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g-NATURAL METRICS OF CONSTANT SECTIONAL CURVATURE ON TANGENT BUNDLES

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Abstract. Let (M, g) be a Riemannian manifold and G a g-natural metric on its tangent bundle TM. In this paper we prove first that the space (TM, G) has constant sectional curvature if and only if it is flat Riemannian, and then we give, for $\dim M \geq 3$, a characterization of flat Riemannian g-natural metrics on tangent bundles.

Introduction

In [1], K.M.T. Abbassi and M. Sarih introduced the notion of g-natural metrics on the tangent bundle TM of a Riemannian manifold (M, g). A metric G on TM is called a g-natural metric if it comes from g by a first order natural operator $S^2_+T^* \rightsquigarrow (S^2T^*)T$, where $S^2_+T^*$ and $(S^2T^*)T$ denote respectively the natural bundle of Riemannian metrics and the natural bundle of $(0, 2)$-tensor fields on the tangent bundles (cf. [6] for the definitions of natural bundles and operators and associated notions). They gave a characterization of g-natural metrics on TM in terms of functions defined on $\mathbb{R}^+$, and obtained a necessary and sufficient conditions for g-natural metrics to be either nondegenerate or Riemannian. But they did not give an explicit expression for the inverse of nondegenerate g-natural metrics although it is important to compute some geometrical analysis tools like the Ricci tensor, the scalar curvature, the Laplace operator, etc ... .

Some geometrical properties could be inherited on the g-natural metrics from the basic metric g and conversely. In [2] the authors proved that if a tangent bundle equipped with a g-natural metric (TM, G) is of constant sectional curvature then the same holds for (M, g). Furthermore, making some restrictions on the Riemannian g-natural metrics on TM, the same authors gave the characterization of flat Riemannian g-natural metrics on TM (cf. [3]).

In this paper we prove that if (M, g) is non flat, its tangent bundle TM equipped with a g-natural metric G has non constant sectional curvature, and also that only flat g-natural metrics are of constant sectional curvature. In the next section 1 we give some preliminaries and some known results on g-natural metrics. In the section 2 we compute explicitly the inverse of any nondegenerate g-natural metric.

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In section 3 using this inverse expression and Koszul's formula, we determine the Levi-Civita connection of any nondegenerate g-natural metric. Finally in section 4, we show that the flat Riemannian g-natural metrics are the only g-natural metrics that have a constant sectional curvature, then we give a characterization of these metrics.

1. **Preliminaries**

Let (M, g) be a Riemannian manifold and ∇ the Levi-Civita connection of g. Then the tangent space of TM at any point $(x, u) \in TM$ splits into the horizontal and vertical subspaces with respect to ∇ :

$$T_{(x,u)}TM = H_{(x,u)}M \oplus V_{(x,u)}M.$$

If $(x, u) \in TM$ is given then, for any vector $X \in T_xM$, there exists a unique vector $X^h \in H_{(x,u)}M$ such that $\pi_* X^h = X$, where $\pi : TM \to M$ is the natural projection. X^h denotes the *horizontal lift* of X at the point $(x, u) \in TM$. The *vertical lift* of a vector $X \in T_xM$ at $(x, u) \in TM$ is a vector $X^v \in V_{(x,u)}M$ such that $X^v.(df) = X.f$, for all functions f on M. Here we consider 1-forms df on M as functions on TM (i.e. $(df)(x, u) = u.f$). Note that the map $X \to X^h$ is an isomorphism between the vector spaces T_xM and $H_{(x,u)}M$. Similarly, the map $X \to X^v$ is an isomorphism between the vector spaces T_xM and $V_{(x,u)}M$. Obviously, each tangent vector $\tilde{Z} \in T_{(x,u)}TM$ can be written in the form
$\tilde{Z} = X^h + Y^v$, where $X, Y \in T_xM$ are uniquely determined vectors.

If φ is a smooth function on M, then

$$X^h(\varphi \circ \pi) = (X\varphi) \circ \pi \text{ and } X^v(\varphi \circ \pi) = 0 \tag{1.1}$$

hold for every vector field X on M.

A system of local coordinates $(U\,; x_i,\ i = 1, \cdots, m)$ in M induces on TM a system of local coordinates $\big(\pi^{-1}(U)\,; x_i, u^i,\ i = 1, \cdots, m\big)$.
Let $X = \sum_{i=1}^m X^i \frac{\partial}{\partial x_i}$ be the local expression in U of a vector field X on M. Then, the horizontal lift X^h and the vertical lift X^v of X are given, with respect to the induced coordinates, by :

$$X^h = \sum_i X^i \frac{\partial}{\partial x_i} - \sum_{i,j,k} \Gamma^i_{jk} u^j X^k \frac{\partial}{\partial u^i} \quad \text{and} \tag{1.2}$$

$$X^v = \sum_i X^i \frac{\partial}{\partial u^i}, \tag{1.3}$$

where the (Γ^i_{jk}) are the Christoffel's symbols of g.

