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MARGINALLY TRAPPED SURFACES AND KALUZA-KLEIN THEORY

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(Communicated by H. Hilmi HACISALİHOĞLU)

Abstract. The concept of *trapped surfaces* introduced by Roger Penrose in [Phys. Rev. Lett. **14** (1965), 57–59] plays an extremely important role in cosmology and general relativity. It is considered as a cornerstone for the achievement of the singularity theorems, the analysis of gravitational collapse, the cosmic censorship hypothesis, Penrose inequality, etc. In term of mean curvature vector, a surface in a space-time (or more generally, in a semi-Riemannian manifold) is marginally trapped if its mean curvature vector is light-like at each point. In this article, we survey recent classification results on marginally trapped surfaces from differential geometric viewpoint. Also, we provide a brief introduction to a closely related subject; namely, the Kaluza-Klein theory. In the final part, we present several different recent approaches to the Kaluza-Klein theory without using compactification of the extra dimensions.

1. Black holes and galaxies.

The idea of an object with gravity strong enough to prevent light from escaping was proposed in 1783 by J. Michell (1724 - 1793), an amateur British astronomer. In 1795, P.-S. Laplace (1749 - 1823), a French physicist independently came to the same conclusion. Black holes, as currently understood, are described by Einstein's general theory of relativity developed in 1916 (cf. [16]).

This theory predicts that when a large enough amount of mass is present in a sufficiently small region of space, all paths through space are warped inwards towards the center of the volume, preventing all matter and radiation within it from escaping. Einstein's theory has important astrophysical applications. It points towards the existence of black holes. In addition, general relativity is the basis of current cosmological models of an expanding universe.

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According to the American Astronomical Society, every large galaxy has a supermassive black hole ($\sim 10^5$ - $10^9\, M_{sun}$) at its center. The black hole's mass is proportional to the mass of the host galaxy, suggesting that the two are linked very closely.

Black holes can't be seen, because everything that falls into them, including light, is trapped. But the swift motions of gas and stars near an otherwise invisible object allows astronomers to calculate that it's a black hole and even to estimate its mass.

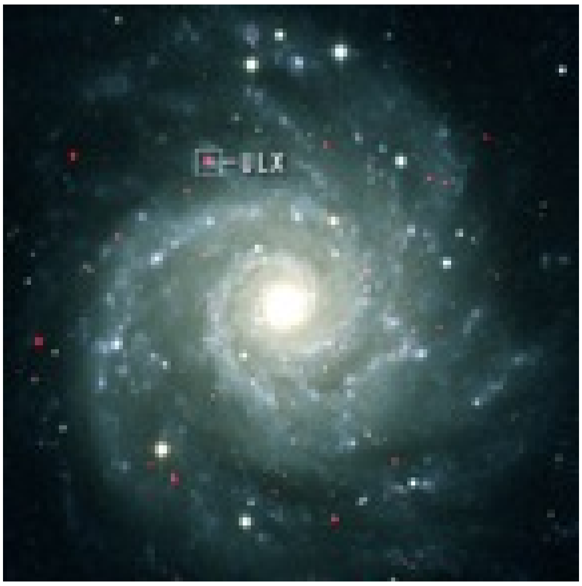
The following are six photos of black holes mostly taken from Hubble telescope and were released by NASA.

Figure 1. Black Hole in Galaxy M87.

Streaming out from the center of the galaxy M87 like a cosmic searchlight is one of nature's most amazing phenomena, a black-hole-powered jet of electrons and other sub-atomic particles traveling at nearly the speed of light. In this Hubble telescope image, the blue jet contrasts with the yellow glow from the combined light of billions of unseen stars and the yellow, point-like clusters of stars that make up this galaxy. Lying at the center of M87, the monstrous black hole has swallowed up matter equal to 2 billion times our Sun's mass. M87 is 50 million light-years from Earth.

Credit: Hubble Space Telescope/NASA/ESA.

Figure 2. An Intermediate-Mass Black Hole in Galaxy M74.



