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# ON TRIGONOMETRIC FUNCTIONS AND COSINE AND SINE RULES IN TAXICAB PLANE

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Abstract. In this study, we try to devolope cosine and sine functions in the taxicab plane by using the reference angle. Also, we give geometrical interpretations by using these functions. Then, analogues of the cosine and sine rules in the taxicab plane are studied.

## 1. Introduction

The taxicab plane is the study of the geometry consisting of Euclidean points, lines and angles in $\mathbb{R}^2$ with the taxicab metric $d_T$

$$d_T((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2| .$$

Taxicab plane trigonometry has been studied by some authors. Different definitions of cosine and sine functions in the taxicab plane are given [1], [2], [3],[8].

In this paper, firstly we try to determine the taxicab cosine function of an angle $\theta$ given with the reference angle $\alpha$, [8]. A taxicab sine function including the taxicab norm is defined and its geometrical interpretation is given. Furthermore, the taxicab sine and cosine indexes are defined and the connections among them are determined. Then, analogues of the cosine and sine rules in the taxicab plane are studied.

## 2. Taxicab Cosine Function Including Reference Angle

Let $\theta$ be the angle between the vectors $OA$ and $OB$ given with the reference angle $\alpha$ as defining in [8]. In [2], the taxicab cosine function of an angle $\theta$, tcos$\theta$, is defined by

$$\text{tcos}\theta = \frac{|OA|\,|OB|}{|OA|_T\,|OB|_T}\cos\theta, \quad 0 \leq \theta \leq \pi. \tag{2.1}$$

In this definition, if one of the vectors $OA$ or $OB$ is parallel to the $x-$axis, then

$$\text{tcos}\theta = \frac{\cos\theta}{|\cos\theta| + |\sin\theta|}. \tag{2.2}$$

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We try to improve t$\cos\theta$ for the situation that $OA$ or $OB$ are not parallel to the $x-$axis by using the reference angle $\alpha$ of the angle $\theta$ (see Figure 1).

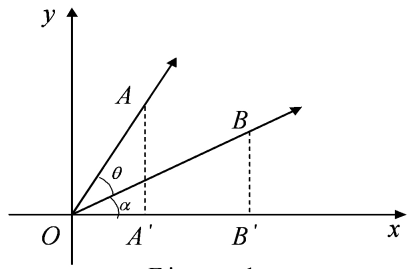


Figure 1

From the equation (2.1) and Figure 1,

$$\begin{aligned}\mathrm{t}\cos\theta &= \frac{|OA|\,|OB|}{|OA|_T\,|OB|_T}\cos\theta \\ &= \frac{\cos\theta}{\left(\dfrac{|OA'|+|AA'|}{|OA|}\right)\left(\dfrac{|OB'|+|BB'|}{|OB|}\right)} \\ &= \frac{\cos\theta}{\left(\dfrac{|OA'|}{|OA|}+\dfrac{|AA'|}{|OA|}\right)\left(\dfrac{|OB'|}{|OB|}+\dfrac{|BB'|}{|OB|}\right)},\end{aligned}$$

where $A'$and $B'$ are the ortogonal projection points of $A$ and $B$ on $x$-axis respectively. Therefore,

$$\mathrm{t}\cos\theta = \frac{\cos\theta}{(|\cos(\theta+\alpha)|+|\sin(\theta+\alpha)|)\,(|\cos\alpha|+|\sin\alpha|)} \tag{2.3}$$

is obtained. If $\alpha = 0$ in (2.3), then

$$\mathrm{t}\cos\theta = \frac{\cos\theta}{|\cos\theta|+|\sin\theta|}$$

and

$$\mathrm{t}\cos\theta = \frac{sgn(\cos\theta)}{1+\dfrac{|\sin\theta|}{|\cos\theta|}}, \quad \theta \neq \frac{\pi}{2}. \tag{2.4}$$

Also, using (2.4), we get $\cos\theta$ in terms of t$\cos\theta$, as following

$$\cos\theta = \frac{\mathrm{t}\cos\theta}{\sqrt{(\mathrm{t}\cos\theta)^2+(sgn(\mathrm{t}\cos\theta)-\mathrm{t}\cos\theta)^2}}. \tag{2.5}$$

**Definition 2.1.** Let $\theta$ be an angle with the reference angle $\alpha = 0$. Then

$$\theta_c = \sqrt{(\mathrm{t}\cos\theta)^2+(sgn(\mathrm{t}\cos\theta)-\mathrm{t}\cos\theta)^2} = \frac{\mathrm{t}\cos\theta}{\cos\theta} \tag{2.6}$$

is called **the taxicab cosine index** $\theta_c$ of $\theta$.