Next, we introduce some notations which will be used to describe vectors obtained from lifted vectors by basic operations on TM. Let T be a tensor field of type $(1, s)$ on M. If $X_1, X_2, \cdots, X_{s-1} \in T_xM$, then $h\{T(X_1, \cdots, u, \cdots, X_{s-1})\}$ (respectively $v\{T(X_1, \cdots, u, \cdots, X_{s-1})\}$) is a horizontal (respectively vertical) vector at (x, u) which is defined by the formula

$$h\{T(X_1, \cdots, u, \cdots, X_{s-1})\} = \sum u^\lambda \left(T(X_1, \cdots, \left(\frac{\partial}{\partial x_\lambda}\right)_x, \cdots, X_{s-1}) \right)^h$$

(resp. $v\{T(X_1, \cdots, u, \cdots, X_{s-1})\} = \sum u^\lambda \left(T(X_1, \cdots, \left(\frac{\partial}{\partial x_\lambda}\right)_x, \cdots, X_{s-1}) \right)^v$).
In particular, if T is the identity tensor of type $(1, 1)$, then we obtain the geodesic

flow vector field at (x,u), $\xi_{(x,u)} = \sum_\lambda u^\lambda \left(\frac{\partial}{\partial x_\lambda}\right)^h_{(x,u)}$, and the canonical vertical vector at (x,u), $\mathcal{U}_{(x,u)} = \sum_\lambda u^\lambda \left(\frac{\partial}{\partial x_\lambda}\right)^v_{(x,u)}$.

Moreover $h\{T(X_1,\cdots,u,\cdots,u,\cdots,X_{s-t})\}$ and $v\{T(X_1,\cdots,u,\cdots,u,\cdots,X_{s-t})\}$ are defined by similar way.

Also let us make the notations

$$h\{T(X_1,\cdots,X_s)\} =: T(X_1,\cdots,X_s)^h \tag{1.4}$$

and

$$v\{T(X_1,\cdots,X_s)\} =: T(X_1,\cdots,X_s)^v. \tag{1.5}$$

Thus $h\{X\} = X^h$ and $v\{X\} = X^v$, for each vector X tangent to M.

From the preceding quantities, one can define vector fields on TU in the following way: If $u = \sum_i u^i \left(\frac{\partial}{\partial x_i}\right)_x$ is a given point in TU and $X_1,\cdots,X_{s-1}$ are vector fields on U, then we denote by

$$h\{T(X_1,\cdots,u,\cdots,X_{s-1})\} \quad \text{(respectively} \quad v\{T(X_1,\cdots,u,\cdots,X_{s-1})\})$$

the horizontal (respectively vertical) vector field on TU defined by

$$h\{T(X_1,\cdots,u,\cdots,X_{s-1})\} = \sum_\lambda u^\lambda T(X_1,\cdots,\frac{\partial}{\partial x_\lambda},\cdots,X_{s-1})^h$$

$$(\text{ resp.} \quad v\{T(X_1,\cdots,u,\cdots,X_{s-1})\} = \sum_\lambda u^\lambda T(X_1,\cdots,\frac{\partial}{\partial x_\lambda},\cdots,X_{s-1})^v\,).$$

Moreover, for vector fields $X_1,\cdots,X_{s-t}$ on U, where $s,\, t \in \mathbb{N}^*$ $(s > t)$, the vector fields $h\{T(X_1,\cdots,u,\cdots,u,\cdots,X_{s-t})\}$ and $v\{T(X_1,\cdots,u,\cdots,u,\cdots,X_{s-t})\}$, on TU, are defined by similar way.

The Riemannian curvature of g is defined by

$$R(X,Y) = [\nabla_X\,,\,\nabla_Y] - \nabla_{[X,Y]}. \tag{1.6}$$

Now, for $(r,s) \in \mathbb{N}^2$, we denote by $\pi_M : TM \to M$ the natural projection and F the natural bundle defined by

$$\begin{aligned} FM &= \pi_M^*(\underbrace{T^*\otimes\cdots\otimes T^*}_{r \text{ times}} \otimes \underbrace{T\otimes\cdots\otimes T}_{s \text{ times}})M \to M, \\ Ff(X_x,S_x) &= (Tf.X_x, (T^*\otimes\cdots\otimes T^*\otimes T\otimes\cdots\otimes T)f.S_x) \end{aligned} \tag{1.7}$$

for all manifolds M, local diffeomorphisms f of M, $X_x \in T_xM$ and $S_x \in (T^*\otimes\cdots\otimes T^*\otimes T\otimes\cdots\otimes T)_xM$. We call the sections of the canonical projection $FM \to M$ F-tensor fields of type (r,s). So, if $\oplus$ denotes the fibered product of fibered manifolds, then the F-tensor fields are mappings $A : TM \oplus \underbrace{TM\oplus\cdots\oplus TM}_{s \text{ times}} \to \sqcup_{x\in M} \otimes^r T_xM$ which are linear in the last s summands and such that $\pi_2 \circ A = \pi_1$, where π_1 and π_2 are respectively the natural projections of the source and target fiber bundles of A. For $r = 0$ and $s = 2$, we obtain the classical notion of F-metrics. So, F-metrics are mappings $TM \oplus TM \oplus TM \to \mathbb{R}$ which are linear in the second and the third arguments.

