(X\) is a \(F, \mathcal{F}_2\)-Cauchy double sequence under assumption that it is an \(F, \mathcal{F}_2\)-Cauchy double sequence. Let \((x_{mn})\) be an \(F, \mathcal{F}_2\)-Cauchy double sequence in \(X\). Then, there exists \(s = s(\varepsilon), t = t(\varepsilon) \in \mathbb{N}\) such that for every \(\varepsilon > 0\),
\[ A(\varepsilon) = \{ (m, n) \in \mathbb{N} \times \mathbb{N} : \|x_{mn} - x_{st}\|_0^+ \geq \varepsilon \} \in \mathcal{F}_2. \]
Let
\[ P_i = \{ (m, n) \in \mathbb{N} \times \mathbb{N} : \|x_{mn} - x_{s_i t_i}\|_0^+ < \frac{1}{i} \}, \]
where \(s_i = s(1/i), (i \in \mathbb{N}), t_i = t(1/i)\). It is clear that \(P_i \in \mathcal{F}(\mathcal{F}_2)\) for all \(i \in \mathbb{N}\). Since \(\mathcal{F}_2\) has the property (AP2), then by Lemma 1.1 there exists a set \(P \subset \mathbb{N} \times \mathbb{N}\) such that \(P \in \mathcal{F}(\mathcal{F}_2)\) and \(P \setminus P_i\) is finite for all \(i \in \mathbb{N}\). Now we show that
\[ \lim_{m,n,s,t \to \infty} \|x_{mn} - x_{st}\|_0^+ = 0, \]
for \((m, n), (s, t) \in P\). To prove this, let \(\varepsilon > 0\) and \(j \in \mathbb{N}\) such that \(j > 2/\varepsilon\). If \((m, n), (s, t) \in P\) then \(P \setminus P_i\) is a finite set, so there exists \(N = N(j)\) such that \((m, n), (s, t) \in P_j\) for all \(m, n, s, t > N(j)\). Therefore,
\[ \|x_{mn} - x_{s_it_i}\|_0^+ < \frac{1}{j} \text{ and } \|x_{st} - x_{s_it_i}\|_0^+ < \frac{1}{j}, \]
for all $m, n, s, t > N(j)$. Hence it follows that

$$\|x_{mn} - x_{st}\|_0^+ \leq \|x_{mn} - x_{st}\|_0^+ + \|x_{st} - x_{st}\|_0^+$$

$$\leq \frac{1}{j} + \frac{1}{j} = \frac{2}{j} < \varepsilon,$$

for all $m, n, s, t > N(j)$. Thus, for any $\varepsilon > 0$ there exists $N = N(\varepsilon)$ such that for $m, n, s, t > N(j)$ and $(m, n), (s, t) \in P$ we have

$$\|x_{mn} - x_{st}\|_0^+ < \varepsilon.$$

This shows that the double sequence $(x_{mn})$ in $X$ is an $F, F^*_2$-Cauchy double sequence in fuzzy normed spaces.

References