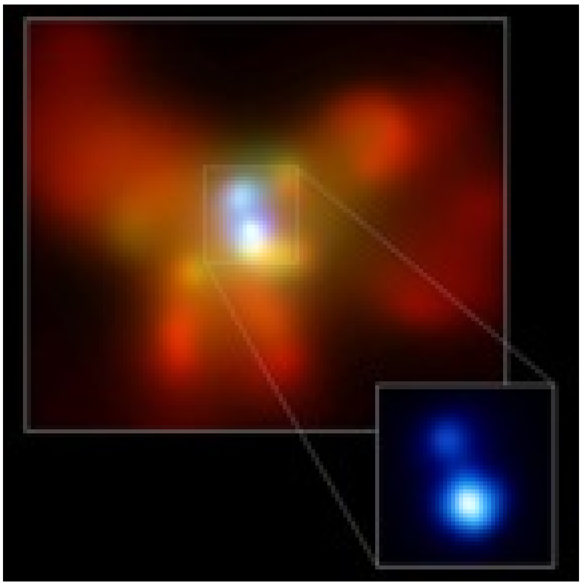
This intermediate mass black hole ($\sim 10^3$ - $10^4\, M_{sun}$) *in galaxy M74 uncovered by Hubble has a mass of about 10,000 Suns.*

Credit: NASA/CXC/UM.

Figure 3. A 300-Million-Solar-Mass Black Hole.

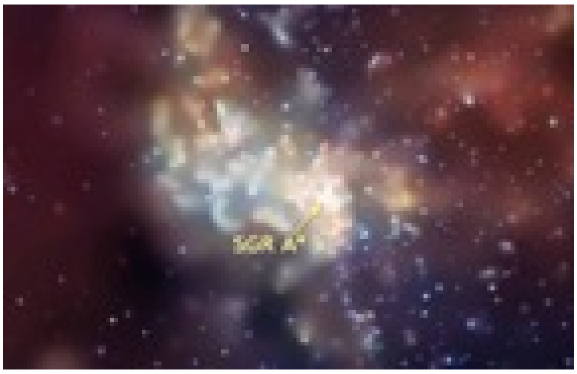
A 3,700-light-year-wide dust disk encircles a 300-million-solar-mass black hole in the center of the elliptical galaxy NGC7052.

Figure 4. Two Supermassive Black Holes in Same Galaxy.



This image of NGC6240, a butterfly-shaped galaxy that is the product of the collision of two smaller galaxies, revealed that the central region of the galaxy (inset) contains not one, but two active giant black holes.

Figure 5. Our Galaxy Milky Way's Giant Black Hole.



Recently, using NASA, Japanese, and European X-ray satellites, a team of Japanese astronomers has discovered that our galaxy's central black hole let loose a powerful flare three centuries ago. (Released on April 16, 2008).

Figure 6. Second Black Hole Found in Milky Way's Center.

Astronomers think they have found a rare if not unique black hole very near the center of our galaxy, the Milky Way. This newly detected object appears to be an intermediate mass black hole, packing about 1,300 solar masses.

2. Space-times.

For physical reasons, a *space-time* is mathematically defined as a 4-dimensional, smooth, connected pseudo-Riemannian manifold with a smooth Lorentz metric of signature $(-,+,+,+)$. By combining space and time into a single manifold, physicists have significantly simplified a large number of physical theories, as well as described in a more uniform way the workings of the universe at both the supergalactic and subatomic levels.

Formerly, from experiments at slow speeds, time was believed to be a constant, which progressed at a fixed rate; however, later high-speed experiments revealed that time slowed down at higher speeds (with such slowing called "time dilation"). Many experiments have confirmed the slowing from time dilation, such as atomic clocks onboard a Space Shuttle running slower than synchronized Earth-bound clocks. Since time varies, it is treated as a variable within the space-time coordinate grid, and time is no longer assumed to be a constant, independent of the location in space.

The geometry of space-time in special relativity is described by the Minkowski metric on $\mathbf{R}^4$. This space-time is called Minkowski space-time $\mathbb{E}^4_1$.

Besides Minkowski space-time, there are two other space-times which are of constant curvature; namely, the de Sitter space-time S^4_1 (or dS_4 by many physicists) and the anti-de Sitter space-times H^4_1 (or AdS_4).

De Sitter space-time can be defined as a hypersurface of Minkowski space. Take Minkowski space-time $\mathbb{E}^5_1$ with the standard metric

$$g = -dt^2 + dx^2 + dy^2 + dz^2,$$

the de Sitter space-time is the submanifold described by the hyperboloid

$$-t^2 + x^2 + y^2 + z^2 = c^2.$$

where c is some positive constant. The metric on de Sitter space-time is the metric induced from the ambient Minkowski metric.