Proposition 1.1. [1] *Let (M, g) be a Riemannian manifold and G a g-natural metric on TM. Then if $\dim M \geq 2$, there exists six functions $\alpha_i,\ \beta_i : \mathbb{R}^+ \to \mathbb{R},\ i = 1, 2, 3,$ such that for any $x \in M$ and all vectors $u,\ X,\ Y \in T_xM$, we have*

$$\begin{cases} G_{(x,u)}\left(X^h, Y^h\right) &= (\alpha_1 + \alpha_3)(t) g_x(X, Y) \\ & \quad + (\beta_1 + \beta_3)(t) g_x(X, u) g_x(Y, u), \\ G_{(x,u)}\left(X^h, Y^v\right) &= \alpha_2(t) g_x(X, Y) + \beta_2(t) g_x(X, u) g_x(Y, u), \\ G_{(x,u)}\left(X^v, Y^h\right) &= \alpha_2(t) g_x(X, Y) + \beta_2(t) g_x(X, u) g_x(Y, u), \\ G_{(x,u)}\left(X^v, Y^v\right) &= \alpha_1(t) g_x(X, Y) + \beta_1(t) g_x(X, u) g_x(Y, u), \end{cases}$$

where $t = g_x(u, u)$, X^h and X^v are respectively the horizontal lift and the vertical lift of the vector $X \in T_xM$ at the point $(x, u) \in TM$.

For $\dim M = 1$, the same holds with $\beta_i = 0, \quad i = 1, 2, 3.$

Notation 1.1.

- $\phi_i(t) = \alpha_i(t) + t\beta_i(t)$,
- $\alpha(t) = \alpha_1(t)(\alpha_1 + \alpha_3)(t) - \alpha_2^2(t)$,
- $\phi(t) = \phi_1(t)(\phi_1 + \phi_3)(t) - \phi_2^2(t)$,

for all $t \in \mathbb{R}^+$.

Proposition 1.2. [1] *A g-natural metric G on the tangent bundle of a Riemannian manifold $(M,\ g)$ is :*

(i) *nondegenerate if and only if the functions $\alpha_i,\ \beta_i,\ i = 1, 2, 3,$ of Proposition* 1.1 *defining G, satisfy*

$$\alpha(t)\phi(t) \neq 0 \tag{1.8}$$

for all $t \in \mathbb{R}^+$.

(ii) *Riemannian if and only if the functions $\alpha_i,\ \beta_i,\ i = 1, 2, 3,$ of Proposition* 1.1 *defining G, satisfy the inequalities*

$$\begin{cases} \alpha_1(t) > 0, & \phi_1(t) > 0, \\ \alpha(t) > 0, & \phi(t) > 0, \end{cases} \tag{1.9}$$

for all $t \in \mathbb{R}^+$.

For $\dim M = 1$, this system reduces to $\alpha_1(t) > 0$ and $\alpha(t) > 0$, for all $t \in \mathbb{R}^+$.

The following lemmas will be useful in the sequel.

Lemma 1.1. [5] *Let (M, g) be a Riemannian manifold, ∇ be the Levi-Civita connection and R be the Riemannian curvature of g. Then the Lie bracket on the tangent bundle TM of M satisfies*

(1) $\left[X^h, Y^h\right] = [X, Y]^h - v\left\{R(X, Y)u\right\}$,
(2) $\left[X^h, Y^v\right] = \left(\nabla_X Y\right)^v$,
(3) $\left[X^v, Y^v\right] = 0$,

for all X , Y , $Z \in \mathfrak{X}(M)$.

Lemma 1.2. [2] *Let (M,g) be a Riemannian manifold, $(x,u)\in TM$ and $X,Y,Z\in \mathfrak{X}(M)$, f a function defined from $\mathbb{R}$ to $\mathbb{R}$, and denote by F_Y the function on TM defined by $F_Y(u)=g_x(Y_x,u)$, for all $(x,u)\in TM$. Then we have:*

(1) $X^h_{(x,u)}.f(|u|^2)=0$,
(2) $X^v_{(x,u)}.f(|u|^2)=2f'(|u|^2)g_x(X_x,u)$,
(3) $X^h_{(x,u)}.F_Y=g_x((\nabla_X Y)_x,u)=F_{\nabla_X Y}(x,u)$,
(4) $X^h_{(x,u)}.(g(Y,Z)\circ\pi)=X_x.(g(Y,Z))$,
(5) $X^v_{(x,u)}.(g(Y,Z)\circ\pi)=0$,
(6) $X^v_{(x,u)}.F_Y=g_x(X,Y)$,
where $|u|^2=g_x(u,u)$.

From now on, whenever we consider an arbitrary Riemannian g-natural metric G on TM, we implicitly assume that it is defined by the functions $\alpha_i,\beta_i:\mathbb{R}^+\longrightarrow\mathbb{R},\ i=1,2,3$, given in Proposition 1.1 .

