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Hyperbolic Fibonacci Sequence

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Abstract

In this paper, we investigate the hyperbolic Fibonacci sequence and the hyperbolic Fibonacci numbers. Furthermore, we give recurrence relations, the golden ratio and Binet's formula for the hyperbolic Fibonacci sequence and Lorentzian inner product, cross product and mixed product for the hyperbolic Fibonacci vectors.

1. Introduction

For the Fibonacci sequence

$$1\,,\,1\,,\,2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ 55,\ 89,\ 144,\ 233,\ldots,F_n,\ldots$$

defined by the recurrence relation

$$F_n = F_{n-1} + F_{n-2}\,,\quad (n \geq 3),$$

with $F_1 = F_2 = 1$, it is well known that the n-th term of the Fibonacci sequence (F_n) [1]-[3]. Some recent generalizations have produced a variety of new and extended results,[4]-[8].
Hyperbolic numbers have applications in different areas of mathematics and theoretical physics. In particular, they are related to the Lorentz-Minkowski (Space-time) geometry in the plane as well as complex numbers are to Euclidean one [9] . The work on the function theory for hyperbolic numbers can be found in [10]-[15]. The set of hyperbolic numbers $\mathbb{H}$ can be described in the form as

$$\mathbb{H} = \{\, z = x + h\,y \,|\, h \notin \mathbb{R}\,,\ h^2 = 1\,,\ x,y \in \mathbb{R}\}. \tag{1.1}$$

Addition, substraction and multiplication of any two hyperbolic numbers z_1 and z_2 are defined by

$$\begin{aligned} z_1 \pm z_2 =& (x_1 + h\,y_1) \pm (x_2 + h\,y_2) = (x_1 \pm x_2) + h\,(y_1 \pm y_2),\\ z_1 \times z_2 =& (x_1 + h\,y_1) \times (x_2 + h\,y_2) = x_1\,x_2 + y_1\,y_2 + h\,(x_1\,y_2 + y_1\,x_2). \end{aligned} \tag{1.2}$$

On the other hand, the division of two hyperbolic numbers are given by

$$\begin{aligned} &\frac{z_1}{z_2} = \frac{x_1 + h\,y_1}{x_2 + h\,y_2}\\ &\frac{(x_1 + h\,y_1)(x_2 - h\,y_2)}{(x_2 + h\,y_2)(x_2 - h\,y_2)} = \frac{x_1\,x_2 + y_1\,y_2}{x_2^2 - y_2^2} + h\,\frac{(x_1\,y_2 + y_1\,x_2)}{x_2^2 - y_2^2}. \end{aligned} \tag{1.3}$$

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If $x_2^2 - y_2^2 \neq 0$, then the division $\frac{z_1}{z_2}$ is possible. Therefore, the hyperbolic number system is a non-division algebra.
The hyperbolic conjugation of $z = x + hy$ is defined by

$$\bar{z} = z^{\dagger} = x - hy\,,\ \bar{\bar{z}} = z\,.$$

For any z_1, z_2 hyperbolic numbers, can be written as follows:

$$\begin{aligned} \overline{z_1 + z_2} &= \overline{z_1} + \overline{z_2}, \\ \overline{z_1 \times z_2} &= \overline{z_1} \times \overline{z_2}, \\ \|\overrightarrow{z}\|^2 &= z \times \bar{z} = x^2 - y^2. \end{aligned}$$

where z is time-like if $\|\overrightarrow{z}\|^2 > 0$, light-like if $\|\overrightarrow{z}\|^2 = 0$ and space-like if $\|\overrightarrow{z}\|^2 < 0$. The ring of hyperbolic numbers has zero-divisors. Moreover, these zero-divisors are also idempotent elements $\{e, e^{\dagger}\}$ for hyperbolic numbers, given by

$$e = \frac{1+h}{2},\ e^{\dagger} = \frac{1-h}{2}\,.$$

where $ee^{\dagger} = 0$, $e^2 = e$, $(e^{\dagger})^2 = e^{\dagger}$, $e + e^{\dagger} = 1$ and $e - e^{\dagger} = k$. Then, each hyperbolic number z can be written as follows:

$$z = x + hy = (x+y)\,e + (x-y)\,e^{\dagger} = z_1\,e + z_2\,e^{\dagger}.$$

These numbers are also called double, split, perplex, Lorentz and duplex numbers [12].