Similarly, an anti de Sitter space-time can be realized as a hypersurface of the pseudo-Euclidean space $\mathbb{E}^5_2$ with index 2 described by

$$-t^2 - x^2 + y^2 + z^2 = -c^2.$$

where c is some positive constant.

Another important cosmological model in general relativity is the Robertson-Walker space-time described as the warped product:

$$L^4_1(f,c) := (I \times S, g^c_f), \quad g^c_f = -dt^2 + f^2(t)g_c, \tag{2.1}$$

where (S, g_c) is a 3-dimensional space of constant curvature c. It describes a simply-connected, homogeneous, isotropic expanding or contracting universe.

Robertson-Walker space-times provide good descriptions of our Universe except in the earliest and the final era (cf. [25]).

3. Einstein's general relativity.

The starting point of marginally trapped surfaces and the Kaluza-Klein theory is the General Relativity Theory of Albert Einstein (1879 – 1955) published in 1916.

Einstein's General Relativity is *the geometrical theory of gravitation*, which unifies special relativity and Newton's law of universal gravitation with the insight that *gravitation is not due to a force* but rather is a manifestation of curved space and time, with this curvature being produced by the mass-energy and momentum content of the space-time.

Einstein's general relativity is distinguished from other theories of gravitation by its use of the so-called Einstein field equations:

$$R_{ij} - \frac{1}{2}Rg_{ij} = kT_{ij}$$

to relate mass-energy tensor T_{ij} with Ricci curvature R_{ij} of space-time, where g_{ij} is the metric tensor, R is the scalar curvature, and k is called the Einstein constant of gravitation.

In a vacuum (a region of space-time with no matter), i.e., $T_{ij} = 0$, the space-time is called an Einstein space. In this case, the Ricci tensor R_{ij} is proportional to the metric tensor g_{ij}.

The predictions of general relativity differ significantly from those of classical physics, especially concerning the passage of time, the geometry of space, the motion of bodies in free fall, and the propagation of light.

Examples of such differences include gravitational time dilation, the gravitational redshift of light, and the gravitational time delay. General relativity's predictions have been confirmed in all observations and experiments to date.

4. Trapped surfaces.

The concept of *trapped surfaces*, introduced by Roger Penrose (1931 -) in 1965 [27] plays extremely important role in general relativity and cosmology.

It is considered as a cornerstone for the achievement of the singularity theorems, the analysis of gravitational collapse, the cosmic censorship hypothesis, the Penrose inequality, ... etc.

In the theory of cosmic black holes, if there is a massive source inside the surface, then close enough to a massive enough source, the outgoing light rays may also be converging; a trapped surface. Everything inside is trapped. Nothing can escape, not even light.

It is believed that there will be a marginally trapped surface, separating the trapped surfaces from the untrapped ones, where the outgoing light rays are instantaneously parallel. In terms of the mean curvature vector field; a codimension two surface is marginally trapped if and only if its mean curvature vector field is light-like at each point on the surface.

The surface of a black hole is the marginally trapped surface. As times develops, the marginally trapped surface generates a hypersurface in space-time, a *trapping horizon.*

Although many physicists are interested in marginally trapped surfaces, almost no classification results on marginally trapped surfaces are known from differential geometric point of view until the last few years. Moreover, the issue of differentiability of the boundary of the trapped region is still wide open in general relativity theory. However, for strictly stable outer marginally trapped surfaces, the following two results were shown in [2]:

(i) Local existence of a trapping horizon; and

(ii) Outgoing light rays are converging just inside and diverging just outside such a surface.

5. Cosmic censorship hypothesis.

"Stephen Hawking (1942 -) and Kip Thorne (1940 -) firmly believe that information swallowed by a black hole is forever hidden from the outside universe, and can never be revealed even as the black hole evaporates and completely disappears ..." [quoted from Hawking]

In general relativity, the *cosmic censorship hypothesis* (CCH) is a conjecture about the nature of singularities in space-time.

