



Statistical Convergence of Nets Through Directed Sets

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Abstract

The concept of statistical convergence based on asymptotic density is introduced in this article through nets. Some possible extensions of classical results for statistical convergence of sequences are obtained in this article, with extensions to nets.

1. Introduction

The concept of statistical convergence was introduced independently by H. Fast [10] and by H. Steinhaus in [34] as an applicable concept that generalizes the classical concept of usual convergence. This convergence was studied for sequences of numbers in [11, 12, 31], for sequences of elements in uniform spaces in [4, 21], for sequences of elements in paranormed spaces in [2, 14], for sequences of elements in topological groups in [6], for sequences of elements in metric spaces in [3], for sequences of elements in topological vector spaces in [20], and for sequences of elements in topological vector lattices in [1]. There are articles [26, 27], which study statistical convergence of double sequences and generalized sequences. There are generalizations of this concept through ideals in the articles [15, 16, 17, 32]. Almost all applicable statistical convergence ideas depend on asymptotic densities of sets. These sets may be subsets of N , $N \times N$, $N \times N \times N$, ..., where N represents the set of all natural numbers. So, if the concept of asymptotic density for subsets of directed sets is introduced, then the concept of statistical convergence for nets can be introduced. This is done in the present article. For this purpose, a natural restriction is made on directed sets. The restriction is the following:

For the directed sets (D, \leq) considered in this article, to each $\beta \in D$, the set $\{\alpha \in D : \alpha \leq \beta\}$ is finite and the set $\{\alpha \in D : \alpha \geq \beta\}$ is infinite. It is assumed that all directed sets considered in this article satisfy this condition.

All directed sets considered through N , $N \times N$, $N \times N \times N$, ..., in earlier studies for statistical convergence satisfy this condition. Thus, a common extension is proposed in this article.

There is an article [17], which discusses statistical convergence of nets through ideals, but not through a concept of asymptotic density. The present article presents statistical convergence of nets through a concept of asymptotic density for directed sets.

The articles [7, 8, 9, 18, 19, 22, 23, 24, 25, 28, 29, 35, 36, 37, 38, 39] are related to this study of the concept. There are articles related to summability through statistical convergence (see [11, 33]) and articles for generalizations of asymptotic density (see [5]). Let us first introduce a concept of asymptotic density for our purpose.

2. Asymptotic density

A definition of asymptotic density for a special class of directed sets is presented in this section. This definition considers with the classical definition for the directed set of natural numbers.

Definition 2.1. Let (D, \leq) be a directed set that satisfies the condition mentioned above.

To each $\alpha \in D$, let $D_\alpha = \{\beta \in D : \beta \leq \alpha\}$ and $|D_\alpha|$ denote the cardinality of D_α . The lower asymptotic density of a nonempty subset A of D is defined as the number $\liminf_{\alpha \in D} \frac{|A \cap D_\alpha|}{|D_\alpha|}$ and the upper asymptotic density of A is defined as the number $\limsup_{\alpha \in D} \frac{|A \cap D_\alpha|}{|D_\alpha|}$.

If the upper and lower densities are equal, then the common number is called the asymptotic density of A and it is denoted by $\delta(A; D)$. Thus, $\delta(A; D) = \lim_{\alpha \in D} \frac{|A \cap D_\alpha|}{|D_\alpha|}$, in the real interval $[0, 1]$. If A is an empty subset, it is assumed that $\delta(A; D) = 0$.

Here, for $x_\alpha \in \mathbb{R}$, the real line,

$$\liminf_{\alpha \in D} x_\alpha = \sup_{\beta \in D} \inf_{\alpha \geq \beta} x_\alpha$$

and

$$\limsup_{\alpha \in D} x_\alpha = \inf_{\beta \in D} \sup_{\alpha \geq \beta} x_\alpha.$$

Example 2.2. Let $D = \{(x_1, x_2, x_3) : x_i \in N, i = 1, 2, 3\}$. Define \leq on D by: $(x_1, x_2, x_3) \leq (y_1, y_2, y_3)$ if and only if $x_1 \leq y_1, x_2 \leq y_2$ and $x_3 \leq y_3$. Then, to each $(y_1, y_2, y_3) \in D$, the set $\{(x_1, x_2, x_3) \in D : (x_1, x_2, x_3) \leq (y_1, y_2, y_3)\}$ is finite, and it contains $y_1 \cdot y_2 \cdot y_3$ elements. Let $A = \{(x, x, x) : x \in N\}$. Then, $\delta(A; D) = 0$.