2. Hyperbolic Fibonacci sequence

The hyperbolic Fibonacci sequence defined by

$$\widetilde{F}_n = F_n + hF_{n+1}\,,\ \ (h^2 = 1) \tag{2.1}$$

with $\widetilde{F}_1 = 1 + h$, $\widetilde{F}_2 = 1 + 2h$ where $h^2 = 1$. That is, the hyperbolic Fibonacci sequence $\widetilde{F}_n$ is

$$h,\ 1+h,\ 1+2h,\ 2+3h,\ 3+5h, \ldots, (1+h)F_n + hF_{n-1}, \ldots \tag{2.2}$$

Using the equations (2.1) and (2.2) , it was obtained

$$\begin{aligned} &\widetilde{F}_{n+1} = (1+h)\,F_{n+1} + h\,F_n \\ &\widetilde{F}_{n+2} = (1+2h)\,F_{n+1} + (1+h)\,F_n \\ &\widetilde{F}_{n+3} = (2+3h)\,F_{n+1} + (1+2h)\,F_n \\ &\vdots \\ &\widetilde{F}_{n+r} = (F_n + hF_{n+1})\,F_{r+1} + (F_{n-1} + hF_n)\,F_r \\ &\vdots \end{aligned} \tag{2.3}$$

For the hyperbolic Fibonacci sequence, it was obtained the following properties:

$$\begin{aligned} &\widetilde{F}_{n+1}^2 + \widetilde{F}_n^2 = 2\widetilde{F}_{2n+1} + F_{2n+2}\,, \\ &\widetilde{F}_{n+1}^2 - \widetilde{F}_{n-1}^2 = 2\widetilde{F}_{2n} + F_{2n+1}\,, \\ &\widetilde{F}_{n+r} = \widetilde{F}_n F_{r+1} + \widetilde{F}_{n-1} F_r \quad (n \geq 3) \\ &\widetilde{F}_{n-1}\widetilde{F}_{n+1} - \widetilde{F}_n^2 = h(-1)^n\,, \\ &\widetilde{F}_{n-r}\widetilde{F}_{n+r} - \widetilde{F}_n^2 = h(-1)^{n-r+1}\,F_r^2\,, \\ &\widetilde{F}_n^2 + hF_{n+1}^2 = \widetilde{F}_{2n+1}\,, \\ &\widetilde{F}_n\widetilde{F}_m + \widetilde{F}_{n+1}\widetilde{F}_{m+1} = 2\widetilde{F}_{n+m+1} + F_{n+m+2}\,, \\ &\widetilde{F}_n\widetilde{F}_{m+1} - \widetilde{F}_{n+1}\widetilde{F}_m = h(-1)^m\,F_{n-m}\,, \\ &\tfrac{\widetilde{F}_{n+r} + (-1)^r\,\widetilde{F}_{n-r}}{\widetilde{F}_n} = L_r = F_{r+1} + (-1)^r F_{r-1}\,. \end{aligned}$$

Theorem 2.1. *If $\widetilde{F}_n$ is the hyperbolic Fibonacci number, then*

$$\lim_{n\to\infty} \frac{\widetilde{F}_{n+1}}{\widetilde{F}_n} = \frac{\alpha^2}{\alpha^2 - 1}$$

where $\alpha = (1+\sqrt{5})\big/2 = 1.618033..$ is the golden ratio.

Proof. We have for the Fibonacci number F_n,

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \alpha$$

where $\alpha = (1+\sqrt{5})\big/2 = 1.618033..$ is the golden ratio [3].

Then, using this limit value for the hyperbolic Fibonacci number $\widetilde{F}_n$, we obtain

$$\begin{aligned}
\lim_{n\to\infty}\frac{\widetilde{F}_{n+1}}{\widetilde{F}_n} = \lim_{n\to\infty}\frac{F_{n+1}+hF_{n+2}}{F_n+hF_{n+1}} &= \lim_{n\to\infty}\frac{(F_{n+1}+hF_{n+2})(F_n-hF_{n+1})}{F_n^2-F_{n+1}{}^2} \\
&\lim_{n\to\infty}\frac{F_{n+1}(F_n-F_{n+2})+h(F_nF_{n+2}-F_{n+1}^2)}{(F_n^2-F_{n+1}^2)} \\
&\lim_{n\to\infty}\frac{-F_{n+1}^2}{F_n^2-F_{n+1}^2}+h\lim_{n\to\infty}\frac{(-1)^{n+1}}{F_n^2-F_{n+1}^2} \\
&= \frac{-\alpha^2}{1-\alpha^2}+0 \\
&= \frac{\alpha^2}{\alpha^2-1}
\end{aligned}$$

where the identities $F_{n+2}=F_n+F_{n+1}$ and $F_{n+1}F_{n-1}-F_n{}^2=(-1)^n$ are used. □