Singularities that arise in the solutions of Einstein's equations are typically hidden within event horizons, and therefore cannot be seen from the rest of space-time. Singularities which are not so hidden are called *naked.*

5.1. **Weak cosmic censorship hypothesis.** In 1969, R. Penrose formulated the *weak CCH conjecture*:

"*No naked singularities other than the Big Bang singularity exist in the universe.*"

This hypothesis is not stated in a completely formal way. In a sense it is more of a research program proposal: part of the research is to find a proper formal statement that is physically reasonable and that can be proved to be true or false, and sufficiently general to be interesting.

It is not difficult to construct space-times which have naked singularities, but which are not "physically reasonable"; the canonical example is given by H. Reissner and G. Nordstrom (in 1990's), which contains a naked singularity.

5.2. **Strong cosmic censorship hypothesis.** Later, R. Penrose reformulated a stronger version of the cosmic censorship hypothesis (known as the *strong cosmic censorship hypothesis*) to exclude these situations.

Roughly speaking, the strong CCH asserts that

"No singularity is ever visible to any observer".

In 1991 John Preskill and Kip Thorne bet against Stephen Hawking that the original weak hypothesis was false. Due to the special situations just mentioned; they won the bet for a T-shirt to cover the winner's nakedness.

Hawking later reformulated the bet to exclude those technicalities.

5.3. **Penrose's inequality.** The *Penrose inequality* is a (conjectured) lower bound for the total mass of a space-time, provided by the area A of an apparent horizon.

In 1973, R. Penrose presented an argument that the total mass of a space-time which contains black holes with event horizons of total area A should be at least $\sqrt{A/16\pi}$.

An important special case of this physical statement translates into a very beautiful mathematical inequality in Riemannian geometry known as the *Riemannian Penrose inequality* [19].

The Riemannian Penrose inequality was first established by G. Huisken and T. Ilmanen [19] in 1997 for a single black hole and by H. Bray [3] in 1999 for any number of black holes.

6. Some classification theorems of marginally trapped surfaces.

In this section we present some classification theorems for marginally trapped surfaces from differential geometric viewpoint.

6.1. **The earlier classification result for marginally trapped surfaces.** Let $L : M \to \mathbb{E}^4_s$ be an isometric immersion from a pseudo-Riemannian surface M into $\mathbb{E}^4_s$. Denote by Δ the Laplacian on M. Then $L : M \to \mathbb{E}^4_s$ is minimal if and only if L is harmonic, i.e., $\Delta L = 0$. An immersion $L : M \to \mathbb{E}^4_s$ is called **biharmonic** if and only if we have $\Delta^2 L = 0$ (cf. [4]).

As far as I know, the first classification result on marginally trapped surfaces from differential geometric viewpoint is related with biharmonic surfaces, which was obtained as follows by Chen and Ishikawa in [9].

Theorem 6.1. *Let $L : M \to \mathbb{E}^4_1$ be a biharmonic surface in Minkowski space-time $\mathbb{E}^4_1$ with flat normal connection. Then L is marginally trapped if and only if up to rigid motions of $\mathbb{E}^4_1$, $L : M \to \mathbb{E}^4_1$ is given by*

$$L(u, v) = (\varphi(u, v), u, v, \varphi(u, v)),$$

where φ is proper biharmonic function on M, i.e., $\Delta\varphi \neq 0$ and $\Delta^2\varphi = 0$.

6.2. **Marginally trapped surfaces with positive relative nullity.** For marginally trapped spatial surfaces with positive nullity in the Minkowski space-time $\mathbb{E}^4_1$, we have the following classification result obtained in [12].

Theorem 6.2. *Up to Minkowskian motions, there exist two families of marginally trapped spatial surfaces with positive relative nullity in $\mathbb{E}^4_1$:*

(1) A surface defined by $L(x, y) = \big(f(x), x, y, f(x)\big)$, where $f(x)$ is an arbitrary differentiable function with $f''(x)$ being nowhere zero.

(2) *A surface defined by*

$$L(x,y)=\bigg(\int_0^x r(x)q'(x)dx+q(x)y, y\cos x-\int_0^x r(x)\sin x dx,$$
$$y\sin x+\int_0^x r(x)\cos x dx, \int_0^x r(x)q'(x)dx+q(x)y\bigg),$$

where q and r are defined on an open interval $I\ni 0$ satisfying $q''(x)+q(x)\neq 0$ for each $x\in I$.

