Complex Multivariate Montgomery Type Identity Leading to Complex Multivariate Ostrowski and Grüss Inequalities

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Abstract
We give a general complex multivariate Montgomery type identity which is a representation formula for a complex multivariate function. Using it we produce general tight complex multivariate high order Ostrowski and Grüss type inequalities. The estimates involve $L_p$ norms, any $1 \leq p \leq \infty$. We include also applications.

Keywords: Multivariate complex integral, Multivariate complex continuous functions, Multivariate complex analytic functions, Multivariate complex Montgomery identity, Multivariate complex Ostrowski and Grüss inequalities.


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1. Introduction

Our motivation comes from the following results:

Theorem 1.1. (A. Ostrowski, 1938 [1]). Let $f : [a, b] \rightarrow \mathbb{R}$ be continuous on $[a, b]$ and differentiable on $(a, b)$ such that $f' : (a, b) \rightarrow \mathbb{R}$ is bounded on $(a, b)$, i.e., $\|f'\|_\infty := \sup_{t \in (a, b)} |f'(t)| < \infty$. Then

$$\left| \frac{1}{b-a} \int_a^b f(t) \, dt - f(x) \right| \leq \frac{1}{4} + \left( \frac{x - \frac{a+b}{2}}{b-a} \right)^2 \|f'\|_\infty (b-a),$$

for all $x \in [a,b]$ and the constant $\frac{1}{4}$ is the best possible.

Theorem 1.2. (G. Grüss, 1934 [2]). Let $f, g : [a, b] \rightarrow \mathbb{R}$ be Lebesgue integrable functions, and $m, M, n, N \in \mathbb{R}$ such that: $-\infty < m \leq f \leq M < \infty$, $-\infty < n \leq g \leq N < \infty$, a.e. on $[a, b]$. Then

$$\left| \frac{1}{b-a} \int_a^b f(t) g(t) \, dt - \left( \frac{1}{b-a} \int_a^b f(t) \, dt \right) \left( \frac{1}{b-a} \int_a^b g(t) \, dt \right) \right| \leq \frac{1}{4} (M - m) (N - n),$$

with the constant $\frac{1}{4}$ being the best possible.
Let \( f \in C^1([a,b]) \) and the kernel \( p : [a, b]^2 \rightarrow \mathbb{R} \) be such that
\[
p(x, t) := \begin{cases} t - a, & \text{if } t \in [a, x], \\ t - b, & \text{if } t \in (x, b]. \end{cases}
\]

Then, we have the basic Montgomery integral identity [3, p. 565],
\[
f(x) = \frac{1}{b-a} \int_a^b f(t) \, dt + \frac{1}{b-a} \int_a^b p(x, t) f'(t) \, dt, \quad \forall x \in [a, b].
\]

In order to describe complex extensions of Ostrowski and Grüss inequalities using the complex integral we need the following preparation.

Suppose \( \gamma \) is a smooth path parametrized by \( z(t), t \in [a, b] \) and \( f \) is a complex function which is continuous on \( \gamma \). Put \( \gamma(a) = u \) and \( \gamma(b) = w \) with \( u, w \in \mathbb{C} \). We define the integral of \( f \) on \( \gamma_{a, w} = \gamma \) as
\[
\int_{\gamma} f(z) \, dz = \int_{\gamma_{a, w}} f(z) \, dz := \int_{a}^{b} f(z(t)) \gamma'(t) \, dt.
\]

We observe that the actual choice of parametrization of \( \gamma \) does not matter.

This definition immediately extends to paths that are piecewise smooth. Suppose \( \gamma \) is parametrized by \( z(t), t \in [a, b] \), which is differentiable on the intervals \([a, c]\) and \([c, b]\), then assuming that \( f \) is continuous on \( \gamma \) we define
\[
\int_{\gamma_{a, w}} f(z) \, dz := \int_{\gamma_{a, c}} f(z) \, dz + \int_{\gamma_{c, w}} f(z) \, dz,
\]
where \( v := z(c) \). This can be extended for a finite number of intervals.

We also define the integral with respect to arc-length
\[
\int_{\gamma_{a, w}} f(z) |dz| := \int_{a}^{b} f(z(t)) |\gamma'(t)| \, dt
\]
and the length of the curve \( \gamma \) is then
\[
\ell(\gamma) = \int_{\gamma_{a, w}} |dz| := \int_{a}^{b} |\gamma'(t)| \, dt.
\]

Let \( f \) and \( g \) be holomorphic in \( G \), an open domain and suppose \( \gamma \subset G \) is a piecewise smooth path from \( \gamma(a) = u \) to \( \gamma(b) = w \). Then we have the integration by parts formula
\[
\int_{\gamma_{a, w}} f(z) g'(z) \, dz = f(w) g(w) - f(u) g(u) - \int_{\gamma_{a, w}} f'(z) g(z) \, dz.
\]

We recall also the triangle inequality for the complex integral, namely
\[
\left| \int_{\gamma} f(z) \, dz \right| \leq \int_{\gamma} |f(z)| |dz| \leq \|f\|_{\gamma, \infty} \ell(\gamma),
\]
where \( \|f\|_{\gamma, \infty} := \sup_{z \in \gamma} |f(z)| \).

We also define the \( p \)-norm with \( p \geq 1 \) by
\[
\|f\|_{\gamma, p} := \left( \int_{\gamma} |f(z)|^p |dz| \right)^{\frac{1}{p}}.
\]

For \( p = 1 \) we have
\[
\|f\|_{\gamma, 1} := \int_{\gamma} |f(z)| |dz|.
\]

If \( p, q > 1 \) with \( \frac{1}{p} + \frac{1}{q} = 1 \), then by Hölder’s inequality we have
\[
\|f\|_{\gamma, 1} \leq [\ell(\gamma)]^{\frac{1}{q}} \|f\|_{\gamma, p}.
\]
First, we mention a Complex extension of Ostrowski inequality to the complex integral by providing upper bounds for the quantity

$$
\left| f(v)(w-u) - \int_{v}^{w} f(z) \, dz \right|
$$

under the assumption that $\gamma$ is a smooth path parametrized by $z(t), t \in [a,b], u = z(a), v = z(x)$ with $x \in (a,b)$ and $w = z(b)$ while $f$ is holomorphic in $G$, an open domain and $\gamma \subset G$.

Secondly, we mention a Complex extension of Grüss inequality:

Suppose $\gamma \subset \mathbb{C}$ is a piecewise smooth path parametrized by $z(t), t \in [a,b]$ from $z(a) = u$ to $z(b) = w$ with $w \neq u$. If $f$ and $g$ are continuous on $\gamma$, we consider the complex Čebyšev functional defined by

$$
\mathcal{D}_{\gamma}(f,g) := \frac{1}{w-u} \int_{v}^{w} f(z) g(z) \, dz - \frac{1}{w-u} \int_{v}^{w} f(z) \, dz \frac{1}{w-u} \int_{v}^{w} g(z) \, dz.
$$

We display upper bounds to $|\mathcal{D}_{\gamma}(f,g)|$.