All real functions $\alpha_i,\beta_i,\phi_i,\alpha,$ and ϕ and their derivatives are evaluated at $t:=g_x(u,u)$, $u\in T_xM$, unless otherwise stated.

2. Inverse of nondegenerate g-natural metrics

Let $(a,b)\in\mathbb{R}^2,\ m\in\mathbb{N}^*,\ u=(u^1,\ \cdots\ ,u^m)\in\mathbb{R}^m$ and denote by $\mu(a,b,u)$ the following square matrix of order $m\in\mathbb{N}^*$:

$$\mu(a,b,u)=\begin{pmatrix} a+b(u^1)^2 & & bu^iu^j \\ & \ddots & \\ bu^iu^j & & a+b(u^m)^2 \end{pmatrix}, \tag{2.1}$$

that is, $[\mu(a,b,u)]_{ij}=a\delta_{ij}+bu^iu^j$.

We establish the following lemma which is easy to check by straightforward computation:

Lemma 2.1. *If* $a(a+b|u|^2)\neq 0$, *then $\mu(a,b,u)$ is invertible and its inverse $\mu(a,b,u)^{-1}$ is given by*

$$\mu(a,b,u)^{-1}_{ij} \quad = \quad \frac{\delta_{ij}}{a}-\frac{b}{a(a+b|u|^2)}u^iu^j, \tag{2.2}$$

where $\mu(a,b,u)^{-1}_{ij}$ is the element of i^{th} line and of j^{th} column of the matrix $\mu(a,b,u)^{-1}$ and $|u|^2=\sum_{i=1}^m(u^i)^2$.

Next, we are going to determine the inverse of a nondegenerate g-natural metric G .

Let $(U,x_i,i=1,\cdots,m)$ be a normal coordinates system of (M,g) centred at $p\in M$, and $(\pi^{-1}(U);x_i,u^i,i=1,\cdots,m)$ its induced coordinates system on TM. For $l=1,2,3$, let us consider the matrix-valued functions on $\pi^{-1}(U)$ defined by

$$M_l=(\alpha_l g_{ij}+\beta_l u_iu_j)\,, \tag{2.3}$$

where g_{ij} and u_i are the functions on $\pi^{-1}(U)$ given by $g_{ij}=g\circ\pi(\partial_{x_i},\partial_{x_j})$, $u_i=g_{ik}u^k$ and $\partial_{x_i}=\frac{\partial}{\partial_{x_i}}$; $i,j=1,\cdots,m$.

So $\begin{pmatrix} (M_1+M_3) & M_2 \\ M_2 & M_1 \end{pmatrix}$ is the matrix-valued functions of $G_{|_{\pi^{-1}(U)}}$

with respect to the local frame $(\partial^h_{x_i}, \partial^v_{x_i})_{i=1,\cdots,m}$ on $\pi^{-1}(U)$. We shall denote

$$G \equiv \begin{pmatrix} (M_1+M_3) & M_2 \\ M_2 & M_1 \end{pmatrix}. \tag{2.4}$$

If G is nondegenerate, its inverse G^{-1} has the form

$$G^{-1} \equiv \begin{pmatrix} \Lambda & \Theta \\ \Theta & \Omega \end{pmatrix}, \tag{2.5}$$

where $\Lambda = (\lambda^{ij})_{1\le i,j\le m}$, $\Theta = (\theta^{ij})_{1\le i,j\le m}$, and $\Omega = (\omega^{ij})_{1\le i,j\le m}$ are square matrix-valued functions of order m, defined on $\pi^{-1}(U)$.

Therefore we have the following proposition:

Proposition 2.1. *If*

$$\begin{cases} \alpha(t)\phi(t) \neq 0, \\ \alpha_1(t)(\alpha_1+\alpha_3)(t) \neq 0, \\ \phi_1(t)(\phi_1+\phi_3)(t) \neq 0, \end{cases} \tag{2.6}$$

for any $t \in \mathbb{R}^+$, then the blocks of the matrix-valued functions in (2.5) satisfy :

$$\Lambda(p,u) \equiv \left(\lambda^{ij}(p,u)\right)_{1\le i\,,\, j\le m} \text{ with} \tag{2.7}$$

$$\lambda^{ij}(p,u) = \frac{\alpha_1(|u|^2)}{\alpha(|u|^2)}\delta_{ij} - \psi_\lambda(|u|^2)u^i u^j, \tag{2.8}$$

$$\Theta(p,u) \equiv \left(\theta^{ij}(p,u)\right)_{1\le i\,,\, j\le m} \text{ with} \tag{2.9}$$

$$\theta^{ij}(p,u) = -\frac{\alpha_2(|u|^2)}{\alpha(|u|^2)}\delta_{ij} - \psi_\theta(|u|^2)u^i u^j, \tag{2.10}$$

$$\Omega(p,u) \equiv \left(\omega^{ij}(p,u)\right)_{1\le i\,,\, j\le m} \text{ with} \tag{2.11}$$