Conversely, every marginally trapped spatial surface with positive relative nullity in the Minkowski space-time $\mathbb{E}^4_1$ is congruent to an open portion of a surface obtained from the two families.

We also have the following two classification theorems from [12] for marginally trapped spatial surfaces with positive relative nullity in the de Sitter space-time $S^4_1(1)$ and the anti-de Sitter space-time $H^4_1(-1)$.

Theorem 6.3. *Up to rigid motions of $S^4_1(1)$, there exist two families of marginally trapped spatial surfaces with positive relative nullity in the de Sitter space-time $S^4_1(1)\subset\mathbb{E}^5_1$:*

(1) *A surface given by*

$$L(x,y)=\Big(f(x)\cos y, \sin x\cos y, \sin y, \cos x\cos y, f(x)\cos y\Big),$$

where f is an arbitrary function defined on an open interval I satisfying $f''+f\neq 0$ at each point in I.

(2) *A surface given by*

$$L(s,y)=\big(p(s),\eta_1(s),\eta_2(s),\eta_3(s),p(s)\big)\cos y$$
$$-\Big(b-\int_0^s r(s)p'(s)ds, \xi_1(s),\xi_2(s),\xi_3(s), b-\int_0^s r(s)p'(s)ds\Big)\sin y,$$

where b is a real number, p and r are defined on an open interval $I\ni 0$ such that r is non-constant, $\eta=(\eta_1,\eta_2,\eta_3)$ is a unit speed curve in $S^2(1)\subset\mathbb{E}^3$ with geodesic curvature $\kappa_g=r$, and $\xi=(\xi_1,\xi_2,\xi_3)$ is the unit normal of η in $S^2(1)$.

Conversely, every marginally trapped spatial surface with positive relative nullity in the de Sitter space-time S^4_1 is congruent to an open portion of a surface obtained from the two families.

Theorem 6.4. *Up to rigid motions of $H^4_1(-1)$, there exist five families of marginally trapped spatial surfaces with positive relative nullity in the anti-de Sitter space-time $H^4_1(-1)$:*

(1) $L(x,y)=\Big(f(x)\cosh y, \cosh x\cosh y, \sinh y, \sinh x\cosh y, f(x)\cosh y\Big)$, *where $f(x)$ is defined on an open interval I such that $f''(x)-f(x)\neq 0$ at each $x\in I$.*

(2) $L(x,y)=\Big(f(x)\sinh y, \cosh y, \cos x\sinh y, \sin x\sinh y, f(x)\sinh y\Big)$, *where $f(x)$ is defined on an open interval I such that $f''(x)+f(x)\neq 0$ at each $x\in I$.*

(3) $L(x,y)=\big(x^2e^y, \frac{3e^y}{2}-2\sinh y, e^y-2\sinh y, xe^y, x^2e^y-\frac{e^y}{2}\big)$.

(4)$L(x,y)=\Big(\sinh y-\frac{x^2e^y}{2}-e^y, f(x)e^y, xe^y, f(x)e^y, \sinh y-\frac{x^2e^y}{2}\Big)$, *where $f(x)$ is defined on an open interval I such that $f''(x)\neq 0$ at each $x\in I$.*

(5) *A surface defined by*

$$L(s,y)=\big((p(s),\eta_1(s),\eta_2(s),\eta_3(s),p(s)\big)\cosh y$$
$$-\left(b-\int_0^s r(s)p'(s)ds,\xi_1(s),\xi_2(s),\xi_3(s),b-\int_0^s r(s)p'(s)ds\right)\sinh y,$$

where b is a real number, p and r are defined on an open interval $I\ni 0$ such that r is non-constant, $\eta=(\eta_1,\eta_2,\eta_3)$ is a unit speed curve in $H^2(-1)\subset\mathbb{E}^3_1$ with geodesic curvature $\kappa_g=r$, and $\xi=(\xi_1,\xi_2,\xi_3)$ is the unit normal of η in $H^2(-1)$.

Conversely, every marginally trapped spatial surfaces with positive relative nullity in the anti-de Sitter space-time H^4_1 is congruent to an open portion of a surface obtained from the five families.