We have the following results for functions of a complex variable:

\textbf{Theorem 1.3.} (S. Dragomir, 2019 [4]). Let $f$ be holomorphic in $G$, an open domain and suppose $\gamma \subset G$ is a smooth path from $z(a) = u$ to $z(b) = w$. If $v = z(x)$ with $x \in (a,b)$, then $\gamma_{u,v} = \gamma_{a,v} \cup \gamma_{v,w}$,

$$
\left| f(v)(w-u) - \int_{v}^{w} f(z) \, dz \right| \leq \|f'\|_{\gamma_{u,v};\infty} \int_{\gamma_{u,v}} |z-u| \, |dz| + \|f'\|_{\gamma_{v,w};\infty} \int_{\gamma_{v,w}} |z-w| \, |dz| \leq
$$

$$
\left\{ \int_{\gamma_{u,v}} |z-u| \, |dz| + \int_{\gamma_{v,w}} |z-w| \, |dz| \right\} \|f'\|_{\gamma_{u,v};\infty},
$$

and

$$
\left| f(v)(w-u) - \int_{v}^{w} f(z) \, dz \right| \leq \max_{z \in \gamma_{u,v}} |z-u| \|f'\|_{\gamma_{u,v};1} + \max_{z \in \gamma_{v,w}} |z-w| \|f'\|_{\gamma_{v,w};1} \leq
$$

$$
\max \left\{ \max_{z \in \gamma_{u,v}} |z-u|, \max_{z \in \gamma_{v,w}} |z-w| \right\} \|f'\|_{\gamma_{u,v};1}.
$$

If $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then

$$
\left| f(v)(w-u) - \int_{v}^{w} f(z) \, dz \right| \leq \left( \int_{\gamma_{u,v}} |z-u|^q \, |dz| \right)^{\frac{1}{q}} \|f'\|_{\gamma_{u,v};p} + \left( \int_{\gamma_{v,w}} |z-w|^q \, |dz| \right)^{\frac{1}{q}} \|f'\|_{\gamma_{v,w};p} \leq
$$

$$
\left( \int_{\gamma_{u,v}} |z-u|^q \, |dz| + \int_{\gamma_{v,w}} |z-w|^q \, |dz| \right)^{\frac{1}{q}} \|f'\|_{\gamma_{u,v};p}.
$$

Suppose $\gamma \subset \mathbb{C}$ is a piecewise smooth path parametrized by $z(t), t \in [a,b]$ from $z(a) = u$ to $z(b) = w$. Now, for $\phi, \Phi \in \mathbb{C}$ define the set of complex-valued functions

$$
\Lambda_{\gamma}(\phi, \Phi) := \left\{ f : \gamma \to \mathbb{C} \mid \left| f(z) - \frac{\phi + \Phi}{2} \right| \leq \frac{1}{2} |\Phi - \phi| \text{ for each } z \in \gamma \right\}.
$$

We have the following complex Grüss type inequalities:

\textbf{Theorem 1.4.} (S. Dragomir, 2018 [5]). Suppose $\gamma \subset \mathbb{C}$ is a piecewise smooth path parametrized by $z(t), t \in [a,b]$ from $z(a) = u$ to $z(b) = w$ with $w \neq u$. If $f$ and $g$ are continuous on $\gamma$ and there exist $\phi, \Phi, \psi, \Psi \in \mathbb{C}$, $\phi \neq \Phi, \psi \neq \Psi$ such that $f \in \Lambda_{\gamma}(\phi, \Phi)$ and $g \in \Lambda_{\gamma}(\psi, \Psi)$ then

$$
|\mathcal{D}_{\gamma}(f,g)| \leq \frac{1}{4} \|\Phi - \phi\| |\Psi - \psi| \frac{L^{2}(\gamma)}{|w-u|^{2}}.
$$
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If the path $\gamma$ is a segment $[u, w]$ connecting two distinct points $u$ and $w$ in $\mathbb{C}$ then we write $\int_{\gamma} f(z) \, dz$ as $\int_{u}^{w} f(z) \, dz$.

If $f, g$ are continuous on $[u, w]$ and there exists $\phi, \Phi, \psi, \Psi \in \mathbb{C}$, $\phi \neq \Phi$, $\psi \neq \Psi$ such that $f \in L_{[u, w]}(\phi, \Phi)$ and $g \in L_{[u, w]}(\psi, \Psi)$ then

$$\left| \frac{1}{w-u} \int_{u}^{w} f(z) g(z) \, dz - \frac{1}{w-u} \int_{u}^{w} f(z) \, dz \frac{1}{w-u} \int_{u}^{w} g(z) \, dz \right| \leq \frac{1}{4} |\Phi - \phi| |\Psi - \psi|.$$ 

We will use the complex Montgomery identity which follows:

**Theorem 1.5.** (S. Dragomir, 2018 [4]) Let $f$ be holomorphic in $G$, an open domain and suppose $\gamma \subset G$ is a smooth path from $z(a) = u$ to $z(b) = w$. If $v = z(t)$ with $t \in [a, b]$, then $\gamma_{uw} = \gamma_{uv} \cup \gamma_{vw}$, and

$$f(v) = \frac{1}{w-u} \int_{\gamma} f(z) \, dz + \frac{1}{w-u} \int_{\gamma_{uv}} (z-u) f'(z) \, dz + \frac{1}{w-u} \int_{\gamma_{vw}} (z-w) f'(z) \, dz.$$ 

Define

$$p(v, z) := \begin{cases} z-u, & \text{if } z \in \gamma_{uv} \\ z-w, & \text{if } z \in \gamma_{vw}. \end{cases}$$

Thus, it holds

$$f(v) = \frac{1}{w-u} \int_{\gamma} f(z) \, dz + \frac{1}{w-u} \int_{\gamma} p(v, z) f'(z) \, dz, \quad (1.1)$$

a form which we will use a lot in this article.

**Representation formula (1.1) is the main inspiration to write this article.**

We will use (1.1) to derive a multivariate Complex Montgomery type identity then based on it, we will produce Complex multivariate Ostrowski and Grüss type inequalities.

For the last we need:

**Definition 1.6.** Here we extend the notion of line (curve) integral into multivariate case. Let $\gamma_j, j = 1, \ldots, m$, be a smooth path parametrized by $z_j(t_j), t_j \in [a_j, b_j]$ and $f$ is a complex valued function which is continuous on $\prod_{j=1}^{m} \gamma_j \subseteq \mathbb{C}^m$. Put $z_j(a_j) = u_j$ and $z_j(b_j) = w_j$, with $u_j, w_j \in \mathbb{C}, j = 1, \ldots, m$.