Example 2.3. Let $D = N$ be the directed set with the usual order relation. Then, to each $\alpha \in D$, $D_\alpha = \{\beta \in D : \beta \leq \alpha\}$ has precisely α elements. The asymptotic density introduced in Definition 2.1 for D coincides with the classical asymptotic density for subsets of N .

Definition 2.4. Let $(D^{(1)}, \leq^{(1)})$ and $(D^{(2)}, \leq^{(2)})$ be two directed sets. Let $D = D^{(1)} \times D^{(2)}$. Define the product order \leq in D by: $(x_1, x_2) \leq (y_1, y_2)$ if and only if $x_1 \leq^{(1)} y_1$ and $x_2 \leq^{(2)} y_2$. Observe again that to each $\alpha \in D$, the set $D_\alpha = \{\beta \in D : \beta \leq \alpha\}$ is finite. This definition can be extended to any Cartesian product of a finite number of directed sets.

Remark 2.5. If $A \subseteq D^{(1)}$, and if $\delta(A; D^{(1)})$ exists then $\delta(A \times D^{(2)}; D) = \delta(A; D^{(1)})$; for the notations used in the previous Definition 2.4. Moreover, if $B \subseteq D^{(2)}$ and $\delta(A; D^{(1)}) = 0$ then $\delta(A \times B; D) = \delta(A; D^{(1)})$.

Proposition 2.6. Let D be one among the directed sets $N, N \times N, N \times N \times N, \dots$, when N is endowed with the usual order, and the other sets are endowed with the corresponding product orders. Then, to each $\gamma \in D$,

$$\delta(\{\alpha \in D : \alpha \text{ not greater than or equal to } \gamma\}; D) = 0, \text{ and hence } \delta(\{\alpha \in D : \alpha \geq \gamma\}; D) = 1$$

Proof. It is easy to verify the relation $\delta(\{x \in D : \alpha \geq \gamma\}; D) = 1$. □

Example 2.7. Consider the set N with the following different order relation. $m \leq n$ if and only if m divides n . Then, N is a directed set with the properties mentioned in the introduction. Fix $k \in N \setminus \{1\}$. Let $A = \{n \in N : n \text{ is not greater than or equal to } k\} = N \setminus \{k, 2k, 3k, \dots\} = N \setminus kN$ (say).

If $m \in N \setminus kN$ and if $D_m = \{n \in N : n \leq m\}$, then, for $i \in D_m, i \in N \setminus kN$ and hence $A \cap D_m = D_m$. This shows that $\limsup_{m \rightarrow \infty} \frac{|A \cap D_m|}{|D_m|} = 1$.

If $m = k^i$ for some $i > 1$, then $D_m = \{n \in N : n \leq m\} = \{1, k, 2k, \dots, k^i\}$, when $A \cap D_m = \{1\}$. This shows that $\liminf_{m \rightarrow \infty} \frac{|A \cap D_m|}{|D_m|} = 0$.

In particular, $\delta(A; N)$ does not exist. However, if $A = \{n \in N : n \text{ is not greater than or equal to } 1\} = N \setminus \{1\}$, then $\delta(A; N) = 1$. Now, let $D = N \setminus \{1\}$ and consider the order relation defined above. To each $\beta \in D$, let $D_\beta = \{\alpha \in D : \alpha \leq \beta\}$. For a fixed $\gamma \in D$, let $B = \{\alpha \in D : \alpha \geq \gamma\}$. Then, $\limsup_{\beta \in D} \frac{|B \cap D_\beta|}{|D_\beta|} = 1 > 0$.