Theorem 2.2. *The Binet formula* [1] *for the hyperbolic Fibonacci sequence is as follows;*

$$\widetilde{F}_n=\frac{1}{\alpha-\beta}\left(\bar{\alpha}\,\alpha^n-\bar{\beta}\,\beta^n\right).$$

Proof. If we use definition of the hyperbolic Fibonacci sequence and substitute first equation in footnote, then we get

$$\begin{aligned}
\widetilde{F}_n &= F_n+hF_{n+1} \\
&= \left(\frac{\alpha^n-\beta^n}{\alpha-\beta}\right)+h\left(\frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}\right) \\
&= \frac{\alpha^n(1+h\alpha)-\beta^n(1+h\beta)}{\alpha-\beta} \\
&= \frac{\overline{\alpha}\,\alpha^n-\overline{\beta}\,\beta^n}{\alpha-\beta}
\end{aligned}$$

where $\overline{\alpha}=1+h\alpha$ and $\overline{\beta}=1+h\beta$. □

3. Hyperbolic Fibonacci vectors

Let $\overrightarrow{z_1}=(x_1,x_2,x_3)$ and $\overrightarrow{z_2}=(y_1,y_2,y_3)$ be vectors in $\mathbb{R}^3$. The Lorentzian inner product of z_1 and z_2 is defined as [16]

$$z_1.z_2=\langle\overrightarrow{z_1},\overrightarrow{z_2}\rangle_L=x_1y_1+x_2y_2-x_3y_3.$$

This space denote by $\mathbb{L}^{2,1}$ or $Lorentz\ 3-space\ \mathbb{L}^3$.
A hyperbolic Fibonacci vector is defined by

$$\overrightarrow{\widetilde{\mathrm{F}}_{\mathrm{n}}}=(\widetilde{\mathrm{F}}_n,\widetilde{\mathrm{F}}_{n+1},\widetilde{\mathrm{F}}_{n+2})$$

Also, from equations (2.1) and (2.3) it can be expressed as

$$\overrightarrow{\widetilde{\mathrm{F}}}_{\mathrm{n}}=\overrightarrow{\mathrm{F}}_{\mathrm{n}}+h\overrightarrow{\mathrm{F}}_{\mathrm{n+1}}$$

where $\overrightarrow{\mathrm{F}}_{\mathrm{n}}=(F_n,F_{n+1},F_{n+2})$ and $\overrightarrow{\mathrm{F}}_{\mathrm{n+1}}=(F_{n+1},F_{n+2},F_{n+3})$ are the hyperbolic Fibonacci vectors.
The product of the hyperbolic Fibonacci vector $\overrightarrow{\widetilde{\mathrm{F}}_{\mathrm{n}}}$ and the scalar $\lambda\in\mathbb{R}$ is given by

$$\lambda\overrightarrow{\widetilde{\mathrm{F}}_{\mathrm{n}}}=\lambda\overrightarrow{\mathrm{F}}_{\mathrm{n}}+h\lambda\overrightarrow{\mathrm{F}}_{\mathrm{n+1}}$$

and $\overrightarrow{\widetilde{\mathrm{F}}_{\mathrm{n}}}$ and $\overrightarrow{\widetilde{\mathrm{F}}_{\mathrm{m}}}$ are equal if and only if

$$\begin{aligned}
F_n &= F_m \\
F_{n+1} &= F_{m+1} \\
F_{n+2} &= F_{m+2}.
\end{aligned}$$

[1] Binet formula is the explicit formula to obtain the n-th Fibonacci and Lucas numbers. It is well known that for the Fibonacci and Lucas numbers, Binet formulas are

$$F_n=\frac{\alpha^n-\beta^n}{\alpha-\beta}$$

and

$$L_n=\alpha^n+\beta^n$$

respectively, where $\alpha+\beta=1$, $\alpha-\beta=\sqrt{5}$, $\alpha\beta=-1$ and $\alpha=(1+\sqrt{5})\big/2$, $\beta=(1-\sqrt{5})\big/2$,[7],[8].