We define the complex multivariate integral of $f$ on $\prod_{j=1}^{m} \gamma_j := \prod_{j=1}^{m} \gamma_{u_j, w_j}$ as

$$\int_{\gamma_1} \ldots \int_{\gamma_m} f(z_1, \ldots, z_m) \, dz_1 \ldots dz_m := \int_{\gamma_1} \int_{\gamma_2} \ldots \int_{\gamma_m} f(z_1, \ldots, z_m) \, dz_1 \ldots dz_m :=$$

$$\int_{\gamma_{u_1, w_1}} \cdots \int_{\gamma_{u_m, w_m}} f(z_1, \ldots, z_m) \, dz_1 \ldots dz_m := \int_{\gamma_{u_1, w_1}} \int_{\gamma_{u_2, w_2}} \ldots \int_{\gamma_{u_m, w_m}} f(z_1, \ldots, z_m) \, dz_1 \ldots dz_m :=$$

$$\int_{a_1}^{b_1} \int_{a_2}^{b_2} \ldots \int_{a_m}^{b_m} f(z_1(t_1), \ldots, z_m(t_m)) \prod_{j=1}^{m} z'_j(t_j) \, dt_1 \ldots dt_m. \quad (1.2)$$

We make

**Remark 1.7.** Clearly here $z_j \in C^1([a_j, b_j], \mathbb{C}), j = 1, \ldots, m$. The integrand in (1.2) is a continuous complex valued function over $\prod_{j=1}^{m} [a_j, b_j]$. Therefore $|f(z_1(t_1), \ldots, z_m(t_m))| \prod_{j=1}^{m} |z'_j(t_j)|$ is also continuous but from $\prod_{j=1}^{m} [a_j, b_j]$ into $\mathbb{R}$, hence it is bounded. Consequently it holds

$$\int_{\prod_{j=1}^{m} [a_j, b_j]} |f(z_1(t_1), \ldots, z_m(t_m))| \prod_{j=1}^{m} |z'_j(t_j)| \prod_{j=1}^{m} dt_j < +\infty.$$
Therefore, by Fubini’s theorem, the order integration in (1.2) is immaterial.
Clearly it holds

\[ \left| \int_{a_1}^{b_1} \cdots \int_{a_m}^{b_m} f(z_1(t_1), \ldots, z_m(t_m)) \prod_{j=1}^{m} \frac{dz_j}{|dz_j|} dt_1 \cdots dt_m \right| \leq \int_{a_1}^{b_1} \cdots \int_{a_m}^{b_m} |f(z_1(t_1), \ldots, z_m(t_m))| \prod_{j=1}^{m} |f_j'(t_j)| dt_1 \cdots dt_m. \tag{1.3} \]

We also define the integral with respect to arc-lengths

\[ \int_{\prod_{j=1}^{m} \gamma_j, w_j} f(z_1, \ldots, z_m) \, |dz_1| \, |dz_2| \cdots |dz_m| := \int_{\prod_{j=1}^{m} [a_j, b_j]} f(z_1(t_1), \ldots, z_m(t_m)) \prod_{j=1}^{m} |f_j'(t_j)| \, dt_1 \cdots dt_m. \tag{1.4} \]

It holds (by (1.3), (1.4))

\[ \left| \int_{\prod_{j=1}^{m} \gamma_j, w_j} f(z_1, \ldots, z_m) \, |dz_1| \, |dz_2| \cdots |dz_m| \right| \leq \int_{\prod_{j=1}^{m} \gamma_j, w_j} |f(z_1, \ldots, z_m)| \, |dz_1| \, |dz_2| \cdots |dz_m| \leq \|f\|_{\prod_{j=1}^{m} \gamma_j, w_j} \prod_{j=1}^{m} l(\gamma_j), \]

where

\[ \|f\|_{\prod_{j=1}^{m} \gamma_j, w_j} := \sup_{(z_1, \ldots, z_m) \in \prod_{j=1}^{m} \gamma_j} |f(z_1, \ldots, z_m)|, \]

and

\[ l(\gamma_j) = \int_{\gamma_j, w_j} |dz_j| = \int_{a_j}^{b_j} |f_j'(t_j)| \, dt_j, \quad j = 1, \ldots, m. \]

We also define the p-norm with \( p \geq 1 \) by

\[ \|f\|_{\prod_{j=1}^{m} \gamma_j, w_j} := \left( \int_{\prod_{j=1}^{m} \gamma_j} |f(z_1, \ldots, z_m)|^p \, |dz_1| \, |dz_2| \cdots |dz_m| \right)^{\frac{1}{p}}. \]

For \( p = 1 \) we have

\[ \|f\|_{\prod_{j=1}^{m} \gamma_j, w_j} := \int_{\prod_{j=1}^{m} \gamma_j} |f(z_1, \ldots, z_m)| \, |dz_1| \, |dz_2| \cdots |dz_m|. \]

If \( p, q > 1 \) with \( \frac{1}{p} + \frac{1}{q} = 1 \), then by Hölder’s inequality we have

\[ \|f\|^{\frac{1}{p}}_{\prod_{j=1}^{m} \gamma_j, w_j} \leq \left( \prod_{j=1}^{m} l(\gamma_j) \right)^{\frac{1}{q}} \|f\|_{\prod_{j=1}^{m} \gamma_j, w_j}. \]

\section{2. Main results}

We start by presenting a complex trivariate Montgomery type representation identity of complex functions:

\textbf{Theorem 2.1.} Let \( f : \prod_{j=1}^{3} D_j \subseteq \mathbb{C}^3 \rightarrow \mathbb{C} \) be a continuous function that is analytic per coordinate on the domain \( D_j, \ j = 1, 2, 3, \)
and \( x = (x_1, x_2, x_3) \in \prod_{j=1}^{3} D_j. \) For \( j = 1, 2, 3, \) suppose \( \gamma_j \subset D_j \) is a smooth path parametrized by \( z_j(t_j), \ t_j \in [a_j, b_j] \) with \( z_j(a_j) = u_j, z_j(t_j) = x_j \) and \( z_j(b_j) = w_j \), where \( u_j, w_j \in D_j, u_j \neq w_j. \) Assume also that all partial derivatives of \( f \) up to order three are continuous functions on \( \prod_{j=1}^{3} D_j. \)
Here we define the kernels for $i = 1, 2, 3$, $p_i : \mathbb{C}^2 \rightarrow \mathbb{C}$

\[ p_i(x_i, s_i) := \begin{cases} 
    s_i - u_i, & \text{if } s_i \in \gamma_{i,x_i}, \\
    s_i - w_i, & \text{if } s_i \in \gamma_{i,w_i}.
\end{cases} \]

Then

\[
f(x_1, x_2, x_3) = \frac{1}{\prod_{i=1}^{3} (w_i - u_i)} \left\{ \int_{\gamma_1} \int_{\gamma_2} \int_{\gamma_3} f(s_1, s_2, s_3) ds_3 ds_2 ds_1 + \sum_{j=1}^{3} \left( \int_{\gamma_1} \int_{\gamma_2} \int_{\gamma_3} p_j(x_j, s_j) \frac{\partial f(s_1, s_2, s_3)}{\partial s_j} ds_3 ds_2 ds_1 \right) \right. \\
\left. + \sum_{j=1}^{3} \left( \int_{\gamma_1} \int_{\gamma_2} \int_{\gamma_3} p_j(x_j, s_j) \frac{\partial^2 f(s_1, s_2, s_3)}{\partial s_k \partial s_j} ds_3 ds_2 ds_1 \right) \right\}.
\]

(2.1)

Above $l$ counts $(j, k) : j < k; j, k \in \{1, 2, 3\}$.