$$\omega^{ij}(p,u) = \frac{(\alpha_1+\alpha_3)(|u|^2)}{\alpha(|u|^2)}\delta_{ij} - \psi_\omega(|u|^2)u^i u^j, \tag{2.12}$$

for all $u = \sum_{i=1}^m u^i \partial_{x_i\,|_p} \in T_pM$, where

$$\begin{aligned} \psi_\lambda &= \frac{\alpha_1[(\beta_1+\beta_3)\phi_1 - \beta_2\phi_2] - \alpha_2(\alpha_1\beta_2 - \alpha_2\beta_1)}{\alpha\phi}, \\ \psi_\theta &= \frac{-\alpha_2[(\beta_1+\beta_3)\phi_1 - \beta_2\phi_2] + (\alpha_1+\alpha_3)(\alpha_1\beta_2 - \alpha_2\beta_1)}{\alpha\phi}, \\ \psi_\omega &= \frac{(\alpha_1+\alpha_3)[\beta_1(\phi_1+\phi_3) - \beta_2\phi_2] + \alpha_2[\alpha_2(\beta_1+\beta_3) - \beta_2(\alpha_1+\alpha_3)]}{\alpha\phi}. \end{aligned} \tag{2.13}$$

Proof. According to (2.4) and (2.5), the product of the matrix-valued functions G and G^{-1} block per block gives:

$$\begin{pmatrix} (M_1+M_3)\Lambda + M_2\Theta & (M_1+M_3)\Theta + M_2\Omega \\ M_2\Lambda + M_1\Theta & M_2\Theta + M_1\Omega \end{pmatrix}, \tag{2.14}$$

and so we have the identities:

$$(M_1+M_3)\Lambda + M_2\Theta = Id, \tag{2.15}$$

$$(M_1+M_3)\Theta + M_2\Omega = 0, \tag{2.16}$$

$$M_2\Lambda + M_1\Theta = 0, \tag{2.17}$$

$$M_2\Theta + M_1\Omega = Id. \tag{2.18}$$

Furthermore, for any $u \in T_pM$, since $(U; x_i, i = 1, \cdots, m)$ is a normal coordinates system centred at p, we have

$$\begin{aligned} (M_1+M_3)(p,u) &= \mu((\alpha_1+\alpha_3)(|u|^2)\,, (\beta_1+\beta_3)(|u|^2)\,, u), \\ M_2(p,u) &= \mu(\alpha_2(|u|^2)\,, \beta_2(|u|^2), u), \\ M_1(p,u) &= \mu(\alpha_1(|u|^2)\,, \beta_1(|u|^2), u), \end{aligned}$$

where $u \equiv (u^i)_{i=1,\cdots,m}$. Then according to the system (2.6) and Lemma 2.1, the matrix-valued functions M_1 and $(M_1 + M_3)$ at (p, u) are invertible. It follows that at (p, u), the identities (2.17) and (2.16) give respectively

$$\Theta = -M_1^{-1}M_2\Lambda \tag{2.19}$$

and

$$\Theta = -(M_1+M_3)^{-1}M_2\Omega. \tag{2.20}$$

Combining the identities (2.19) and (2.15), we obtain at (p, u)

$$(M_1 + M_3 - M_2M_1^{-1}M_2)\Lambda = Id.$$

So $\Lambda(p, u)$ is invertible with

$$\Lambda(p,u) = (M_1 + M_3 - M_2M_1^{-1}M_2)^{-1}_{|_{(p,u)}}. \tag{2.21}$$

Next we compute the elements of the matrix-valued function $(M_1+M_3-M_2M_1^{-1}M_2)$ at (p, u), and we obtain

$$[(M_1 + M_3 - M_2M_1^{-1}M_2)(p,u)]_{ij} = \lambda_1(|u|^2)\delta_{ij} + \lambda_2(|u|^2)u^iu^j, \tag{2.22}$$

where

$$\begin{aligned} \lambda_1 &= \frac{\alpha}{\alpha_1} \quad \text{and} \\ \lambda_2 &= \frac{\phi_1[\alpha_1(\beta_1+\beta_3) - \alpha_2\beta_2 - \phi_2\beta_2] + \beta_1\phi_2^2}{\alpha_1\phi_1}, \end{aligned} \tag{2.23}$$

with

$$\lambda_1 \neq 0 \text{ and } \lambda_1 + t\lambda_2 = \frac{\phi}{\phi_1} \neq 0\ , \text{ everywhere.} \tag{2.24}$$

So by Lemma 2.1, we obtain the inverse $\Lambda = (\lambda^{ij})_{1\leq i,j\leq m}$ of $[(M_1+M_3)-M_2M_1^{-1}M_2]$ at (p,u), with

$$\begin{aligned}(2.25)\qquad \lambda^{ij}(p,u) &= \frac{\delta_{ij}}{\lambda_1(|u|^2)} - \frac{\lambda_2(|u|^2)}{\lambda_1(|u|^2)[\lambda_1(|u|^2)+|u|^2\lambda_2(|u|^2)]}u^iu^j \\ &= \frac{\alpha_1(|u|^2)}{\alpha(|u|^2)}\delta_{ij} - \psi_\lambda(|u|^2)u^iu^j.\end{aligned}$$

Next, according to (2.19), we compute

$$(2.26)\qquad \theta^{ij}(p,u) = -[M_1^{-1}M_2\Lambda]_{ij_{|(p,u)}},$$

and we obtain (2.9).