Some examples of the hyperbolic Fibonacci vectors can be given easily as;

$$\begin{aligned}\overrightarrow{\widetilde{\mathrm{F}}_1} &= (\widetilde{F}_1, \widetilde{F}_2, \widetilde{F}_3)\\ &= (F_1, F_2, F_3) + h(F_2, F_3, F_4)\\ &= (1+h, 1+2h, 2+3h)\end{aligned}$$

$$\begin{aligned}\overrightarrow{\widetilde{\mathrm{F}}_2} &= (\widetilde{F}_2, \widetilde{F}_3, \widetilde{F}_4)\\ &= (F_2, F_3, F_4) + h(F_3, F_4, F_5)\\ &= (1+2h, 2+3h, 3+5h)\end{aligned}$$

Theorem 3.1. *Let $\overrightarrow{\widetilde{\mathrm{F}}_\mathrm{n}}$ and $\overrightarrow{\widetilde{\mathrm{F}}_\mathrm{m}}$ be two hyperbolic Fibonacci vectors. The Lorentzian inner product of $\overrightarrow{\widetilde{\mathrm{F}}_\mathrm{n}}$ and $\overrightarrow{\widetilde{\mathrm{F}}_\mathrm{m}}$ is given by*

$$\left\langle \overrightarrow{\widetilde{\mathrm{F}}_\mathrm{n}}, \overrightarrow{\widetilde{\mathrm{F}}_\mathrm{m}} \right\rangle_L = (F_{n+m+1} - F_{n+m+4}) + h(3F_{n+m+2} + 2F_{n+m+3} - F_{n+1}F_{m+1}). \tag{3.1}$$

Proof. The Lorentzian inner product of $\overrightarrow{\widetilde{\mathrm{F}}}_\mathrm{n} = (\widetilde{F}_n, \widetilde{F}_{n+1}, \widetilde{F}_{n+2})$ and $\overrightarrow{\widetilde{\mathrm{F}}}_\mathrm{m} = (\widetilde{F}_m, \widetilde{F}_{m+1}, \widetilde{F}_{m+2})$ defined by

$$\begin{aligned}\left\langle \overrightarrow{\widetilde{\mathrm{F}}_\mathrm{n}}, \overrightarrow{\widetilde{\mathrm{F}}_\mathrm{m}} \right\rangle_L &= \widetilde{F}_n\widetilde{F}_m + \widetilde{F}_{n+1}\widetilde{F}_{m+1} - \widetilde{F}_{n+2}\widetilde{F}_{m+2}\\ &= \left\langle \overrightarrow{\mathrm{F}}_\mathrm{n}, \overrightarrow{\mathrm{F}}_\mathrm{m} \right\rangle + \left\langle \overrightarrow{\mathrm{F}}_\mathrm{n+1}, \overrightarrow{\mathrm{F}}_\mathrm{m+1} \right\rangle\\ &\quad + h[\left\langle \overrightarrow{\mathrm{F}}_\mathrm{n}, \overrightarrow{\mathrm{F}}_\mathrm{m+1} \right\rangle + \left\langle \overrightarrow{\mathrm{F}}_\mathrm{n+1}, \overrightarrow{\mathrm{F}}_\mathrm{m} \right\rangle]\end{aligned}$$

where $\overrightarrow{\mathrm{F}_\mathrm{n}} = (F_n, F_{n+1}, F_{n+2})$ is the hyperbolic Fibonacci vector. Also, the equations (1.1), (1.2) and (1.3), we obtain

$$\left\langle \overrightarrow{\mathrm{F}}_\mathrm{n}, \overrightarrow{\mathrm{F}}_\mathrm{m} \right\rangle = F_nF_m + F_{n+1}F_{m+1} - F_{n+2}F_{m+2} \tag{3.2}$$

$$\left\langle \overrightarrow{\mathrm{F}}_\mathrm{n+1}, \overrightarrow{\mathrm{F}}_\mathrm{m+1} \right\rangle = F_{n+1}F_{m+1} + F_{n+2}F_{m+2} - F_{n+3}F_{m+3} \tag{3.3}$$

$$\left\langle \overrightarrow{\mathrm{F}}_\mathrm{n}, \overrightarrow{\mathrm{F}}_\mathrm{m+1} \right\rangle = F_nF_{m+1} + F_{n+1}F_{m+2} - F_{n+2}F_{m+3} \tag{3.4}$$

and

$$\left\langle \overrightarrow{\mathrm{F}}_\mathrm{n+1}, \overrightarrow{\mathrm{F}}_\mathrm{m} \right\rangle = F_{n+1}F_m + F_{n+2}F_{m+1} - F_{n+3}F_{m+2} \tag{3.5}$$