**Proof.** Here we apply (1.1) repeatedly.

First we see that

\[ f(x_1, x_2, x_3) = A_0 + B_0, \]

where

\[ A_0 := \frac{1}{w_1 - u_1} \int_{\gamma_1} f(s_1, x_2, x_3) ds_1, \]

and

\[ B_0 := \frac{1}{w_1 - u_1} \int_{\gamma_1} p_1(x_1, s_1) \frac{\partial f(s_1, x_2, x_3)}{\partial s_1} ds_1. \]

Furthermore we have

\[ f(s_1, x_2, x_3) = A_1 + B_1, \]

where

\[ A_1 := \frac{1}{w_2 - u_2} \int_{\gamma_2} f(s_1, s_2, x_3) ds_2, \]

and

\[ B_1 := \frac{1}{w_2 - u_2} \int_{\gamma_2} p_2(x_2, s_2) \frac{\partial f(s_1, s_2, x_3)}{\partial s_2} ds_2. \]

Also we find that

\[ f(s_1, s_2, x_3) = \frac{1}{w_3 - u_3} \int_{\gamma_3} f(s_1, s_2, s_3) ds_3 + \]

\[ \frac{1}{w_3 - u_3} \int_{\gamma_3} p_3(x_3, s_3) \frac{\partial f(s_1, s_2, s_3)}{\partial s_3} ds_3. \]

Next we put things together, and we derive

\[ A_1 = \frac{1}{(w_2 - u_2)(w_3 - u_3)} \int_{\gamma_2} f(s_1, s_2, s_3) ds_3 ds_2 + \frac{1}{(w_2 - u_2)(w_3 - u_3)} \int_{\gamma_3} p_3(x_3, s_3) \frac{\partial f(s_1, s_2, s_3)}{\partial s_3} ds_3 ds_2. \]
And we get
\[ A_0 = \frac{1}{\prod_{i=1}^{3} (w_i - u_i)} \int_{\gamma_1} \int_{\gamma_2} \int_{\gamma_3} f(s_1, s_2, s_3) ds_3 ds_2 ds_1 + \frac{1}{\prod_{i=1}^{3} (w_i - u_i)} \int_{\gamma_1} \int_{\gamma_2} \int_{\gamma_3} p_3(x_3, s_3) \frac{\partial f(s_1, s_2, s_3)}{\partial s_3} ds_3 ds_2 ds_1 \]
\[ + \frac{1}{(w_1 - u_1) (w_2 - u_2)} \int_{\gamma_1} \int_{\gamma_2} p_2(x_2, s_2) \frac{\partial f(s_1, s_2, s_3)}{\partial s_2} ds_2 ds_1. \]

Also we obtain
\[ \frac{\partial f(s_1, s_2, x_3)}{\partial s_2} = \frac{1}{w_3 - u_3} \int_{\gamma_3} \frac{\partial f(s_1, s_2, s_3)}{\partial s_3} ds_3 + \frac{1}{w_3 - u_3} \int_{\gamma_3} p_3(x_3, s_3) \frac{\partial^2 f(s_1, s_2, s_3)}{\partial s_3 \partial s_2} ds_3. \]

Therefore we get
\[ A_0 = \frac{1}{\prod_{i=1}^{3} (w_i - u_i)} \int_{\gamma_1} \int_{\gamma_2} \int_{\gamma_3} f(s_1, s_2, s_3) ds_3 ds_2 ds_1 + \frac{1}{\prod_{i=1}^{3} (w_i - u_i)} \int_{\gamma_1} \int_{\gamma_2} \int_{\gamma_3} p_3(x_3, s_3) \frac{\partial f(s_1, s_2, s_3)}{\partial s_3} ds_3 ds_2 ds_1 \]
\[ + \frac{1}{\prod_{i=1}^{3} (w_i - u_i)} \int_{\gamma_1} \int_{\gamma_2} \int_{\gamma_3} p_2(x_2, s_2) \frac{\partial f(s_1, s_2, s_3)}{\partial s_2} ds_2 ds_1 + \frac{1}{\prod_{i=1}^{3} (w_i - u_i)} \int_{\gamma_1} \int_{\gamma_2} \int_{\gamma_3} p_2(x_2, s_2) p_3(x_3, s_3) \frac{\partial^2 f(s_1, s_2, s_3)}{\partial s_3 \partial s_2} ds_3 ds_2 ds_1. \]

Similarly we obtain that
\[ B_0 = \frac{1}{\prod_{i=1}^{3} (w_i - u_i)} \int_{\gamma_1} \int_{\gamma_2} \int_{\gamma_3} p_1(x_1, s_1) \frac{\partial f(s_1, s_2, s_3)}{\partial s_1} ds_3 ds_2 ds_1 + \]
\[ \frac{1}{\prod_{i=1}^{3} (w_i - u_i)} \int_{\gamma_1} \int_{\gamma_2} \int_{\gamma_3} p_1(x_1, s_1) p_3(x_3, s_3) \frac{\partial^2 f(s_1, s_2, s_3)}{\partial s_3 \partial s_1} ds_3 ds_2 ds_1 + \]
\[ \frac{1}{\prod_{i=1}^{3} (w_i - u_i)} \int_{\gamma_1} \int_{\gamma_2} \int_{\gamma_3} p_1(x_1, s_1) p_2(x_2, s_2) \frac{\partial^2 f(s_1, s_2, s_3)}{\partial s_2 \partial s_1} ds_3 ds_2 ds_1 + \]
\[ \frac{1}{\prod_{i=1}^{3} (w_i - u_i)} \int_{\gamma_1} \int_{\gamma_2} \int_{\gamma_3} p_1(x_1, s_1) p_2(x_2, s_2) p_3(x_3, s_3) \frac{\partial^3 f(s_1, s_2, s_3)}{\partial s_3 \partial s_2 \partial s_1} ds_3 ds_2 ds_1. \]

We have proved (2.1).

Next comes the general complex multivariate Montgomery type representation identity of complex functions:

**Theorem 2.2.** Let \( f : \prod_{j=1}^{m} D_j \subseteq \mathbb{C}^m \rightarrow \mathbb{C} \) be a continuous function that is analytic per coordinate on the domain \( D_j, j = 1, \ldots, m, \)

and \( x = (x_1, \ldots, x_m) \in \prod_{j=1}^{m} D_j. \) For \( j = 1, \ldots, m, \) suppose \( \gamma_j \subset D_j \) is a smooth path parametrized by \( z_j(t_j), t_j \in [a_j, b_j] \) with

\( z_j(a_j) = u_j, z_j(b_j) = w_j, \)

where \( u_j, w_j \in D_j, u_j \neq w_j. \) Assume also that all partial derivatives of \( f \) up to order

\( m \in \mathbb{N} \) are continuous functions on \( \prod_{j=1}^{m} D_j. \)