Definition 2.8. A directed set is said to satisfy the condition (*), if to each fixed $\gamma \in D$, for the set $B = \{\alpha \in D : \alpha \geq \gamma\}$, it is true that $\limsup_{\beta \in D} \frac{|B \cap D_\beta|}{|D_\beta|} > 0$, when $D_\beta = \{\alpha \in D : \alpha \leq \beta\}$.

3. Statistical convergence

The classical concept of statistical convergence is generalized in this section. Some new fundamental properties are derived.

Definition 3.1. Let $(x_\alpha)_{\alpha \in D}$ be a net in a topological space (X, τ) and let $x \in X$. Let us say that $(x_\alpha)_{\alpha \in D}$ converges statistically to x in (X, τ) , if, to each $U \in \tau$ such that $x \in U$, the relation $\delta(\{\alpha \in D : x_\alpha \notin U\}; D) = 0$ is true.

Let us first verify the uniqueness of statistical limits in Hausdorff spaces.

Proposition 3.2. Suppose $(x_\alpha)_{\alpha \in D}$ be a net in a Hausdorff space (X, τ) such that it converges statistically to x and y in X . Then, $x = y$.

Proof. Suppose $x \neq y$. Then, there are disjoint open sets U and V such that $x \in U$ and $y \in V$. Then,

$$\{\alpha \in D : x_\alpha \notin U\} \cup \{\alpha \in D : x_\alpha \notin V\} = \{\alpha \in D : x_\alpha \notin U \cap V\} = D.$$

But, $\delta(\{\alpha \in D : x_\alpha \notin U\} \cup \{\alpha \in D : x_\alpha \notin V\}; D) = 0$ and $\delta(D; D) = 1$; which is a contradiction. Therefore, $x = y$. Observe that $\delta(A \cup B; D) = 0$, whenever $\delta(A; D) = 0$ and $\delta(B; D) = 0$, for subsets A and B of D . □

Proposition 3.3. Let $(D^{(1)}, \leq^{(1)})$, $(D^{(2)}, \leq^{(2)})$ and (D, \leq) be as in Definition 2.4. Let (X, τ_X) and (Y, τ_Y) be given topological spaces. Let τ be the product topology on $X \times Y$. Let $(x_\alpha)_{\alpha \in D^{(1)}}$ and $(y_\beta)_{\beta \in D^{(2)}}$ be two nets in X and Y respectively. Then, $((x_\alpha, y_\beta))_{(\alpha, \beta) \in D}$ converges statistically to some (x, y) in $(X \times Y, \tau)$ if and only if $(x_\alpha)_{\alpha \in D^{(1)}}$ converges statistically to x in (X, τ_X) and $(y_\beta)_{\beta \in D^{(2)}}$ converges statistically to y in (Y, τ_Y) .

Proof. Suppose, $(x_\alpha)_{\alpha \in D^{(1)}}$ converges statistically to x in (X, τ_X) and $(y_\beta)_{\beta \in D^{(2)}}$ converges statistically to y in (Y, τ_Y) . Fix $U \in \tau_X$ and $V \in \tau_Y$ such that $x \in U$ and $y \in V$.

Then,

$$\delta(\{\alpha \in D^{(1)} : x_\alpha \notin U\}; D^{(1)}) = 0 \text{ and } \delta(\{\beta \in D^{(2)} : y_\beta \notin V\}; D^{(2)}) = 0$$

By Remark 2.5,

$$\delta(\{\alpha \in D^{(1)} : x_\alpha \notin U\} \times D^{(2)} \cup D^{(1)} \times \{\beta \in D^{(2)} : y_\beta \notin V\}; D) = 0.$$

Thus,

$$\delta(\{(\alpha, \beta) \in D : (x_\alpha, y_\beta) \notin U \times V\}; D) = 0.$$