Then from equation (3.2), (3.3), (3.4) and (3.5), we have the equation (3.1). □

Special Case-1: For the Lorentzian inner product of the hyperbolic Fibonacci vectors $\overrightarrow{\widetilde{\mathrm{F}}_\mathrm{n}}$ and $\overrightarrow{\widetilde{\mathrm{F}}}_\mathrm{n+1}$, we get

$$\begin{aligned}\left\langle \overrightarrow{\widetilde{\mathrm{F}}}_\mathrm{n}, \overrightarrow{\widetilde{\mathrm{F}}}_\mathrm{n+1} \right\rangle_L &= \widetilde{F}_n\widetilde{F}_{n+1} + \widetilde{F}_{n+1}\widetilde{F}_{n+2} - \widetilde{F}_{n+2}\widetilde{F}_{n+3}\\ &= \left\langle \overrightarrow{\mathrm{F}}_\mathrm{n}, \overrightarrow{\mathrm{F}}_\mathrm{n+1} \right\rangle + \left\langle \overrightarrow{\mathrm{F}}_\mathrm{n+1}, \overrightarrow{\mathrm{F}}_\mathrm{n+2} \right\rangle\\ &\quad + h[\left\langle \overrightarrow{\mathrm{F}}_\mathrm{n}, \overrightarrow{\mathrm{F}}_\mathrm{n+2} \right\rangle + \left\langle \overrightarrow{\mathrm{F}}_\mathrm{n+1}, \overrightarrow{\mathrm{F}}_\mathrm{n+1} \right\rangle]\\ &= (F_{2n+2} - F_{2n+5}) + h(2F_{2n+3} + F_{2n+5} - F_{n+2}F_{n+3})\end{aligned}$$

and

$$\begin{aligned}\left\langle \overrightarrow{\widetilde{\mathrm{F}}}_\mathrm{n}, \overrightarrow{\widetilde{\mathrm{F}}}_\mathrm{n} \right\rangle_L &= \widetilde{F}_n^2 + \widetilde{F}_{n+1}^2 - \widetilde{F}_{n+2}^2\\ &= \left\langle \overrightarrow{\mathrm{F}}_\mathrm{n}, \overrightarrow{\mathrm{F}}_\mathrm{n} \right\rangle + \left\langle \overrightarrow{\mathrm{F}}_\mathrm{n+1}, \overrightarrow{\mathrm{F}}_\mathrm{n+1} \right\rangle + h\left\langle \overrightarrow{\mathrm{F}}_\mathrm{n}, \overrightarrow{\mathrm{F}}_\mathrm{n+1} \right\rangle\\ &= (F_{2n+1} - F_{2n+4}) + 2h(F_{2n+2} - F_{n+2}F_{n+3}).\end{aligned}$$

Then for the Lorentzian inner product of the hyperbolic vector [2], we have, using identities of the Fibonacci numbers

$$\begin{aligned}F_n^2 + F_{n+1}^2 &= F_{2n+1}\\ F_{n+3}^2 - F_{n+1}^2 &= F_{2n+2}\\ F_nF_m + F_{n+1}F_{m+1} &= F_{n+m+1}\end{aligned}$$

(see, [11]), we have

$$\begin{aligned}\left\|\overrightarrow{\widetilde{\mathrm{F}}_\mathrm{n}}\right\|^2 &= \left\langle \overrightarrow{\widetilde{\mathrm{F}}_\mathrm{n}}, \overrightarrow{\widetilde{\mathrm{F}}_\mathrm{n}} \right\rangle_L = \widetilde{F}_n^2 + \widetilde{F}_{n+1}^2 - \widetilde{F}_{n+2}^2\\ &= (F_{2n+1} - F_{2n+2}) + 2h(F_{2n+2} - 2F_{n+2}F_{n+3}).\end{aligned}$$

[2]Lorentzian inner product of hyperbolic number as follows:

$$\langle \overrightarrow{z}, \overrightarrow{z} \rangle_L = {x_1}^2 + {x_2}^2 - {x_3}^2, \text{ [11]}.$$