Geometrically, $\theta_c$ is equal to the ratio of the product Euclidean vector lengths to the product of the taxicab vector lengths. Also it is known that the ratio of Euclidean vector lengths is equal to the ratio of the taxicab vector lengths, [4].

## 3. Taxicab Sine Function Including Reference Angle

**Definition 3.1.** Let $\theta$ be the angle between any two vectors $OA, OB$. Then, the taxicab sine function, tsin$\theta$, is defined by

$$\text{tsin}\theta = \frac{|OA|\,|OB|}{|OA|_T\,|OB|_T}\sin\theta, \quad 0 \le \theta \le \pi. \tag{3.1}$$

From (3.1),

$$|OA|_T\,|OB|_T\,\text{tsin}\theta = |OA|\,|OB|\,\text{tsin}\theta = |OA \times OB|_T\,. \tag{3.2}$$

Hence, the cross product can be interpreted in the taxicab space as in the Euclidean space. Furthermore,

$$|OA \times OB|_T = |OA|_T\,|OB|_T\,\text{tsin}\theta$$

and

$$\text{tsin}\theta = \frac{|OA \times OB|_T}{|OA|_T\,|OB|_T}.$$

**Geometrical Interpretation.** It is well known that $|OA \times OB|$ is the area of the parallelogram with two sides $OA$ and $OB$ in the Euclidean plane. Similarly, the following equality

$$|OA \times OB|_T = |OA|_T\,|OB|_T\,\text{tsin}\theta$$

is interpreted as the area of the parallelogram with two sides $OA$ and $OB$.

Now, consider $\theta$ with the reference angle $\alpha$ as in Figure 1. Then, from (3.1)

$$\begin{aligned}
\text{tsin}\theta &= \frac{|OA|\,|OB|}{|OA|_T\,|OB|_T}\sin\theta \\
&= \frac{\sin\theta}{\left(\dfrac{|OA'|+|AA'|}{|OA|}\right)\left(\dfrac{|OB'|+|BB'|}{|OB|}\right)} \\
&= \frac{\sin\theta}{\left(\dfrac{|OA'|}{|OA|}+\dfrac{|AA'|}{|OA|}\right)\left(\dfrac{|OB'|}{|OB|}+\dfrac{|BB'|}{|OB|}\right)}
\end{aligned}$$

and so,

$$\text{tsin}\theta = \frac{\sin\theta}{(|\cos(\theta+\alpha)| + |\sin(\theta+\alpha)|)(|\cos\alpha| + |\sin\alpha|)} \tag{3.3}$$

is obtained. If $\alpha = 0$ in (3.3), then

$$\text{tsin}\theta = \frac{\sin\theta}{|\cos\theta| + |\sin\theta|}$$

and

$$\text{tsin}\theta = \frac{sgn(\sin\theta)}{1 + \dfrac{|\cos\theta|}{|\sin\theta|}}, \quad \theta \neq \pi. \tag{3.4}$$

Also, using (3.4), we get $\sin\theta$ in terms of tsin$\theta$ as follows

$$\sin\theta = \frac{\text{tsin}\theta}{\sqrt{(\text{tsin}\theta)^2 + (sgn(\text{tsin}\theta) - \text{tsin}\theta)^2}}. \tag{3.5}$$

**Definition 3.2.** Let $\theta$ be an angle with the reference angle $\alpha = 0$. Then

$$\theta_s = \sqrt{(\mathrm{tsin}\theta)^2 + (sgn(\mathrm{tsin}\theta) - \mathrm{tsin}\theta)^2} = \frac{\mathrm{tsin}\theta}{\sin\theta} \tag{3.6}$$

is called **the taxicab sine index** $\theta_s$ of $\theta$.

Geometrically, $\theta_s$ is equal to the ratio of the product of Euclidean vector lengths to the product of the taxicab vector lengths.