Remark 1. A conformal representation formula of Weierstrass-Bryant type was obtained in [1] for the class of marginally trapped surfaces M in the Minkowski space-time $\mathbb{E}^4_1$ which satisfy the following two additional conditions:

(a) M has flat normal connection in $\mathbb{E}^4_1$, and

(2) M is locally isometric either to a minimal surface in $\mathbb{E}^3$ or to a maximal surface in $\mathbb{E}^3_1$.

6.3. **Marginally trapped surfaces in Robertson-Walker space-times.** For marginally trapped surfaces in a Robertson-Walker space-time, we have the following [13].

Theorem 6.5. *Let $L^4_1(f,c)=I\times_f S$ be a Robertson-Walker space-time which contains no open subsets of constant curvature. Then $L^4_1(f,c)$ does not admit any marginally trapped surface M with positive relative nullity.*

Remark 2. If we do not assume M to have positive relative nullity, there exist marginally trapped surfaces in Robertson-Walker space-times of non-constant sectional curvature.

6.4. **Boost invariant marginally trapped surfaces in $\mathbb{E}^4_1$.** The *boost group* in $\mathbb{E}^4_1$ is defined by

$$G=\left\{\begin{pmatrix}\cosh\theta & \sinh\theta & 0 & 0\\ \sinh\theta & \cosh\theta & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}:\theta\in\mathbf{R}\right\}.$$

Recently, marginally trapped surfaces in $\mathbb{E}^4_1$ invariant under boost group are classified by Haesen and Ortega [17].

6.5. **Marginally trapped surfaces in strictly stationary space-times.** A space-time is called *strictly stationary* it contains a Killing vector field which is time-like everywhere.

Mars and Senovilla [23] proved the following non-existence result for strictly stationary space-time

Theorem 6.6. *There do not exist closed marginally trapped surfaces in strictly stationary space-times.*

6.6. **Some other recent classification results.** Recently, together with Van der Veken we have completely classified in [14] marginally trapped surfaces which lie in a light cones in Minkowski, de Sitter or anti-de Sitter space-times. In particular, we proved that such marginally trapped surfaces must have parallel mean curvature vector.

Completely classification of marginally trapped surfaces with parallel mean curvature vector in Minkowski, de Sitter and anti-de Sitter space-times are also obtained in the same article.

Several families of marginally trapped surfaces in Lorentzian complex space forms have also been classified recently; namely, we are able to achieve the following:

(a) *Classification of marginally trapped surfaces in Lorentzian complex plane* $\mathbf{C}_1^2$;

(b) *Classifications of several families of marginally trapped surfaces in Lorentzian complex projective plane* CP_1^2 *and in Lorentzian complex hyperbolic plane* CH_1^2.

Some of the results for (b) are joint results with F. Dillen in [8] and with I. Mihai in [11].

6.7. **Two challenging open problems on marginally trapped surfaces.** The following two problems are challenging.

Problem 1. *Completely classify marginally trapped surfaces in Minkowski, de Sitter, anti-de Sitter and Robertson-Walker space-times.*

Problem 2. *Completely classify marginally trapped surfaces in* CP_1^2 *or in* CH_1^2.

7. Unified field theory.

A physical field can be thought of as the assignment of a physical quantity at each point of space-time. For example, on weather forecasts, the wind velocity during a day over a country is described by assigning a vector at each point of space (with moving arrows representing the change in wind velocity during the day).

Unified field theory is an attempt to unify all the fundamental forces (gravitation, electromagnetism, weak interaction, strong interaction) and the interactions between elementary particles into a single theoretical framework. The term was coined by Einstein who attempted to reconcile the general theory of relativity with electromagnetism in a single field theory.

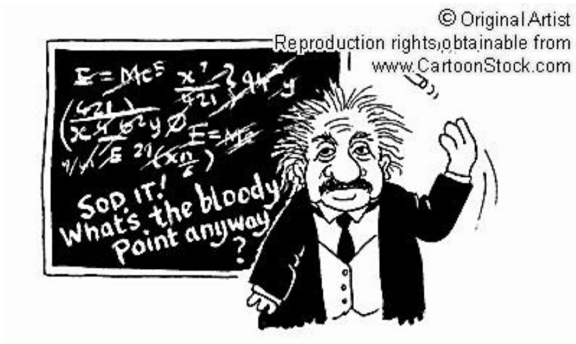
However, Einstein's attempt was unsuccessful!