We define the kernels \( p_j : \gamma_j^2 \rightarrow \mathbb{C} \)

\[ p_j(x_i, s_i) := \begin{cases} 
    s_i - u_i, & \text{if } s_i \in \gamma_{i_{u_i}}, \\
    s_i - w_i, & \text{if } s_i \in \gamma_{i_{w_i}},
\end{cases} \]

for \( i = 1, 2, \ldots, m. \)
Then

\[
f(x_1, x_2, \ldots, x_m) = \frac{1}{\prod_{i=1}^{m} (w_i - u_i)} \left\{ \int_{\prod_{i=1}^{m} \mathbb{P}} f(s_1, s_2, \ldots, s_m) \, ds_1 \ldots ds_m + \sum_{j=1}^{m} \left( \int_{\prod_{i=1}^{m} \mathbb{P}} p_j(x_j, s_j) \frac{\partial f(s_1, s_2, \ldots, s_m)}{\partial s_j} \, ds_1 \ldots ds_m \right) \right\} +
\]

\[
\left( \sum_{l=1}^{\frac{m}{2}} \left( \sum_{j<k} \left( \int_{\prod_{i=1}^{m} \mathbb{P}} p_j(x_j, s_j) p_k(x_k, s_k) \frac{\partial^2 f(s_1, s_2, \ldots, s_m)}{\partial s_j \partial s_k} \, ds_1 \ldots ds_m \right) \right) \right)_{(l)} +
\]

\[
\left( \sum_{l=1}^{\frac{m}{3}} \left( \sum_{j<k<r} \left( \int_{\prod_{i=1}^{m} \mathbb{P}} p_1(x_1, s_1) \ldots p_l(x_l, s_l) \ldots p_m(x_m, s_m) \frac{\partial^{m-1} f(s_1, \ldots, s_m)}{\partial s_1 \partial s_2 \ldots \partial s_l} \, ds_1 \ldots ds_l \right) \right) \right)_{(l)} + \ldots +
\]

\[
\left( \sum_{l=1}^{m} \left( \int_{\prod_{i=1}^{m} \mathbb{P}} p_l(x_l, s_l) \frac{\partial^m f(s_1, \ldots, s_m)}{\partial s_1 \partial s_2 \ldots \partial s_l} \, ds_1 \ldots ds_l \right) \right) \right) + \int_{\prod_{i=1}^{m} \mathbb{P}} \left( \int_{\prod_{i=1}^{m} \mathbb{P}} p_l(x_l, s_l) \frac{\partial^m f(s_1, \ldots, s_m)}{\partial s_1 \partial s_2 \ldots \partial s_l} \, ds_1 \ldots ds_l \right) \right) \right\}.
\]

(2.2)

Above \(l_1\) counts \((j, k) : j < k; j, k \in \{1, 2, \ldots, m\}\), also \(l_2\) counts \((j, k, r) : j < k < r; j, k, r \in \{1, 2, \ldots, m\}\), etc. Also \(p_l(x_l, s_l)\) and \(\hat{s}_l\) means that \(p_l(x_l, s_l)\) and \(\hat{s}_l\) are missing, respectively.

Proof. Similar to Theorem 2.1.

\(\square\)

We make

Remark 2.3. (on Theorems 2.1, 2.2)

By (2.1) we get

\[
E_f(x_1, x_2, x_3) := f(x_1, x_2, x_3) - \frac{1}{\prod_{i=1}^{3} (w_i - u_i)} \left\{ \int_{\prod_{i=1}^{3} \mathbb{P}} f(s_1, s_2, s_3) \, ds_1 ds_2 ds_3 \right\}
\]

\[
= \frac{1}{\prod_{i=1}^{3} (w_i - u_i)} \left( \int_{\prod_{i=1}^{3} \mathbb{P}} p_1(x_1, s_1) \frac{\partial f(s_1, s_2, s_3)}{\partial s_1} \, ds_1 ds_2 ds_3 \right) - \sum_{j=1}^{3} \left( \int_{\prod_{i=1}^{3} \mathbb{P}} p_j(x_j, s_j) \frac{\partial f(s_1, s_2, s_3)}{\partial s_j} \, ds_1 ds_2 ds_3 \right)_{(l)}
\]

Above \(l\) counts \((j, k) : j < k; j, k \in \{1, 2, 3\}\).

Similarly, by (2.2) we find

\[
E_f(x_1, x_2, \ldots, x_m) = f(x_1, x_2, \ldots, x_m) -
\]

\[
= \frac{1}{\prod_{i=1}^{m} (w_i - u_i)} \left\{ \int_{\prod_{i=1}^{m} \mathbb{P}} f(s_1, \ldots, s_m) \, ds_1 \ldots ds_m - \sum_{j=1}^{m} \left( \int_{\prod_{i=1}^{m} \mathbb{P}} p_j(x_j, s_j) \frac{\partial f(s_1, \ldots, s_m)}{\partial s_j} \, ds_1 \ldots ds_m \right) \right\} - \]
Theorem 2.4. By (2.3) and generalized Hölder’s inequality.

Above $l_1$ counts $(j, k): j < k; l_2$ counts $(j, k, r): j < k < r; j, k, r \in \{1, 2, ..., m\}$, etc. Also $p_i(x_i, s_i)$ and $\partial s_i$ means that $p_j(x_j, s_j)$ and $\partial s_j$ is missing, respectively.

Hence it holds

$$|E_f(x_1, x_2, x_3)| \leq \frac{1}{3} \prod_{i=1}^3 |w_i - u_i| \times \left( \int_{\prod_{i=1}^3 \gamma_i} \int_{\prod_{i=1}^3 \gamma_i} \left| \frac{\partial f(x_1, x_2, x_3)}{\partial x_1, \partial x_2, \partial x_3} \right| ds_1 |ds_2| |ds_3| \right),$$

(2.3)

and

$$|E_f(x_1, ..., x_m)| \leq \frac{1}{m} \prod_{i=1}^m |w_i - u_i| \times \left( \int_{\prod_{i=1}^m \gamma_i} \left| \prod_{i=1}^m p_i(x_i, s_i) \right| |ds_1| ... |ds_m| \right).$$

(2.4)

We give the following complex multivariate Ostrowski type inequalities:

Theorem 2.4. All as in Theorem 2.1. Here $r_1, r_2, r_3, r_4 > 0: \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + \frac{1}{r_4} = 1$. Then

$$|E_f(x_1, x_2, x_3)| \leq \frac{1}{3} \prod_{i=1}^3 |w_i - u_i| \times \min \left\{ \left( \int_{\prod_{i=1}^3 \gamma_i} \left| p_i(x_i, s_i) \right| ds_i \right) \left( \frac{\partial^3 f}{\partial s_3 \partial s_2 \partial s_1} \right)_{[m]} \gamma_j, \right\},$$

$$\left( \prod_{i=1}^3 \|p_i(x_i, s_i)\|_{r_j, \gamma_j} \right) \left( \prod_{j=1}^3 \left( \prod_{i=1}^3 \gamma_i \right) \right)^\frac{1}{2} \left( \frac{\partial^3 f}{\partial s_3 \partial s_2 \partial s_1} \right)_{[r_4]} \gamma_j,$$

$$\left( \sup_{(x_1, x_2, x_3) \in \prod_{i=1}^3 \gamma_i} \left( \int_{\prod_{i=1}^3 \gamma_i} \left| p_i(x_i, s_i) \right| ds_i \right) \right) \left( \frac{\partial^3 f}{\partial s_3 \partial s_2 \partial s_1} \right)_{[1]} \gamma_j,$$