**Identities.** If any angle $\theta$ is given with the reference angle $\alpha$, the following relations can be obtained easily.

$$\begin{array}{ll} \textbf{i)} & \theta_c = \theta_s, \text{ that is } \dfrac{\mathrm{tcos}\theta}{\cos\theta} = \dfrac{\mathrm{tsin}\theta}{\sin\theta} \\ \textbf{ii)} & \dfrac{|\mathrm{tcos}\theta| + |\mathrm{tsin}\theta|}{|\cos\theta| + |\sin\theta|} = \alpha_c.(\theta + \alpha)_c. \end{array} \tag{3.7}$$

If it is taken $\alpha = 0$ in the last equality then

$$|\mathrm{tcos}\theta| + |\mathrm{tsin}\theta| = 1.$$

is obtained.

## 4. Taxicab Cosine Rule

Let $ABC$ be a triangle with side lengths $a_T = d_T(B, C)$, $b_T = d_T(A, C)$ and $c_T = d_T(A, B)$ in the taxicab plane. The following lemmas and theorems give a taxicab analogue of the cosine rule in the Euclidean plane in some special cases.

**Lemma 4.1.** *If one side of a triangle $ABC$, say $AB$, is parallel to one of the coordinate axes and none of the angles is an obtuse angle, then*

$$\begin{aligned} a_T &= b_T + c_T - 2b_T tcosA \\ b_T &= a_T + c_T - 2a_T tcosB \\ c_T &= \frac{a_T^2 + b_T^2 - 2a_T b_T tcosC}{a_T + b_T}. \end{aligned}$$

*Proof.* Consider any triangle $ABC$, where the side $AB$, is parallel to the $x-$ axis.

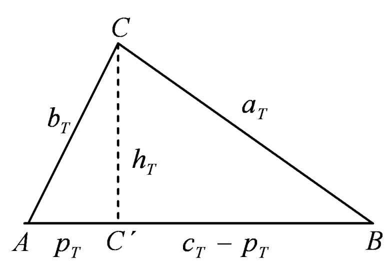


Figure 2

Let $h_T = d_T(C, AB)$ and $p_T = d_T(A, C')$, where $C'$ denotes the foot of the altitude from $C$ (Figure 2). Now we calculate the side lengths $a_T$, $b_T$ and $c_T$ of a triangle $ABC$ in terms of tcos$A$, tcos$B$ and tcos$C$ respectively, by using the triangles $AC'C$ and $C'BC$.

*i*) It is easily seen, from the triangles $AC'C$ and $C'BC$ that,

$$h_T = b_T - p_T \text{ and } h_T = a_T - (c_T - p_T). \tag{4.1}$$

Then

$$a_T = b_T + c_T - 2p_T \tag{4.2}$$

is obtained. For the angle $A$ of the triangle $AC'C$, from $tcos$A$=\dfrac{p_T}{b_T}$ one gets

$$p_T = b_T \text{tcos} A. \tag{4.3}$$

Using (4.3) in (4.2), one obtains

$$a_T = b_T + c_T - 2b_T \text{tcos} A. \tag{4.4}$$

*ii*) As in (*i*), one gets

$$b_T = a_T - c_T + 2p_T. \tag{4.5}$$

So, $tcos$B$=\dfrac{c_T - p_T}{a_T}$ and

$$p_T = c_T - a_T \text{tcos} B. \tag{4.6}$$

Using (4.6) in (4.5), one obtains

$$b_T = a_T + c_T - 2a_T \text{tcos} B. \tag{4.7}$$

(*iii*) Similarly (*i*), $c_T - p_T = \dfrac{a_T + c_T - b_T}{2}$, $p_T = \dfrac{b_T + c_T - a_T}{2}$ and

$$h_T = \frac{a_T + b_T - c_T}{2}. \tag{4.8}$$

So, $\text{tcos} C = \dfrac{a_T^2 + b_T^2 - (a_T + b_T)c_T}{2a_T b_T}$. Thus we find

$$c_T = \frac{a_T^2 + b_T^2 - 2a_T b_T \text{tcos} C}{a_T + b_T} \tag{4.9}$$

which completes the proof.

**Lemma 4.2.** *If one side of a triangle $ABC$, say $AB$, is parallel to one of the coordinate axes and the angle $A$ is not an acute angle, then*

$$\begin{aligned} a_T &= h_T + c_T - b_T tcosA \\ b_T &= h_T - c_T + a_T tcosB \\ c_T &= \frac{a_T b_T}{h_T}(1 - tcosC) \end{aligned}$$

*where $h_T = d_T(C, AB)$.*

*Proof.* The proof can be made easily as in Lemma 4.1.