Furthermore by combining (2.20) and (2.18) we obtain at (p,u)

$$(2.27)\qquad [-M_2(M_1+M_3)^{-1}M_2+M_1]\Omega = Id.$$

This shows that the matrix-valued function $\left[-M_2(M_1+M_3)^{-1}M_2+M_1\right]$ is invertible at (p,u), and

$$(2.28)\qquad \Omega = [M_1 - M_2(M_1+M_3)^{-1}M_2]^{-1} \text{ at } (p,u).$$

Finally, as in the proof of (2.25), we obtain

$$(2.29)\quad [M_1-M_2(M_1+M_3)^{-1}M_2]_{ij_{|(p,u)}} = \omega_1(|u|^2)\delta_{ij} + \omega_2(|u|^2)u^iu^j,$$

where $\omega_1 = \frac{\alpha}{\alpha_1+\alpha_3} \neq 0$ and $\omega_2 = \frac{(\phi_1+\phi_3)[\beta_1(\alpha_1+\alpha_3)-\alpha_2\beta_2-\beta_2\phi_2]+\phi_2^2(\beta_1+\beta_3)}{(\alpha_1+\alpha_3)(\phi_1+\phi_3)}$,

$$(2.30)\qquad \text{with } \omega_1 + t\omega_2 = \frac{\phi}{\phi_1+\phi_3} \neq 0, \text{ everywhere.}$$

So by using again Lemma 2.1, we prove (2.11).

□

Besides we have the following lemma:

Lemma 2.2. *If $\alpha(t)\phi(t) \neq 0, \forall t \in \mathbb{R}^+$, then the functions ψ_λ, ψ_θ, ψ_ω defined respectively in* (2.7)*,* (2.9) *and* (2.11) *satisfy on $\mathbb{R}^+$ the following identities:*

$$\begin{aligned}(2.31)\qquad \phi_2\psi_\lambda + \phi_1\psi_\theta &= \frac{\alpha_1\beta_2-\alpha_2\beta_1}{\alpha},\\ (2.32)\qquad (\phi_1+\phi_3)\psi_\lambda + \phi_2\psi_\theta &= \frac{\alpha_1(\beta_1+\beta_3)-\alpha_2\beta_2}{\alpha},\\ (2.33)\qquad \phi_2\psi_\theta + \phi_1\psi_\omega &= \frac{(\alpha_1+\alpha_3)\beta_1-\alpha_2\beta_2}{\alpha},\\ (2.34)\qquad (\phi_1+\phi_3)\psi_\theta + \phi_2\psi_\omega &= \frac{(\alpha_1+\alpha_3)\beta_2-\alpha_2(\beta_1+\beta_3)}{\alpha}.\end{aligned}$$

The proof of the identities of Lemma 2.2 is not very difficult and can be obtained by straightforward computations .

Proposition 2.2. *If G is nondegenerate, the elements of the matrix-valued functions in* (2.5) *are given on $\pi^{-1}(U)$ by*

$$(2.35)\qquad \lambda^{ij} = \frac{\alpha_1}{\alpha}g^{ij} - \psi_\lambda u^iu^j,$$

$$(2.36)\qquad \theta^{ij} = -\frac{\alpha_2}{\alpha}g^{ij} - \psi_\theta u^iu^j,$$

$$\omega^{ij} = \frac{\alpha_1+\alpha_3}{\alpha} g^{ij} - \psi_\omega u^i u^j, \tag{2.37}$$

where $\left(g^{ij}\right)_{1\le i,j\le m}$ *denotes the inverse of* $g \equiv (g_{ij})_{1\le i,j\le m}$.

Proof. Let us set

$$L = \begin{pmatrix} (M_1+M_3) & M_2 \\ M_2 & M_1 \end{pmatrix} \begin{pmatrix} (\lambda^{ij})_{1\le i,j\le m} & (\theta^{ij})_{1\le i,j\le m} \\ (\theta^{ij})_{1\le i,j\le m} & (\omega^{ij})_{1\le i,j\le m} \end{pmatrix},$$

with $L = (L_{ij})_{1\le i,j\le 2m}$.