There is no accepted unified field theory yet, and this remains an open line of research.

Einstein's quest proved elusive and a unified field theory, sometimes referred to as the

"Theory of Everything",

has remained the holy grail for physicists, the long-sought theory which would explain the nature and behavior of all matter.



Einstein develops his theory of negativity.

8. Maxwell equations.

The Maxwell equations describing electromagnetism can be understood to be the Hodge equations of circle bundle with fiber $U(1)$, or a principal $U(1)$-bundle $\pi : P \to M$.

The electromagnetic field F is a harmonic 2-form in the space $\Omega^2(M)$ of differentiable 2-forms on the manifold M. In the absence of charges and currents, the so-called free-field Maxwell equations are

$$dF = 0 \quad \text{and} \quad d^*F = 0 \tag{8.1}$$

where * is the Hodge star operator.

9. Kaluza-Klein theory.

Shortly after the publication of Einstein's theory of General Relativity, T. Kaluza (1885 - 1954) noticed in 1919 that when he solved Einstein's equations using five dimensions, Maxwell's equations for electromagnetism emerged spontaneously.

Kaluza wrote to Einstein who encouraged him to publish. This very influential Kaluza's 7 pages paper was published in 1921 ([20]).

In order to explain *why the extra 5^{th} dimensional is unobservable*, Oskar Klein (1894 - 1977) suggested in 1926 that this extra 5th dimension would be compactified and unobservable on experimentally accessible energy scale.

Klein proposed that *the 4th spatial dimension is curled up in a circle of very small radius*, so that a particle moving a short distance along that axis would return to where it began. However, Kaluza and Klein's work was neglected for many years as attention was directed towards quantum mechanics.

This idea that fundamental forces can be explained by additional dimensions did not re-emerge until string theory was developed in 1960's. This strategy of using higher dimensions to unify different forces is now a very active area of research in particle physics (also known as high energy physics).

This idea of compactifying the extra dimension has also dominated the search for a unified theory and led to many new development in string theory; for instance, it leads to the 10D string theory and more recently 11D superstring M-theory.

10. Geometric interpretation of Kaluza-Klein theory.

The Kaluza-Klein theory is striking because it has a particularly elegant presentation in terms of differential geometry. In a certain sense, it looks just like ordinary gravity in free space, except that it is phrased in 5D instead of 4D.

To build the Kaluza-Klein theory from differential geometric viewpoint, one picks an invariant metric on the circle S^1 that is the fiber of the $U(1)$-bundle of electromagnetism. An invariant metric g^* is simply one that is invariant under rotations of the circle.

Suppose this metric gives the circle S^1 a *total length* Λ, then considers metrics on the bundle P that are consistent with both the fiber metric, and the metric on the underlying 4D space-time M so that $\pi : P \to M$ is a Riemannian submersion.

The original Kaluza-Klein theory identifies Λ with the fiber metric g_{55}, and allows Λ to vary from fiber to fiber. In this case, the coupling between gravity and the electromagnetic field is not constant, but has its own dynamical field, known as the *radion* in physics.

In the formation given above, the extra 5th dimension in Kaluza-Klein theory can be understood to be the circle group $U(1)$, as electromagnetism can essentially be formulated as a gauge theory on a circle bundle with gauge group $U(1)$. Once this geometrical interpretation of Kaluza-Klein theory is understood, it is relatively straightforward to replace $U(1)$ by a general Lie group. Such generalizations are often called Yang-Mills theories.

11. Several recent approaches to Kaluza-Klein theory.

In recent years, several different approaches to Kaluza-Klein theory have also been developed, instead of compactifying the extra dimensions.

11.1. **Randall and Sundrum's approach.** There is an approach of Kaluza-Klein theory proposed by Lisa Randall (1962 -) and Raman Sundrum (1963 -). Randall and Sundrum's two very influential articles were published in 1999:

1. *Large mass hierarchy from a small extra dimension*, Phys. Rev. Letters **83** (1999), 3370-3373.

2. *An alternative to compactification*, Phys. Rev. Letters **83** (1999), 4690-4693.

Randall-Sundrum's model attempts to address the hierarchy problem:

"Why is gravity so puny, so many billion on billions of times weaker compared with the other forces-electromagnetism and the weak and strong nuclear forces?"