$\forall (x_1, x_2, x_3) \in \prod_{j=1}^3 \gamma_j.$

Proof. By (2.3) and generalized Hölder’s inequality. □
Theorem 2.5. All as in Theorem 2.2. Here \( r_1, r_2, \ldots, r_m, r_{m+1} > 0 : \sum_{i=1}^{m+1} \frac{1}{r_i} = 1. \) Then

\[
|E_f(x_1, \ldots, x_m)| \leq \frac{1}{\prod_{i=1}^{m} |w_i - u_i|} \times \min \left\{ \left( \prod_{i=1}^{m} \int_{\gamma_i} |p_i(x_i, s_i)| |ds_i| \right) \left\| \frac{\partial^m f}{\partial s_{m+1} \cdots \partial s_1} \right\|_{m, j=1}^{m+1} \gamma_j \right.,
\]

\[
\left( \prod_{i=1}^{m} \left\| p_i(x_i, s_i) \left\|_{r_i, \gamma_i} \right. \right) \right\} \left\| \frac{\partial^m f}{\partial s_{m+1} \cdots \partial s_1} \right\|_{r_{m+1}, j=1}^{m+1} \gamma_j,
\]

\[
\left( \sup_{(s_1, \ldots, s_m) \in \prod_{j=1}^{m} \gamma_j} \left( \prod_{i=1}^{m} |p_i(x_i, s_i)| \right) \right) \left\| \frac{\partial^m f}{\partial s_{m+1} \cdots \partial s_1} \right\|_{1, j=1}^{m+1} \gamma_j \right\},
\]

\[
\forall (x_1, \ldots, x_m) \in \prod_{j=1}^{m} \gamma_j.
\]

Proof. By (2.4) and generalized Hölder’s inequality. \( \square \)

We make

Remark 2.6. Working further on (2.1) we call:

\[
A_f^{(3)} := A_f^{(3)} (x_1, x_2, x_3) := \sum_{j=1}^{3} \left( \int_{\prod_{i=1}^{3} \gamma_i} p_j(x_j, s_j) \frac{\partial f(s_1, s_2, s_3)}{\partial s_j} ds_1 ds_2 ds_3 \right)
\]

\[
+ \sum_{j < k}^{3} \left( \int_{\prod_{i=1}^{3} \gamma_i} p_j(x_j, s_j) p_k(x_k, s_k) \frac{\partial^2 f(s_1, s_2, s_3)}{\partial s_j \partial s_k} ds_1 ds_2 ds_3 \right) (l),
\]

and

\[
B_f^{(3)} := B_f^{(3)} (x_1, x_2, x_3) := \int_{\prod_{i=1}^{3} \gamma_i} \left( \prod_{i=1}^{3} p_i(x_i, s_i) \right) \frac{\partial^3 f(s_1, s_2, s_3)}{\partial s_1^3} ds_1 ds_2 ds_3,
\]

Set also

\[
T_f^{(3)} := T_f^{(3)} (x_1, x_2, x_3) := A_f^{(3)} + B_f^{(3)}.
\]

Thus, we have \((x = (x_1, x_2, x_3))\)

\[
f(x) = f(x_1, x_2, x_3) = \frac{1}{3} \prod_{i=1}^{3} (w_i - u_i) \int_{\prod_{i=1}^{3} \gamma_i} f(s_1, s_2, s_3) ds_1 ds_2 ds_3 + \frac{1}{3} \prod_{i=1}^{3} (w_i - u_i) \left( A_f^{(3)} + B_f^{(3)} \right) =
\]

\[
\frac{1}{3} \prod_{i=1}^{3} (w_i - u_i) \int_{\prod_{i=1}^{3} \gamma_i} f(s_1, s_2, s_3) ds_1 ds_2 ds_3 + \frac{1}{3} \prod_{i=1}^{3} (w_i - u_i) T_f^{(3)}.
\]

Working further on (2.2) we call:

\[
A_f^{(m)} := A_f^{(m)} (x_1, \ldots, x_m) := \sum_{j=1}^{m} \left( \int_{\prod_{i=1}^{m} \gamma_i} p_j(x_j, s_j) \frac{\partial f(s_1, \ldots, s_m)}{\partial s_j} ds_1 \cdots ds_m \right) +
\]
Thus, we have (x = (x₁, ..., xₘ))

\[
\sum_{j_1=1}^{m} \left( \int_{\mathbb{R}^n} p_j(x_j, s_j) p_k(x_k, s_k) \frac{\partial^2 f(s_1, ..., s_m)}{\partial s_k \partial s_j} ds_1...ds_m \right)_{(l_1)} + \\
\sum_{j_2=1}^{m} \left( \int_{\mathbb{R}^n} p_j(x_j, s_j) p_k(x_k, s_k) p_r(x_r, s_r) \frac{\partial^3 f(s_1, ..., s_m)}{\partial s_r \partial s_j \partial s_k} ds_1...ds_m \right)_{(l_2)} + ... + \\
\sum_{l=1}^{m-1} \left( \int_{\mathbb{R}^n} p_1(x_1, s_1) ... p_l(x_l, s_l) ... p_m(x_m, s_m) \frac{\partial^{m-1} f(s_1, ..., s_m)}{\partial s_m ... \partial s_l ... \partial s_1} ds_1...ds_l...ds_m \right),
\]

and

\[
B_f^{(m)} := B_f^{(m)}(x_1, ..., x_m) := \int_{\mathbb{R}^n} \left( \prod_{i=1}^{m} p_i(x_i, s_i) \right) \frac{\partial^m f(s_1, ..., s_m)}{\partial s_m ... \partial s_1} ds_1...ds_m.
\]

Set also

\[
T_f^{(m)} := T_f^{(m)}(x_1, ..., x_m) := A_f^{(m)} + B_f^{(m)}. \]

Thus, we have (x = (x₁, ..., xₘ))

\[
f(x) = f(x_1, ..., x_m) = \frac{1}{\prod_{i=1}^{m} (w_i - u_i)} \int_{\mathbb{R}^n} f(s_1, ..., s_m) ds_1...ds_m + \frac{1}{\prod_{i=1}^{m} (w_i - u_i)} \left( A_f^{(m)} + B_f^{(m)} \right) = \\
\frac{1}{\prod_{i=1}^{m} (w_i - u_i)} \int_{\mathbb{R}^n} f(s_1, ..., s_m) ds_1...ds_m + \frac{1}{\prod_{i=1}^{m} (w_i - u_i)} T_f^{(m)}. \tag{2.5}
\]

Let function g as in Theorem 2.2. Then as in (2.5) we obtain

\[
g(x) = g(x_1, ..., x_m) = \frac{1}{\prod_{i=1}^{m} (w_i - u_i)} \int_{\mathbb{R}^n} g(s_1, ..., s_m) ds_1...ds_m + \frac{1}{\prod_{i=1}^{m} (w_i - u_i)} \left( A_g^{(m)} + B_g^{(m)} \right) = \\
\frac{1}{\prod_{i=1}^{m} (w_i - u_i)} \int_{\mathbb{R}^n} g(s_1, ..., s_m) ds_1...ds_m + \frac{1}{\prod_{i=1}^{m} (w_i - u_i)} T_g^{(m)}. \tag{2.6}
\]

Above \(A_g^{(m)}, B_g^{(m)}, T_g^{(m)}\) have the obvious meaning.