It suffices to show that $L_{ij} = \delta_{ij}$, for $i, j = 1, \cdots, 2m$. Actually, we have for $i\,,\,j = 1, \cdots, m$:

$$\begin{aligned}
L_{ij} &= \sum_{k=1}^m [(\alpha_1+\alpha_3) g_{ik} + (\beta_1+\beta_3) u_i u_k](\frac{\alpha_1}{\alpha} g^{kj} - \psi_\lambda u^k u^j) \\
&\quad + \sum_{k=1}^m (\alpha_2 g_{ik} + \beta_2 u_i u_k)(-\frac{\alpha_2}{\alpha} g^{kj} - \psi_\theta u^k u^j) \qquad (2.38)\\
&= \frac{\alpha_1(\alpha_1+\alpha_3)}{\alpha} \sum_{k=1}^m g_{ik} g^{kj} - (\alpha_1+\alpha_3)\psi_\lambda u^j \sum_{k=1}^m g_{ik} u^k \\
&\quad + \frac{\alpha_1(\beta_1+\beta_3)}{\alpha} u_i \sum_{k=1}^m u_k g^{kj} - (\beta_1+\beta_3)\psi_\lambda u_i u^j \sum_{k=1}^m u_k u^k \\
&\quad - \frac{\alpha_2^2}{\alpha} \sum_{k=1}^m g_{ik} g^{kj} - \alpha_2 \psi_\theta u^j \sum_{k=1}^m g_{ik} u^k \\
&\quad - \frac{\alpha_2\beta_2}{\alpha} u_i \sum_{k=1}^m u_k g^{kj} - \beta_2 \psi_\theta u_i u^j \sum_{k=1}^m u_k u^k \\
&= \frac{\alpha_1(\alpha_1+\alpha_3)}{\alpha} \delta_{ij} - (\alpha_1+\alpha_3)\psi_\lambda u^j u_i \\
&\quad + \frac{\alpha_1(\beta_1+\beta_3)}{\alpha} u_i u^j - (\beta_1+\beta_3)\psi_\lambda u_i u^j |u|^2 \\
&\quad - \frac{\alpha_2^2}{\alpha} \delta_{ij} - \alpha_2 \psi_\theta u^j u_i \\
&\quad - \frac{\alpha_2\beta_2}{\alpha} u_i u^j - \beta_2 \psi_\theta u_i u^j |u|^2 \\
&= \delta_{ij} + [\frac{\alpha_1(\beta_1+\beta_3) - \alpha_2\beta_2}{\alpha} - (\phi_1+\phi_3)\psi_\lambda - \phi_2\psi_\theta] u_i u^j \\
L_{ij} &= \delta_{ij} \quad \text{by (2.32)},
\end{aligned}$$

$$\begin{aligned}
L_{\{i+m\}j} &= \sum_{k=1}^m (\alpha_2 g_{ik} + \beta_2 u_i u_k)(\frac{\alpha_1}{\alpha} g^{kj} - \psi_\lambda u^k u^j) \qquad (2.39)\\
&\quad + \sum_{k=1}^m (\alpha_1 g_{ik} + \beta_1 u_i u_k)(-\frac{\alpha_2}{\alpha} g^{kj} - \psi_\theta u^k u^j) \\
&= (\frac{\alpha_1\beta_2 - \alpha_2\beta_1}{\alpha} - \phi_2\psi_\lambda - \phi_1\psi_\theta) u_i u^j \\
L_{\{i+m\}j} &= 0 \quad \text{by (2.31)},
\end{aligned}$$

$$\begin{aligned}
(2.40)\quad L_{\{i+m\}\{j+m\}} &= \sum_{k=1}^{m}(\alpha_2 g_{ik}+\beta_2 u_i u_k)(-\frac{\alpha_2}{\alpha}g^{kj}-\psi_\theta u^k u^j)\\
&\quad+\sum_{k=1}^{m}(\alpha_1 g_{ik}+\beta_1 u_i u_k)[\frac{(\alpha_1+\alpha_3)}{\alpha}g^{kj}-\psi_\omega u^k u^j]\\
&= \delta_{ij}+[\frac{(\alpha_1+\alpha_3)\beta_1-\alpha_2\beta_2}{\alpha}-(\phi_1\psi_\omega+\phi_2\psi_\theta)]u_i u^j\\
L_{\{i+m\}\{j+m\}} &= \delta_{ij} \quad \text{by (2.33)},
\end{aligned}$$

$$\begin{aligned}
L_{i\{j+m\}} &= \sum_{k=1}^{m}[(\alpha_1+\alpha_3)g_{ik}+(\beta_1+\beta_3)u_i u_k][-\frac{\alpha_2}{\alpha}g^{kj}-\psi_\theta u^k u^j]\\
(2.41)\quad &\quad+\sum_{k=1}^{m}(\alpha_2 g_{ik}+\beta_2 u_i u_k)[\frac{(\alpha_1+\alpha_3)}{\alpha}g^{kj}-\psi_\omega u^k u^j]\\
&= [\frac{(\alpha_1+\alpha_3)\beta_2-\alpha_2(\beta_1+\beta_3)}{\alpha}-\phi_2\psi_\omega-(\phi_1+\phi_3)\psi_\theta]u_i u^j\\
L_{i\{j+m\}} &= 0 \quad \text{by (2.34)}.
\end{aligned}$$

Hence $L_{ij}=\delta_{ij}$ for $i,j=1,\cdots,2m$; as stated. □

3. **Levi-Civita connection of a nondegenerate g-natural metric**

In [1], the authors have given explicitly (with some sign and parenthesis misprints) the Levi-Civita connection in the case of Riemannian g-natural metrics. In the following we determine the Levi-Civita connection for a nondegenerate g-natural metric in general by using the inverse formula of nondegenerate g-natural metrics.