This implies that, $((x_\alpha, y_\beta))_{(\alpha, \beta) \in D}$ converges statistically to (x, y) in $(X \times Y, \tau)$. Conversely, assume that $((x_\alpha, y_\beta))_{(\alpha, \beta) \in D}$ converges statistically to (x, y) in $(X \times Y, \tau)$. Fix an open neighborhood U of x in (X, τ_X) . Then,

$$\delta(\{(\alpha, \beta) \in D : (x_\alpha, y_\beta) \notin U \times Y\}; D) = 0.$$

So, $\delta(\{\alpha \in D^{(1)} : x_\alpha \notin U\}; D^{(1)}) = 0$. This implies that, $(x_\alpha)_{\alpha \in D^{(1)}}$ converges statistically to x in (X, τ_X) . Similarly, $(y_\beta)_{\beta \in D^{(2)}}$ converges statistically to y in (Y, τ_Y) . □

Proposition 3.4. Let (X, τ_X) , (Y, τ_Y) and $(X \times Y, \tau)$ be as in the previous Proposition 3.3. Let $(x_\alpha)_{\alpha \in D}$ be a net that converges statistically to some x in (X, τ_X) , for some directed set (D, \leq) . Let $(y_\alpha)_{\alpha \in D}$ be a net that converges statistically to some y in (Y, τ_Y) . Then, $((x_\alpha, y_\alpha))_{\alpha \in D}$ converges statistically to (x, y) in $(X \times Y, \tau)$. On the other hand, if $((x_\alpha, y_\alpha))_{\alpha \in D}$ converges statistically to some (x, y) in $(X \times Y, \tau)$ then $(x_\alpha)_{\alpha \in D}$ converges statistically to x in (X, τ_X) and $(y_\alpha)_{\alpha \in D}$ converges statistically to y in (Y, τ_Y) .

Proof. Suppose $(x_\alpha)_{\alpha \in D}$ converges statistically to x and $(y_\alpha)_{\alpha \in D}$ converges statistically to y . Let U be an open neighbourhood of x in X and V be an open neighbourhood of y in Y . Then,

$$\delta(\{\alpha \in D : x_\alpha \notin U\} \cup \{\alpha \in D : y_\alpha \notin V\}; D) = 0.$$

That is

$$\delta(\{\alpha \in D : (x_\alpha, y_\alpha) \notin U \times V\}; D) = 0.$$

So, $((x_\alpha, y_\alpha))_{\alpha \in D}$ converges statistically to (x, y) .

Conversely assume that $((x_\alpha, y_\alpha))_{\alpha \in D}$ converges statistically to (x, y) . Let U be an open neighbourhood of x . Then,

$$\delta(\{\alpha \in D : (x_\alpha, y_\alpha) \notin U \times Y\}; D) = 0.$$

That is

$$\delta(\{\alpha \in D : x_\alpha \notin U\}; D) = 0.$$

This implies that, $(x_\alpha)_{\alpha \in D}$ converges statistically to x . Similarly $(y_\alpha)_{\alpha \in D}$ converges statistically to y . □

Remark 3.5. Proposition 3.3 and Proposition 3.4 can be extended to any Cartesian product of a finite number of spaces.

Proposition 3.6. Let (X, τ_X) and (Y, τ_Y) be topological spaces and let $f : (X, \tau_X) \rightarrow (Y, \tau_Y)$ be a function which is continuous at a point x in X . Let $(x_\alpha)_{\alpha \in D}$ be a net that converges statistically to some x in (X, τ_X) . Then, $(f(x_\alpha))_{\alpha \in D}$ converges statistically to $f(x)$ in (Y, τ_Y) .