**Corollary 4.1.** *Let the side $AB$ of a triangle $ABC$ be parallel to one of the coordinate axes in the taxicab plane. If the angle $A > \dfrac{\pi}{2}$ or $B > \dfrac{\pi}{2}$ then $a_T = b_T + c_T$ or $b_T = a_T + c_T$ respectively,* [5].

The following corollary gives the taxicab version of the Pythagorean Theorem for a triangle $ABC$ with one side parallel to a coordinate axis.

**Corollary 4.2.** *Let the side $AB$ of a right triangle $ABC$ be parallel to one of the coordinate axes in the taxicab plane. If $A = \frac{\pi}{2}$ or $B = \frac{\pi}{2}$ or $C = \frac{\pi}{2}$, then $a_T = b_T + c_T$ or $b_T = a_T + c_T$ or $c_T = \frac{a_T^2 + b_T^2}{a_T + b_T}$ respectively.*

**Theorem 4.1.** *Let $A$ be the vertex, with the smallest ordinate, of any triangle $ABC$. If $\alpha$ is the reference angle of $A$ then,*

$$\begin{aligned}
a_T &= \frac{k_A}{k_B} b_T + \frac{\alpha_c^2}{k_B} c_T - 2\frac{k_A}{k_B} b_T tcosA \\
b_T &= \frac{k_B}{k_A} a_T + \frac{\alpha_c^2}{k_A} c_T - 2\frac{k_B}{k_A} a_T tcosB \\
c_T &= \frac{k_B^2 a_T^2 + k_A^2 b_T^2 - 2k_A k_B a_T b_T tcosC}{\alpha_c^2 (k_B a_T + k_A b_T)},
\end{aligned}$$

*where $\alpha_c$ is the cosine index of $\alpha$ and $k_\theta = |tcos\theta| + |tsin\theta|$, $\theta = A, B$.*

*Proof.* Without lost the generality, we can take the vertex $A$ at the origin, since taxicab lengths are invariant under translations, [7].

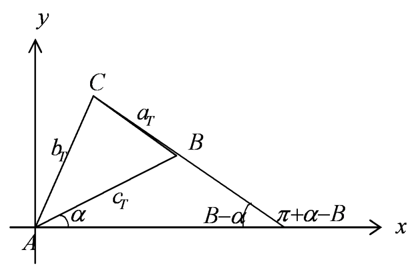


Figure 3

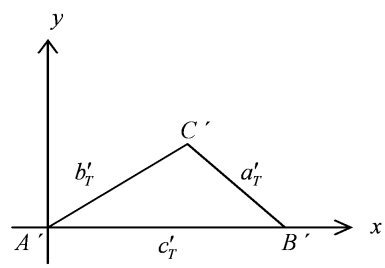


Figure 4

Consider the triangle $ABC$ in Figure 3. If one rotates $ABC$ with angle (-$\alpha$), then the triangle $A'B'C'$ is obtained as in Figure 4. Its position is as in Lemma 4.1. Now we calculate the side lengths $a'_T$, $b'_T$, $c'_T$ after rotation with angle (-$\alpha$). The reference angle of the angle $B$ is $\pi+\alpha-B$. From [6] there is the following relationship between $a_T$ and $a'_T$ ,

$$\frac{a_T}{|\cos(\pi - B + \alpha)| + |\sin(\pi - B + \alpha)|} = \frac{a'_T}{|\cos(\pi - B)| + |\sin(\pi - B)|}$$

and so

$$\frac{a_T}{(|\cos(B - \alpha)| + |\sin(B - \alpha)|)} = \frac{a'_T}{|\cos B| + |\sin B|} \tag{4.10}$$

is obtained.

Multiplying both the numerator and the denominator of the left side of (4.10) with $\cos(B-\alpha)$, and using the equality $\mathrm{tcos}(B - \alpha) = \dfrac{\cos(B-\alpha)}{|\cos(B - \alpha)| + |\sin(B - \alpha)|}$ one gets

$$\frac{a_T \mathrm{tcos}(B - \alpha)}{\cos(B - \alpha)} = \frac{a'_T}{|\cos B| + |\sin B|}. \tag{4.11}$$

Using (2.4) in (4.11),

$$a_T \sqrt{(\mathrm{tcos}(B - \alpha))^2 + (sgn(\mathrm{tcos}(B - \alpha)) - \mathrm{tcos}(B - \alpha))^2} = \frac{a'_T}{|\cos B| + |\sin B|}. \tag{4.12}$$