By (2.5) we get

\[
f(x)g(x) = \frac{g(x)}{\prod_{i=1}^{m} (w_i - u_i)} \int_{\mathbb{R}^n} f(s_1, ..., s_m) ds_1 + \frac{g(x)}{\prod_{i=1}^{m} (w_i - u_i)} T_f^{(m)},
\]

and by (2.6) we get

\[
g(x)f(x) = \frac{f(x)}{\prod_{i=1}^{m} (w_i - u_i)} \int_{\mathbb{R}^n} g(s_1, ..., s_m) ds_1 + \frac{f(x)}{\prod_{i=1}^{m} (w_i - u_i)} T_g^{(m)}.
\]

Consequently after integration we get:
We conclude that (set $d := (s_1, \ldots, s_m)$)

$$
\int_{\prod_{i=1}^m (w_i - u_i)} \prod_{i=1}^m \frac{f(s)g(s)}{d_{i}} \prod_{i=1}^m ds_i = \prod_{i=1}^m g(s) \prod_{i=1}^m ds_i + \frac{1}{\prod_{i=1}^m (w_i - u_i)} \int_{\prod_{i=1}^m (w_i - u_i)} g(s) T_f^{(m)}(s) \prod_{i=1}^m ds_i,  \tag{2.7}
$$

and

$$
\int_{\prod_{i=1}^m (w_i - u_i)} \prod_{i=1}^m \frac{f(s)g(s)}{d_{i}} \prod_{i=1}^m ds_i = \prod_{i=1}^m g(s) \prod_{i=1}^m ds_i + \frac{1}{\prod_{i=1}^m (w_i - u_i)} \int_{\prod_{i=1}^m (w_i - u_i)} f(s) T_g^{(m)}(s) \prod_{i=1}^m ds_i.  \tag{2.8}
$$

By (2.7) and (2.8) we obtain

$$
\int_{\prod_{i=1}^m (w_i - u_i)} \prod_{i=1}^m f(s)g(s) d^m \dd s - \frac{1}{\prod_{i=1}^m (w_i - u_i)} \left( \int_{\prod_{i=1}^m (w_i - u_i)} f(s) d^m \dd s \right) \left( \int_{\prod_{i=1}^m (w_i - u_i)} g(s) d^m \dd s \right) = \frac{1}{2 \left( \prod_{i=1}^m (w_i - u_i) \right)^2} \left[ \int_{\prod_{i=1}^m (w_i - u_i)} f(s) T_g^{(m)}(s) + g(s) T_f^{(m)}(s) \right] d^m \dd s.
$$

Therefore we have

$$
\frac{1}{\prod_{i=1}^m (w_i - u_i)} \int_{\prod_{i=1}^m (w_i - u_i)} f(s) g(s) d^m \dd s - \frac{1}{\prod_{i=1}^m (w_i - u_i)} \left( \int_{\prod_{i=1}^m (w_i - u_i)} f(s) d^m \dd s \right) \frac{1}{\prod_{i=1}^m (w_i - u_i)} \left( \int_{\prod_{i=1}^m (w_i - u_i)} g(s) d^m \dd s \right) = \frac{1}{2 \left( \prod_{i=1}^m (w_i - u_i) \right)^2} \left[ \int_{\prod_{i=1}^m (w_i - u_i)} \left( f(s) A_g^{(m)}(s) + B_g^{(m)}(s) \right) + g(s) \left( A_f^{(m)}(s) + B_f^{(m)}(s) \right) \right] d^m \dd s.
$$

Hence it holds

$$
\Delta(f, g) := \frac{1}{\prod_{i=1}^m (w_i - u_i)} \int_{\prod_{i=1}^m (w_i - u_i)} f(s) g(s) d^m \dd s - \frac{1}{\prod_{i=1}^m (w_i - u_i)} \left( \int_{\prod_{i=1}^m (w_i - u_i)} f(s) d^m \dd s \right) \frac{1}{\prod_{i=1}^m (w_i - u_i)} \left( \int_{\prod_{i=1}^m (w_i - u_i)} g(s) d^m \dd s \right) - \frac{1}{2 \left( \prod_{i=1}^m (w_i - u_i) \right)^2} \left[ \int_{\prod_{i=1}^m (w_i - u_i)} \left( f(s) A_g^{(m)}(s) + g(s) A_f^{(m)}(s) \right) d^m \dd s \right] = \frac{1}{2 \left( \prod_{i=1}^m (w_i - u_i) \right)^2} \left[ \int_{\prod_{i=1}^m (w_i - u_i)} \left( f(s) B_g^{(m)}(s) + g(s) B_f^{(m)}(s) \right) d^m \dd s \right].
$$

Clearly we derive that \(|d^m \dd s| := \prod_{i=1}^m |ds_i|) \)

$$
|\Delta(f, g)| \leq \frac{1}{2 \left( \prod_{i=1}^m |w_i - u_i| \right)^2} \left[ \int_{\prod_{i=1}^m (w_i - u_i)} \left\{ |f(s)| B_g^{(m)}(s) + |g(s)| B_f^{(m)}(s) \right\} d^m \dd s \right] = \tag{2.9}
$$
Theorem 2.7. Let \( f, g \) and all as in Theorem 2.2. Then\[
\left| \frac{1}{\prod_{i=1}^{m} (w_i - u_i)} \int_{\prod_{i=1}^{m} \gamma} f(s) g(s) d\mathbf{s} - \frac{1}{\prod_{i=1}^{m} (w_i - u_i)} \left( \int_{\prod_{i=1}^{m} \gamma} f(s) d\mathbf{s} \right) \right| \left( \int_{\prod_{i=1}^{m} \gamma} g(s) d\mathbf{s} \right) - \frac{1}{\prod_{i=1}^{m} (w_i - u_i)} \left( \int_{\prod_{i=1}^{m} \gamma} f(s) d\mathbf{s} \right) \leq \frac{1}{\prod_{i=1}^{m} (w_i - u_i)} \left[ \int_{\prod_{i=1}^{m} \gamma} \left( f(s) A_g^{(m)}(s) + g(s) A_f^{(m)}(s) \right) d\mathbf{s} \right].
\]

The corresponding \( L_p \) Grüss inequality follows:

Theorem 2.8. Let \( f, g \) and all as in Theorem 2.2 and \( p, q > 1 \) such that \( \frac{1}{p} + \frac{1}{q} = 1 \). Then\[
\left| \frac{1}{\prod_{i=1}^{m} (w_i - u_i)} \int_{\prod_{i=1}^{m} \gamma} f(s) g(s) d\mathbf{s} - \frac{1}{\prod_{i=1}^{m} (w_i - u_i)} \left( \int_{\prod_{i=1}^{m} \gamma} f(s) d\mathbf{s} \right) \right| \left( \int_{\prod_{i=1}^{m} \gamma} g(s) d\mathbf{s} \right) - \frac{1}{\prod_{i=1}^{m} (w_i - u_i)} \left( \int_{\prod_{i=1}^{m} \gamma} f(s) d\mathbf{s} \right) \leq \frac{1}{\prod_{i=1}^{m} (w_i - u_i)} \left[ \int_{\prod_{i=1}^{m} \gamma} \left( f(s) A_g^{(m)}(s) + g(s) A_f^{(m)}(s) \right) d\mathbf{s} \right].
\]

Proof. Use of (2.9) and Hölder inequality.