The Lorentzian cross product [16],[17] of the vectors $\overrightarrow{z_1}$ and $\overrightarrow{z_2}$ in $\mathbb{L}^3$ is

$$\begin{aligned}\overrightarrow{z_1}\times_L\overrightarrow{z_2} &= \begin{vmatrix}-i & -j & k\\ x_1 & x_2 & x_3\\ y_1 & y_2 & y_3\end{vmatrix}\\ &= -i\left(x_2y_3-x_3y_2\right)+j\left(x_1y_3-x_3y_1\right)+k\left(x_1y_2-x_2y_1\right).\end{aligned}$$

Theorem 3.2. *Let $\overrightarrow{\widetilde{\mathrm{F}}_{\mathrm{n}}}$ and $\overrightarrow{\widetilde{\mathrm{F}}_{\mathrm{m}}}$ be two hyperbolic Fibonacci vectors. The Lorentzian cross product of $\overrightarrow{\widetilde{\mathrm{F}}_{\mathrm{n}}}$ and $\overrightarrow{\widetilde{\mathrm{F}}_{\mathrm{m}}}$ is given by*

$$\overrightarrow{\widetilde{\mathrm{F}}_{\mathrm{n}}}\times_L\overrightarrow{\widetilde{\mathrm{F}}_{\mathrm{m}}}=h(-1)^mF_{n-m}(i+j+k). \tag{3.6}$$

Proof. The Lorentzian cross product of $\overrightarrow{\widetilde{\mathrm{F}}_{\mathrm{n}}}=\overrightarrow{\mathrm{F}}_{\mathrm{n}}+h\overrightarrow{\mathrm{F}}_{\mathrm{n+1}}$ and $\overrightarrow{\widetilde{\mathrm{F}}_{\mathrm{m}}}=\overrightarrow{\mathrm{F}}_{\mathrm{m}}+h\overrightarrow{\mathrm{F}}_{\mathrm{m+1}}$ defined by

$$\overrightarrow{\widetilde{\mathrm{F}}_{\mathrm{n}}}\times\overrightarrow{\widetilde{\mathrm{F}}_{\mathrm{m}}}=(\overrightarrow{\mathrm{F}}_{\mathrm{n}}\times\overrightarrow{\mathrm{F}}_{\mathrm{m}})+(\overrightarrow{\mathrm{F}}_{\mathrm{n+1}}\times\overrightarrow{\mathrm{F}}_{\mathrm{m+1}})+h\,(\overrightarrow{\mathrm{F}}_{\mathrm{n}}\times\overrightarrow{\mathrm{F}}_{\mathrm{m+1}}+\overrightarrow{\mathrm{F}}_{\mathrm{n+1}}\times\overrightarrow{\mathrm{F}}_{\mathrm{m}}$$

where $\overrightarrow{\widetilde{\mathrm{F}}}_{\mathrm{n}}$ is the hyperbolic Fibonacci vector and $\overrightarrow{\mathrm{F}}_{\mathrm{n}}\times\overrightarrow{\mathrm{F}}_{\mathrm{m}}$ is the Lorentzian cross product for the hyperbolic Fibonacci vectors $\overrightarrow{\widetilde{\mathrm{F}}}_{\mathrm{n}}$ and $\overrightarrow{\widetilde{\mathrm{F}}}_{\mathrm{m}}$,

Now, we calculate the cross products $\overrightarrow{\mathrm{F}}_{\mathrm{n}}\times\overrightarrow{\mathrm{F}}_{\mathrm{m}}$, $\overrightarrow{\mathrm{F}}_{\mathrm{n+1}}\times\overrightarrow{\mathrm{F}}_{\mathrm{m+1}}$, $\overrightarrow{\mathrm{F}}_{\mathrm{n}}\times\overrightarrow{\mathrm{F}}_{\mathrm{m+1}}$ and $\overrightarrow{\mathrm{F}}_{\mathrm{n+1}}\times\overrightarrow{\mathrm{F}}_{\mathrm{m}}$: Using the property $F_mF_{n+1}-F_{m+1}F_n=(-1)^nF_{m-n}$, we get

$$\overrightarrow{\mathrm{F}}_{\mathrm{n}}\times\overrightarrow{\mathrm{F}}_{\mathrm{m}}=(-1)^mF_{n-m}(i+j+k) \tag{3.7}$$

$$\overrightarrow{\mathrm{F}}_{\mathrm{n+1}}\times\overrightarrow{\mathrm{F}}_{\mathrm{m+1}}=(-1)^{m+1}F_{n-m}(i+j+k) \tag{3.8}$$

$$\overrightarrow{\mathrm{F}}_{\mathrm{n}}\times\overrightarrow{\mathrm{F}}_{\mathrm{m+1}}=(-1)^{m+1}F_{n-m-1}(i+j+k) \tag{3.9}$$

and

$$\overrightarrow{\mathrm{F}}_{\mathrm{n+1}}\times\overrightarrow{\mathrm{F}}_{\mathrm{m}}=(-1)^mF_{n-m+1}(i+j+k). \tag{3.10}$$