Using (3.7) in (4.12),

$$a'_T = a_T \frac{|\mathrm{tcos}B| + |\mathrm{tcos}B|}{\alpha_c} = \frac{k_B a_T}{\alpha_c} \tag{4.13}$$

is obtained. For angles $A$ and $C$ the reference angles are $\alpha$ and $\pi+A+\alpha$ respectively. In the similar way, after rotating with angle (-$\alpha$), the relationships between $b_T$ and $b'_T$, and $c_T$ and $c'_T$ are obtained as the following

$$b'_T = b_T \frac{|\mathrm{tcos}A| + |\mathrm{tsin}A|}{\alpha_c} = \frac{k_a b_T}{\alpha_c} \text{ and } c'_T = c_T \frac{\mathrm{tcos}\alpha}{\cos\alpha} = c_T \alpha_c.$$

Using $a'_T$, $b'_T$ and $c'_T$ instead of $a_T$, $b_T$ and $c_T$ in Lemma 4.1 the proof is completed.

The following corollary gives the taxicab analogue of the Pythagorean Theorem for any triangle $ABC$.

**Corollary 4.3.** *Let $A$ be the vertex, with the smallest ordinate, of any triangle $ABC$ and $\alpha$ be the reference angle of angle $A$. If $A = \frac{\pi}{2}$ or $B = \frac{\pi}{2}$ or $C = \frac{\pi}{2}$, then $a_T = \frac{1}{k_B} b_T + \frac{\alpha_c^2}{k_B} c_T$ or $b_T = \frac{1}{k_A} a_T + \frac{\alpha_c^2}{k_A} c_T$ or $c_T = \frac{k_B^2 a_T^2 + k_A^2 b_T^2}{\alpha_c^2 (k_B a_T + k_A b_T)}$ respectively.*

*If $\alpha = 0$, then one gets Corollary 4.2.*

**Theorem 4.2.** *Let $A$ be the vertex, with the smallest ordinate, of any triangle $ABC$ and $\alpha$ be the reference angle of angle $A$. If $A > \frac{\pi}{2}$ then*

$$\begin{aligned} a_T &= \frac{\alpha_c}{k_A} h_T + \frac{\alpha_c^2}{k_B} c_T - \frac{k_A}{k_B} b_T tcosA \\ b_T &= \frac{\alpha_c}{k_A} h_T - \frac{\alpha_c^2}{k_B} c_T - \frac{k_B}{k_A} a_T tcosB \\ c_T &= \frac{a_T b_T k_A k_B}{\alpha_c^3 h_T} (1 - tcosC), \end{aligned}$$

*where $h_T = \frac{b_T k_A}{\alpha_C}(1+tcosA)$.*

*Proof.* The proof can be shown easily, by using $a'_T$, $b'_T$ and $c'_T$ instead of $a_T$, $b_T$ and $c_T$ in Lemma 4.2.

## 5. Taxicab Sine Rule

Let $ABC$ be a triangle with side lengths $a_T = d_T(B,C)$, $b_T = d_T(A,C)$ and $c_T = d_T(A,B)$ in the taxicab plane. The following theorem gives a taxicab analogue of the sine rule in Euclidean plane.

**Theorem 5.1.** *Let $ABC$ be a triangle with side lengths $a_T = d_T(B,C)$, $b_T = d_T(A,C)$ and $c_T = d_T(A,B)$ in the taxicab plane. Then the equality*

$$\frac{a_T}{tsinA} = \frac{b_T}{tsinB} = \frac{c_T}{tsinC}$$

*is valid.*

*Proof.* The area of the triangle $ABC$ is equal to half of the parallelogram area determined by any two sides of the triangle $ABC$.

$$\text{The area of } ABC = \frac{|AB \times AC|_T}{2} = \frac{|AB \times BC|_T}{2} = \frac{|AC \times BC|_T}{2} \tag{5.1}$$

and

$$\begin{aligned} |AB \times AC|_T &= |AB|_T \,.\, |AC|_T \,.\mathrm{tsin}A \\ |AB \times BC|_T &= |AB|_T \,.\, |BC|_T \,.\mathrm{tsin}B \\ |AB \times AC|_T &= |AC|_T \,.\, |BC|_T \,.\mathrm{tsin}C \end{aligned}$$

are valid. From these equalities and (5.1),

$$\frac{a_T}{\mathrm{tsin}A} = \frac{b_T}{\mathrm{tsin}B} = \frac{c_T}{\mathrm{tsin}C}$$

is obtained.

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