The corresponding \( L_1 \) Grüss inequality follows:

Theorem 2.9. Let \( f, g \) and all as in Theorem 2.2. Then

\[
\left| \frac{1}{\prod_{i=1}^{m} (w_i - u_i)} \int_{\prod_{i=1}^{m} \gamma} f(s) g(s) d\mathbf{s} - \frac{1}{\prod_{i=1}^{m} (w_i - u_i)} \left( \int_{\prod_{i=1}^{m} \gamma} f(s) d\mathbf{s} \right) \right| \left( \int_{\prod_{i=1}^{m} \gamma} g(s) d\mathbf{s} \right) - \frac{1}{\prod_{i=1}^{m} (w_i - u_i)} \left( \int_{\prod_{i=1}^{m} \gamma} f(s) d\mathbf{s} \right) \leq \frac{1}{\prod_{i=1}^{m} (w_i - u_i)} \left[ \int_{\prod_{i=1}^{m} \gamma} \left( f(s) A_g^{(m)}(s) + g(s) A_f^{(m)}(s) \right) d\mathbf{s} \right].
\]
Proof. By Theorem 2.8 for $m=3$.

\[ \frac{1}{2} \left( \prod_{i=1}^{m} (w_i - u_i) \right)^2 \left[ \int_{\prod_{i=1}^{m} \mathbb{Y}} \left( f(s) A_g^{(m)}(s) + g(s) A_f^{(m)}(s) \right) d\mathbf{s} \right] \leq \]

\[ \frac{1}{2} \left( \prod_{i=1}^{m} |w_i - u_i| \right)^2 \left[ \left\| f \right\|_{1, \prod_{i=1}^{m} \mathbb{Y}} \left\| B_g^{(m)} \right\|_{1, \prod_{i=1}^{m} \mathbb{Y}} + \left\| g \right\|_{1, \prod_{i=1}^{m} \mathbb{Y}} \left\| B_f^{(m)} \right\|_{1, \prod_{i=1}^{m} \mathbb{Y}} \right]. \]

\[ \text{Corollary 2.10. Let } f, g \text{ and all as in Theorem 2.1. Then} \]

\[ \frac{1}{3} \prod_{i=1}^{m} (w_i - u_i) \int_{\prod_{i=1}^{m} \mathbb{Y}} f(s) g(s) d\mathbf{s} - \frac{1}{3} \prod_{i=1}^{m} (w_i - u_i) \left( \int_{\prod_{i=1}^{m} \mathbb{Y}} f(s) d\mathbf{s} \right) \frac{1}{3} \prod_{i=1}^{m} (w_i - u_i) \left( \int_{\prod_{i=1}^{m} \mathbb{Y}} g(s) d\mathbf{s} \right) - \]

\[ \frac{1}{2} \left( \prod_{i=1}^{3} (w_i - u_i) \right)^2 \left[ \int_{\prod_{i=1}^{3} \mathbb{Y}} \left( f(s) A_g^{(3)}(s) + g(s) A_f^{(3)}(s) \right) d\mathbf{s} \right] \leq \]

\[ \frac{1}{2} \left( \prod_{i=1}^{3} |w_i - u_i| \right)^2 \left[ \left\| f \right\|_{3, \prod_{i=1}^{3} \mathbb{Y}} \left\| B_g^{(3)} \right\|_{3, \prod_{i=1}^{3} \mathbb{Y}} + \left\| g \right\|_{3, \prod_{i=1}^{3} \mathbb{Y}} \left\| B_f^{(3)} \right\|_{3, \prod_{i=1}^{3} \mathbb{Y}} \right]. \]

Proof. By Theorem 2.7 for $m=3$.

\[ \text{Corollary 2.11. Let } f, g \text{ and all as in Theorem 2.1 and } p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1. \text{ Then} \]

\[ \frac{1}{3} \prod_{i=1}^{m} (w_i - u_i) \int_{\prod_{i=1}^{m} \mathbb{Y}} f(s) g(s) d\mathbf{s} - \frac{1}{3} \prod_{i=1}^{m} (w_i - u_i) \left( \int_{\prod_{i=1}^{m} \mathbb{Y}} f(s) d\mathbf{s} \right) \frac{1}{3} \prod_{i=1}^{m} (w_i - u_i) \left( \int_{\prod_{i=1}^{m} \mathbb{Y}} g(s) d\mathbf{s} \right) - \]

\[ \frac{1}{2} \left( \prod_{i=1}^{3} (w_i - u_i) \right)^2 \left[ \int_{\prod_{i=1}^{3} \mathbb{Y}} \left( f(s) A_g^{(3)}(s) + g(s) A_f^{(3)}(s) \right) d\mathbf{s} \right] \leq \]

\[ \frac{1}{2} \left( \prod_{i=1}^{3} |w_i - u_i| \right)^2 \left[ \left\| f \right\|_{p, \prod_{i=1}^{3} \mathbb{Y}} \left\| B_g^{(3)} \right\|_{q, \prod_{i=1}^{3} \mathbb{Y}} + \left\| g \right\|_{p, \prod_{i=1}^{3} \mathbb{Y}} \left\| B_f^{(3)} \right\|_{q, \prod_{i=1}^{3} \mathbb{Y}} \right]. \]

Proof. By Theorem 2.8 for $m=3$.

\[ \text{Corollary 2.12. Let } f, g \text{ and all as in Theorem 2.1. Then} \]

\[ \frac{1}{3} \prod_{i=1}^{m} (w_i - u_i) \int_{\prod_{i=1}^{m} \mathbb{Y}} f(s) g(s) d\mathbf{s} - \frac{1}{3} \prod_{i=1}^{m} (w_i - u_i) \left( \int_{\prod_{i=1}^{m} \mathbb{Y}} f(s) d\mathbf{s} \right) \frac{1}{3} \prod_{i=1}^{m} (w_i - u_i) \left( \int_{\prod_{i=1}^{m} \mathbb{Y}} g(s) d\mathbf{s} \right) - \]

\[ \frac{1}{2} \left( \prod_{i=1}^{3} (w_i - u_i) \right)^2 \left[ \int_{\prod_{i=1}^{3} \mathbb{Y}} \left( f(s) A_g^{(3)}(s) + g(s) A_f^{(3)}(s) \right) d\mathbf{s} \right] \leq \]

\[ \frac{1}{2} \left( \prod_{i=1}^{3} |w_i - u_i| \right)^2 \left[ \left\| f \right\|_{p, \prod_{i=1}^{3} \mathbb{Y}} \left\| B_g^{(3)} \right\|_{q, \prod_{i=1}^{3} \mathbb{Y}} + \left\| g \right\|_{p, \prod_{i=1}^{3} \mathbb{Y}} \left\| B_f^{(3)} \right\|_{q, \prod_{i=1}^{3} \mathbb{Y}} \right]. \]
\[
\frac{1}{2} \left( \prod_{i=1}^{3} (w_i - u_i) \right)^{-2} \left[ \int_{\prod_{i=1}^{3} \gamma_i} \left( f(s) A^{(3)}_g(s) + g(s) A^{(3)}_f(s) \right) d\gamma \right] \leq \\
\frac{1}{2} \left( \prod_{i=1}^{3} |w_i - u_i| \right)^{-2} \left[ \| f \|_{1, \prod_{i=1}^{3} \gamma_i} \left\| B^{(3)}_g \right\|_{1, \prod_{i=1}^{3} \gamma_i} + \| g \|_{1, \prod_{i=1}^{3} \gamma_i} \left\| B^{(3)}_f \right\|_{1, \prod_{i=1}^{3} \gamma_i} \right].
\]

Proof. By Theorem 2.9 for \( m = 3 \).

References


[2] G. Grüss, "Über das Maximum des absoluten Betrages von \( \int_a^b f(x)g(x) dx - \frac{1}{(b-a)^2} \int_a^b f(x) dx \int_a^b g(x) dx \), Math. Z., 39 (1935), 215-226.