Then from the equations (3.7), (3.8), (3.9) and (3.10), we obtain the equation (3.6). □

Theorem 3.3. *Let $\overrightarrow{\widetilde{\mathrm{F}}_{\mathrm{n}}}$, $\overrightarrow{\widetilde{\mathrm{F}}_{\mathrm{m}}}$ and $\overrightarrow{\widetilde{\mathrm{F}}_{\ell}}$ be the hyperbolic Fibonacci vectors. The Lorentzian mixed product of these vectors is*

$$\left\langle\overrightarrow{\widetilde{\mathrm{F}}_{\mathrm{n}}}\times_L\overrightarrow{\widetilde{\mathrm{F}}_{\mathrm{m}}}\,,\,\overrightarrow{\widetilde{\mathrm{F}}_{\ell}}\right\rangle=0. \tag{3.11}$$

Proof. Using the properties

$$\overrightarrow{\widetilde{\mathrm{F}}_{\mathrm{n}}}\times\overrightarrow{\widetilde{\mathrm{F}}_{\mathrm{m}}}=(\overrightarrow{\mathrm{F}}_{\mathrm{n}}\times\overrightarrow{\mathrm{F}}_{\mathrm{m}})+(\overrightarrow{\mathrm{F}}_{\mathrm{n+1}}\times\overrightarrow{\mathrm{F}}_{\mathrm{m+1}})+h\,(\overrightarrow{\mathrm{F}}_{\mathrm{n}}\times\overrightarrow{\mathrm{F}}_{\mathrm{m+1}}+\overrightarrow{\mathrm{F}}_{\mathrm{n+1}}\times\overrightarrow{\mathrm{F}}_{\mathrm{m}})$$

and

$$\overrightarrow{\widetilde{\mathrm{F}}}_{\ell}=\overrightarrow{\mathrm{F}}_{\ell}+h\overrightarrow{\mathrm{F}}_{\ell+1}$$

we can write,

$$\begin{aligned}\left\langle\overrightarrow{\widetilde{\mathrm{F}}}_{\mathrm{n}}\times_L\overrightarrow{\widetilde{\mathrm{F}}}_{\mathrm{m}},\overrightarrow{\widetilde{\mathrm{F}}}_{\ell}\right\rangle &=\left\langle\overrightarrow{\mathrm{F}}_{\mathrm{n}}\times\overrightarrow{\mathrm{F}}_{\mathrm{m}},\overrightarrow{\mathrm{F}}_{\ell}\right\rangle+\left\langle\overrightarrow{\mathrm{F}}_{\mathrm{n+1}}\times\overrightarrow{\mathrm{F}}_{\mathrm{m+1}},\overrightarrow{\mathrm{F}}_{\ell}\right\rangle\\ &+h[\left\langle\overrightarrow{\mathrm{F}}_{\mathrm{n}}\times\overrightarrow{\mathrm{F}}_{\mathrm{m}},\overrightarrow{\mathrm{F}}_{\ell+1}\right\rangle+\left\langle\overrightarrow{\mathrm{F}}_{\mathrm{n+1}}\times\overrightarrow{\mathrm{F}}_{\mathrm{m+1}},\overrightarrow{\mathrm{F}}_{\ell+1}\right\rangle]\\ &+h[\left\langle\overrightarrow{\mathrm{F}}_{\mathrm{n}}\times\overrightarrow{\mathrm{F}}_{\mathrm{m+1}},\overrightarrow{\mathrm{F}}_{\ell}\right\rangle+\left\langle\overrightarrow{\mathrm{F}}_{\mathrm{n+1}}\times\overrightarrow{\mathrm{F}}_{\mathrm{m}},\overrightarrow{\mathrm{F}}_{\ell}\right\rangle]\\ &+\left\langle\overrightarrow{\mathrm{F}}_{\mathrm{n}}\times\overrightarrow{\mathrm{F}}_{\mathrm{m+1}},\overrightarrow{\mathrm{F}}_{\ell+1}\right\rangle+\left\langle\overrightarrow{\mathrm{F}}_{\mathrm{n+1}}\times\overrightarrow{\mathrm{F}}_{\mathrm{m}},\overrightarrow{\mathrm{F}}_{\ell+1}\right\rangle.\end{aligned}$$