Proof. Let U be an open neighbourhood of $f(x)$ in (Y, τ_Y) . Then, there is an open neighbourhood V of x in (X, τ_X) such that $f(V) \subseteq U$. Then,

$$\{\alpha \in D : f(x_\alpha) \notin U\} \subseteq \{\alpha \in D : x_\alpha \notin V\} \text{ and } \delta(\{\alpha \in D : x_\alpha \notin V\}; D) = 0.$$

So, $\delta(\{\alpha \in D : f(x_\alpha) \notin U\}; D) = 0$. This proves that $(f(x_\alpha))_{\alpha \in D}$ converges statistically to $f(x)$ in (Y, τ_Y) . □

Proposition 3.7. Let $D^{(1)}, D^{(2)}$ and D be as in Proposition 3.3. Let $(x_\alpha)_{\alpha \in D^{(1)}}$ and $(y_\beta)_{\beta \in D^{(2)}}$ be two nets in a topological vector space X over the field of real numbers or the field of complex numbers. Let $(a_\alpha)_{\alpha \in D^{(1)}}$ be a net of scalars. If $(x_\alpha)_{\alpha \in D^{(1)}}$, $(y_\beta)_{\beta \in D^{(2)}}$ and $(a_\alpha)_{\alpha \in D^{(1)}}$ converge statistically to x, y and a respectively, then $(x_\alpha + y_\beta)_{(\alpha, \beta) \in D}$ and $(a_\alpha y_\beta)_{(\alpha, \beta) \in D}$ converge statistically to $x + y$ and ay respectively.

Proof. Use Proposition 3.3 and Proposition 3.6. Observe that, it has been assumed that, the addition and the scalar multiplication in a topological vector space are jointly continuous. □

Proposition 3.8. Let $(x_\alpha)_{\alpha \in D}$ and $(y_\alpha)_{\alpha \in D}$ be two nets in a topological vector space X ; with respect to a common directed set D . Let $(a_\alpha)_{\alpha \in D}$ be a net of scalars. If $(x_\alpha)_{\alpha \in D}$, $(y_\alpha)_{\alpha \in D}$ and $(a_\alpha)_{\alpha \in D}$ converge statistically to x, y and a respectively, then $(x_\alpha + y_\alpha)_{\alpha \in D}$ and $(a_\alpha y_\alpha)_{\alpha \in D}$ converge statistically to $x + y$ and ay respectively.

Proof. Use Proposition 3.4 and Proposition 3.6. □

Remark 3.9. One may derive results similar to Proposition 3.7 and Proposition 3.8 for the structures, topological groups, topological rings, and topological algebras.

4. Statistically Cauchy nets

The concept of statistically Cauchy nets is to be introduced for uniform spaces. For the concepts and notations in uniform spaces, one may refer to the book of Kelley [13] on General topology. The following definition agrees with the known definitions for statistically Cauchy sequences and statistically Cauchy double sequences (see [11, 26, 30]).

Definition 4.1. Let (X, \mathfrak{U}) be a uniform space with a uniformity \mathfrak{U} . A net $(x_\alpha)_{\alpha \in D}$ in X is said to be statistically Cauchy if, for given $U \in \mathfrak{U}$, there is a $\gamma \in D$ such that

$$\delta(\{\alpha \in D : (x_\alpha, x_\gamma) \notin U, \alpha \geq \gamma\}; D) = 0.$$

It is easy to verify that every Cauchy net is a statistically Cauchy net, and hence every converging net is a statistically Cauchy net in a uniform space. It is also possible to prove that statistical convergence implies statistical Cauchyness in a uniform space.

Proposition 4.2. Let D be a directed set. Then, every statistically convergent net $(x_\alpha)_{\alpha \in D}$ in a uniform space is statistically Cauchy.

Proof. Let $(x_\alpha)_{\alpha \in D}$ be a net which converges statistically to x in a uniform space (X, \mathfrak{U}) . Fix $U \in \mathfrak{U}$. Find a symmetric $V \in \mathfrak{U}$ such that $V \circ V \subseteq U$. For this V ,

$$\delta(\{\alpha \in D : (x_\alpha, x) \notin V\}; D) = 0$$

and hence there is a $\gamma \in D$ such that $(x_\gamma, x) \in V$. Then,

$$\{\alpha \in D : (x_\alpha, x_\gamma) \notin U\} \subseteq \{\alpha \in D : (x_\alpha, x) \notin V\}.$$

Thus,

$$\delta(\{\alpha \in D : (x_\alpha, x_\gamma) \notin U, \alpha \geq \gamma\}; D) = 0.$$

This proves that $(x_\alpha)_{\alpha \in D}$ is statistically Cauchy. □

Let us recall the order in product of two directed sets described in Definition 2.4.