In Randall-Sundrum's articles, they suggested that our Universe might have evolved differently in the beginning than it did later. Rather than invoking supersymmetry (interchanges bosons and fermions) that argues for the existence of as yet undetected partners (fermions) to all the known particles. Randall and Sundrum proposed that gravity could reside on a different brane than ours, one separated from us by a 5D space-time in which the extra dimension is 10^{-31} cm wide.

In their model, Randall and Sundrum proposed that all forces and particles stick to our 3-brane except gravity, which is concentrated on the other brane and is free to travel between them across space-time. By the time gravity gets to us gravity is weak; in the other brane it is strong, on a par with the three other forces.

In Randall-Sundrum's models, the warping of the extra dimension is analogous to the warping of space-time in the vicinity of a black hole. This warping, or red-shifting, generates a large ratio of energy scales so that the natural energy scale at one end of the extra dimension is much larger than at the other end.

Mathematically, their warping metric tensor is given by

$$g = \frac{1}{k^2y^2}(dy^2 + \sum_{ij=1}^{4} \eta_{ij} dx^i dx_j), \tag{11.1}$$

where k is some constant and η_{ij} has "$-$+++" signature. This space has boundaries at $y = \frac{1}{k}$ and $y = \frac{1}{Wk}$, with $0 \leq \frac{1}{k} \leq \frac{1}{Wk}$ where k is around the Planck scale (1.616×10^{-35} m) and W is the warping factor.

The Randall-Sundrum scenario has gained a lot of support since 1999 from physics community.

11.2. **Ideal imbedding of Robertson-Walker space-times.** In applications of the embedding theorems one often starts from a given metric and looks for the imbedding space with the minimal dimension or one puts restrictions on the source type.

In [18], S. Haesen and L. Verstraelen provided a different approach by putting a restriction type of ideal imbedding (in the sense of [7]). They proved that the de Sitter space-times, a Robertson-Walker space-times and some anisotropic perfect fluid metrics can be ideally imbedded in some 5-dimensional pseudo-Euclidean space.

11.3. **Space-time-matter theory.** Another variant of Kaluza-Klien theory is the Space-Time-Matter theory (STM-theory for short) or induced matter theory, chiefly promulgated by Paul Wesson and other members of the so-called 5D Space-Time-Matter Consortium.

The Space-Time-Matter theory, introduced by P. Wesson and J. Ponce de Leon in 1992, tries to give a geometric explanation for the occurrence of matter in our Universe by referring to a higher dimension.

In this version of the theory, it is noted that solutions to the *5D Ricci-flat equation:*

$$R_{AB} = 0$$

may be re-expressed so that in four dimensions, these solutions satisfy Einstein's equation:

$$R_{ij} - \frac{1}{2}g_{ij}R = kT_{ij}$$

with the precise form of the T_{ij} following from the Ricci-flat condition on the 5D space. The energy-momentum tensor T_{ij} is normally understood to be due to concentrations of matter in 4D space.

The above result can be interpreted as saying that

"*4D matter is induced from geometry from a Ricci-flat 5D space*".

In the STM-theory, matter is introduced by an extra 5th dimension which is in some case timelike and in an other spacelike, depending on the 4D metric under consideration. However, it is inclined in physics to think that matter should be neither. Therefore, it is natural to seek another approach than STM-theory that if the matter is introduced by an extra 5th dimension, the matter would be neutral, neither space-like nor time-like.

11.4. **My own small attempt.** In [5], I proved that Robertson-Walker space-times can be realized as affine hypersurfaces in such way that the induced affine metric is exactly the Lorentzian metric of the space-times (see [5] for details).

This realization allows us to regard the curved Robertson-Walker space-times in a flat 5D affine space (instead of a 5D Ricci-flat space via STM-theory).

Via the realizations, matters, if they were induced from the extra dimension will be neutral, neither space-like or time-like (contrast to STM-theory). Moreover, the extra 5th dimension is always unobservable since the 5D affine space does not equip with any metric. This allows us to avoid the compactification of the extra 5th dimension into tiny circle as proposed by O. Klein. Also, the realization allows us to investigate geometry and physics of space-times using affine invariants.

A natural challenging open question is the following:

Problem 3. *How to relate physics qualities with the affine invariants on the space-time induced from the 5D flat affine space via the realization?*

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