Then from equations (3.7), (3.8), (3.9)and (3.10) and by using the Lorentzian inner product definition of the hyperbolic number, we obtain

$$\left\langle(i+j+k),\overrightarrow{\mathrm{F}_{\ell}}\right\rangle=F_{\ell}+F_{\ell+1}-F_{\ell+2}=0,$$

$$\left\langle(i+j+k),\overrightarrow{\mathrm{F}_{\ell+1}}\right\rangle=F_{\ell+1}+F_{\ell+2}-F_{\ell+3}=0.$$

Thus, we have the equation (3.11). □

4. Conclusion

The hyperbolic Fibonacci sequence defined by

$$\widetilde{F}_n = F_n + hF_{n+1}\ ,\ \ (h^2 = 1),$$

with $\widetilde{F}_1 = 1+h$, $\widetilde{F}_2 = 1+2h$ where $h^2 = 1,$.

In addition, limit for the hyperbolic Fibonacci sequence and Binet's formula for the hyperbolic Fibonacci sequence is given. Furthermore, vectors and the Lorentzian inner product, cross product and mixed product of these vectors are given.

References

[1] A. F. Horadam, *Complex Fibonacci numbers and Fibonacci quaternions*, American Math. Monthly **70**(1963), 289-291.
[2] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, John Wiley and Sons, Proc., New York-Toronto, 2001.
[3] S. Vajda, *Fibonacci and Lucas Numbers the Golden Section*, Ellis Horrowood Limited Publ., England, 1989.
[4] G. Berzsenyi, *Sums of Product of Generalized Fibonacci Numbers*, Fibonacci Quart., **13**(4), (1975), 343-344.
[5] A. F. Horadam, *A Generalized Fibonacci sequence*, American Math. Monthly, *68*, (1961), 455-459.
[6] A. F. Horadam, *Basic properties of a certain generalized sequence of numbers*, Fibonacci Quart., **3**(3), (1965), 161-176.
[7] M. R. Iyer, *Identities involving generalized Fibonacci numbers*, Fibonacci Quart., **7**(1), (1969), 66-73.
[8] J. E. Walton and A. F. Horadam, *Some further identities for the generalized Fibonacci sequence*, Fibonacci Quart., **12**(3), (1974), 272-280.
[9] F. Catoni , R. Boccaletti, R. Cannata, V. Catoni, E. Nichelatti and P. Zampatti, *The Mathematics of Minkowski Space-Time*, Birkhauser, Basel, 2008.
[10] H. Gargoubi and S. Kossentini, *f-algebra structure on hyperbolic numbers*, Adv. Appl. Clifford Algebr., **26**(4), (2016), 1211-1233.
[11] A. E. Motter and A. F. Rosa, *Hyperbolic calculus*, Adv. Appl. Clifford Algebr., **8**(1), (1998), 109-128.
[12] B. Jancewicz, *The extended Grassmann algebra of R3, in Clifford (Geometric) Algebras with Applications and Engineering*, Birkhauser, Boston, (1996), 389-421.
[13] D. Khadjiev and Y. Göksal, *Applications of hyperbolic numbers to the invariant theory in two-dimensional pseudo-Euclidean space*, Adv. Appl. Clifford Algebr., **26**, (2016), 645-668.
[14] A. N. Güncan and Y. Erbil, *The q-Fibonacci hyperbolic functions*, Appl. Math. Inf. Sci. **8** (1L), (2014), 81-88.
[15] L. Barreira, L. H. Popescu and C. Valls, *Hyperbolic Sequences of Linear Operators and Evolution Maps*, Milan J. Math., **84**, (2016), 203-216.
[16] J. G. Ratcliffe, *Foundations of Hyperbolic Manifolds*, Springer-Verlag, 1994.
[17] K. Akutagawa and S. Nishikawa, *The Gauss Map and Spacelike Surfaces with Prescribed Mean Curvature in Minkowski 3-Space*, Thöko Math., J., **42**, (1990), 67-82.