Proposition 4.3. Let $(x_\alpha)_{\alpha \in D}$ be a net that is statistically Cauchy in a uniform space (X, \mathfrak{U}) . Then, for given $U \in \mathfrak{U}$, there is a $\gamma \in D$ such that

$$\delta(\{(\alpha, \beta) \in D \times D : (x_\alpha, x_\beta) \notin U, \alpha \geq \gamma, \beta \geq \gamma\}; D \times D) = 0.$$

Proof. Fix $U \in \mathfrak{U}$. Find a symmetric $V \in \mathfrak{U}$ such that $V \circ V \subseteq U$. For this V , there is a $\gamma \in D$ such that $\delta(\{\alpha \in D : (x_\alpha, x_\gamma) \notin V, \alpha \geq \gamma\}; D) = 0$. Since

$$\begin{aligned} \{(\alpha, \beta) \in D \times D : (x_\alpha, x_\beta) \notin U, \alpha \geq \gamma, \beta \geq \gamma\} &\subseteq \{(\alpha, \beta) \in D \times D : (x_\alpha, x_\gamma) \notin V \\ &\text{or } (x_\beta, x_\gamma) \notin V, \alpha \geq \gamma, \beta \geq \gamma\} \\ &\subseteq (\{\alpha \in D : (x_\alpha, x_\gamma) \notin V, \alpha \geq \gamma\} \times D) \\ &\cup (D \times \{\beta \in D : (x_\beta, x_\gamma) \notin V, \beta \geq \gamma\}), \end{aligned}$$

by Remark 2.5,

$$\delta(\{(\alpha, \beta) \in D \times D : (x_\alpha, x_\beta) \notin U, \alpha \geq \gamma, \beta \geq \gamma\}; D \times D) = 0. \quad \square$$

Proposition 4.4. Let $D^{(1)}, D^{(2)}$ and D be as in Proposition 3.3. Let (X, \mathfrak{U}_X) and (Y, \mathfrak{U}_Y) be two uniform spaces. Let \mathfrak{U} be the product uniformity on $X \times Y$. Let $(x_\alpha)_{\alpha \in D^{(1)}}$ and $(y_\beta)_{\beta \in D^{(2)}}$ be two nets in X and Y respectively. Then, $((x_\alpha, y_\beta))_{(\alpha, \beta) \in D}$ is statistically Cauchy in $(X \times Y, \mathfrak{U})$ if $(x_\alpha)_{\alpha \in D^{(1)}}$ is statistically Cauchy in (X, \mathfrak{U}_X) and $(y_\beta)_{\beta \in D^{(2)}}$ is statistically Cauchy in (Y, \mathfrak{U}_Y) . Moreover, if $D^{(1)}$ and $D^{(2)}$ satisfy the condition (*) mentioned in Definition 2.8, and $((x_\alpha, y_\beta))_{(\alpha, \beta) \in D}$ is statistically Cauchy in $(X \times Y, \mathfrak{U})$, then $(x_\alpha)_{\alpha \in D^{(1)}}$ is statistically Cauchy in (X, \mathfrak{U}_X) and $(y_\beta)_{\beta \in D^{(2)}}$ is statistically Cauchy in (Y, \mathfrak{U}_Y) .

Proof. The proof follows from the set relation: For $U \in \mathfrak{U}_X, V \in \mathfrak{U}_Y, \gamma_1 \in D^{(1)}$ and for $\gamma_2 \in D^{(2)}$, it is true that

$$\begin{aligned} \{(\alpha, \beta) \in D : ((x_\alpha, x_{\gamma_1}), (y_\beta, y_{\gamma_2})) \notin U \times V, (\alpha, \beta) \geq (\gamma_1, \gamma_2)\} &= (\{\alpha \in D^{(1)} : (x_\alpha, x_{\gamma_1}) \notin U, \alpha \geq \gamma_1\} \times \{\beta \in D^{(2)} : \beta \geq \gamma_2\}) \\ &\cup (\{\alpha \in D^{(1)} : \alpha \geq \gamma_1\} \times \{\beta \in D^{(2)} : (y_\beta, y_{\gamma_2}) \notin V, \beta \geq \gamma_2\}). \end{aligned}$$

□

Proposition 4.5. Let (X, \mathfrak{U}_X) and (Y, \mathfrak{U}_Y) be two uniform spaces. Let \mathfrak{U} be the product uniformity on $X \times Y$. Let $(x_\alpha)_{\alpha \in D}$ and $(y_\alpha)_{\alpha \in D}$ be nets in X and Y respectively. Then, $((x_\alpha, y_\alpha))_{\alpha \in D}$ is statistically Cauchy in $X \times Y$ if and only if $(x_\alpha)_{\alpha \in D}$ is statistically Cauchy in X and $(y_\alpha)_{\alpha \in D}$ is statistically Cauchy in Y .

Proof. Suppose $(x_\alpha)_{\alpha \in D}$ and $(y_\alpha)_{\alpha \in D}$ be statistically Cauchy. Fix $U \in \mathfrak{U}_X$ and $V \in \mathfrak{U}_Y$. Then, there is a $\gamma \in D$ such that

$$\delta(\{\alpha \in D : (x_\alpha, x_\gamma) \notin U, \alpha \geq \gamma\}; D) = 0 \text{ and } \delta(\{\alpha \in D : (y_\alpha, y_\gamma) \notin V, \alpha \geq \gamma\}; D) = 0.$$

The statistically Cauchyness of $((x_\alpha, y_\alpha))_{\alpha \in D}$ follows from the relation:

$$\{\alpha \in D : (x_\alpha, x_\gamma) \notin U \text{ or } (y_\alpha, y_\gamma) \notin V, \alpha \geq \gamma\} \subseteq \{\alpha \in D : (x_\alpha, x_\gamma) \notin U, \alpha \geq \gamma\} \cup \{\alpha \in D : (y_\alpha, y_\gamma) \notin V, \alpha \geq \gamma\}.$$

Conversely, assume that $((x_\alpha, y_\alpha))_{\alpha \in D}$ is statistically Cauchy. Fix $U \in \mathfrak{U}$. Then, there is a $\gamma \in D$ such that

$$\delta(\{\alpha \in D : (x_\alpha, x_\gamma) \notin U, \alpha \geq \gamma\}; D) = \delta(\{\alpha \in D : (x_\alpha, x_\gamma) \notin U \text{ or } (y_\alpha, y_\gamma) \notin Y \times Y, \alpha \geq \gamma\}; D) = 0.$$

This shows that $(x_\alpha)_{\alpha \in D}$ is statistically Cauchy. Similarly, $(y_\alpha)_{\alpha \in D}$ is statistically Cauchy. □

Proposition 4.6. Let $f : (X, \mathfrak{U}) \rightarrow (Y, \mathfrak{V})$ be a uniformly continuous function from a uniform space (X, \mathfrak{U}) into a uniform space (Y, \mathfrak{V}) . Let $(x_\alpha)_{\alpha \in D}$ be a statistically Cauchy net in (X, \mathfrak{U}) . Then, $(f(x_\alpha))_{\alpha \in D}$ is a statistically Cauchy net in (Y, \mathfrak{V}) .

Proof. Fix $V \in \mathfrak{V}$. Find a $U \in \mathfrak{U}$ such that $(f(x), f(y)) \in V$, whenever $(x, y) \in U$. Find a $\gamma \in D$ such that $\delta(\{\alpha \in D : (x_\alpha, x_\gamma) \notin U, \alpha \geq \gamma\}; D) = 0$.

Then,

$$\delta(\{\alpha \in D : (f(x_\alpha), f(x_\gamma)) \notin V, \alpha \geq \gamma\}; D) = 0,$$

because

$$\{\alpha \in D : (f(x_\alpha), f(x_\gamma)) \notin V, \alpha \geq \gamma\} \subseteq \{\alpha \in D : (x_\alpha, x_\gamma) \notin U, \alpha \geq \gamma\}.$$

□

Remark 4.7. Let (X, τ) be a topological vector space. The usual uniformity on X implies the following: A net $(x_\alpha)_{\alpha \in D}$ is Cauchy in X if and only if for every neighbourhood U of 0 there is a $\gamma \in D$ such that

$$\delta(\{\alpha \in D : x_\alpha - x_\gamma \notin U, \alpha \geq \gamma\}; D) = 0.$$

One can derive the following Proposition 4.8 and Proposition 4.9 which are similar to Proposition 3.7 and Proposition 3.8.

Proposition 4.8. Let $D^{(1)}, D^{(2)}, D, (x_\alpha)_{\alpha \in D^{(1)}}, (y_\beta)_{\beta \in D^{(2)}}, (a_\alpha)_{\alpha \in D^{(1)}}$ and X be as in Proposition 3.7. Let $x \in X$ and a be a scalar. If $(x_\alpha)_{\alpha \in D^{(1)}}, (y_\beta)_{\beta \in D^{(2)}}$ and $(a_\alpha)_{\alpha \in D^{(1)}}$ are statistically Cauchy, then $(x_\alpha + y_\beta)_{(\alpha, \beta) \in D}, (a_\alpha x)_{\alpha \in D^{(1)}}$ and $(ax_\alpha)_{\alpha \in D^{(1)}}$ are statistically Cauchy.

Proof. Use Proposition 4.6 and Proposition 4.4. □

Proposition 4.9. Let $(x_\alpha)_{\alpha \in D}, (y_\alpha)_{\alpha \in D}$ and X be as in Proposition 3.8. If $(x_\alpha)_{\alpha \in D}$ and $(y_\alpha)_{\alpha \in D}$ are statistically Cauchy, then $(x_\alpha + y_\alpha)_{\alpha \in D}$ is statistically Cauchy.

Proof. Use Proposition 4.5 and Proposition 4.6. □

5. Net Spaces

Corresponding to sequence spaces, net spaces can be constructed. The following construction is similar to the construction given in [31]. The following construction uses the Propositions 3.7, 3.8, 4.8 and 4.9. Since, verifications part is a direct one, it is omitted.

Let (X, τ) be a topological vector space with the natural uniformity \mathfrak{U} that induces the topology τ . Let D be a fixed directed set. Let $M = \{(x_\alpha)_{\alpha \in D} : \{x_\alpha : \alpha \in D\} \text{ is a bounded subset of } X\}$.

$$\text{Let } M_{cy} = \{(x_\alpha)_{\alpha \in D} \in M : (x_\alpha)_{\alpha \in D} \text{ is statistically Cauchy}\}.$$

$$\text{Let } M_{ct} = \{(x_\alpha)_{\alpha \in D} \in M : (x_\alpha)_{\alpha \in D} \text{ converges statistically in } X\}.$$

$$\text{Let } M_0 = \{(x_\alpha)_{\alpha \in D} \in M : (x_\alpha)_{\alpha \in D} \text{ converges statistically to zero in } X\}.$$

To each balanced neighbourhood U of zero in X , define a function p_U on M by

$$p_U((x_\alpha)_{\alpha \in D}) = \sup\{\lambda \geq 0 : \lambda x_\alpha \in U, \forall \alpha \in D\},$$

and define a subset N_U of M by

$$N_U = \{(x_\alpha)_{\alpha \in D} \in M : p_U((x_\alpha)_{\alpha \in D}) < 1\}.$$

Then, the collection of the sets of the form N_U forms a local base for M that makes M into a topological vector space under pointwise algebraic operations. Also, M_{cy} is a closed linear subspace of M . If (X, \mathfrak{U}) is a complete topological vector space, then M is a complete topological vector space and M_{ct} and M_0 are closed linear subspaces of M .

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