Notation 3.1. *For a Riemannian manifold (M,g), we set :*

$$(3.1)\quad \begin{array}{ll}
T^1(u;X_x,Y_x)=R(X_x,u)Y_x, & T^2(u;X_x,Y_x)=R(Y_x,u)X_x,\\
T^3(u;X_x,Y_x)=R(X_x,Y_x)u, & T^4(u;X_x,Y_x)=g(R(X_x,u)Y_x,u)u,\\
T^5(u;X_x,Y_x)=g(X_x,u)Y_x, & T^6(u;X_x,Y_x)=g(Y_x,u)X_x,\\
T^7(u;X_x,Y_x)=g(X_x,Y_x)u, & T^8(u;X_x,Y_x)=g(X_x,u)g(Y_x,u)u,
\end{array}$$

where $(x,u)\in TM$, $X_x,Y_x\in T_xM$ and R is the Riemannian curvature of g.

Let ∇ be the Levi-Civita connection of g and $\bar\nabla$ the Levi-Civita connection of a nondegenerate g-natural metric G defined by the functions $\alpha_i,\ \beta_i,\ i=1,2,3$, in Proposition 1.1. We have:

Proposition 3.1. *Let $(x,u)\in TM$ and $X,Y\in\mathfrak{X}(M)$, we have*

$$\begin{aligned}
(3.2)\ \left(\bar\nabla_{X^h}Y^h\right)_{(x,u)} &= (\nabla_X Y)^h_{(x,u)}+h\{A(u;X_x,Y_x)\}+v\{B(u;X_x,Y_x)\},\\
(3.3)\ \left(\bar\nabla_{X^h}Y^v\right)_{(x,u)} &= (\nabla_X Y)^v_{(x,u)}+h\{C(u;X_x,Y_x)\}+v\{D(u;X_x,Y_x)\},\\
(3.4)\ \left(\bar\nabla_{X^v}Y^h\right)_{(x,u)} &= h\{C(u;Y_x,X_x)\}+v\{D(u;Y_x,X_x)\},\\
(3.5)\ \left(\bar\nabla_{X^v}Y^v\right)_{(x,u)} &= h\{E(u;Y_x,X_x)\}+v\{F(u;Y_x,X_x)\},
\end{aligned}$$

where $P(u;X_x,Y_x)=\sum_{i=1}^{8} f_i^P(|u|^2)T^i(u;X_x,Y_x)$, *for* $P=A,B,C,D,E,F$, *with*

$$
\begin{aligned}
&f_1^A = f_2^A = -\tfrac{\alpha_1\alpha_2}{2\alpha}\,, && f_3^A = 0,\\
&f_4^A = \alpha_2\psi_\lambda\,, && f_5^A = f_6^A = \tfrac{\alpha_2(\beta_1+\beta_3)}{2\alpha},\\
&f_7^A = (\alpha_1+\alpha_3)'\tfrac{\phi_2}{\phi}, && f_8^A = (\beta_1+\beta_3)'\tfrac{\phi_2}{\phi} + (\beta_1+\beta_3)\psi_\theta;
\end{aligned}
\tag{3.6}
$$

$$
\begin{aligned}
&f_1^B = \tfrac{\alpha_2^2}{\alpha}\,, && f_2^B = 0,\\
&f_3^B = -\tfrac{\alpha_1(\alpha_1+\alpha_3)}{2\alpha}\,, && f_4^B = \alpha_2\psi_\theta,\\
&f_5^B = f_6^B = -\tfrac{(\alpha_1+\alpha_3)(\beta_1+\beta_3)}{2\alpha}, && f_7^B = -(\alpha_1+\alpha_3)'\tfrac{(\phi_1+\phi_3)}{\phi},\\
&f_8^B = -(\beta_1+\beta_3)'\tfrac{(\phi_1+\phi_3)}{\phi} + (\beta_1+\beta_3)\psi_\omega;
\end{aligned}
\tag{3.7}
$$

$$
\begin{aligned}
&f_1^C = 0\,, && f_2^C = -\tfrac{\alpha_1^2}{2\alpha},\\
&f_3^C = 0\,, && f_4^C = \tfrac{\alpha_1\psi_\lambda}{2},\\
&f_5^C = +\tfrac{\alpha_1(\beta_1+\beta_3)}{2\alpha}\,, && f_6^C = (\alpha_1+\alpha_3)'\tfrac{\alpha_1}{\alpha} - \tfrac{\alpha_2}{2\alpha}(2\alpha_2'-\beta_2)\,,\\
&f_7^C = \tfrac{(\beta_1+\beta_3)\phi_1}{2\phi} + \tfrac{1}{2}(2\alpha_2'-\beta_2)\tfrac{\phi_2}{\phi}, && f_8^C = (\beta_1+\beta_3)'\tfrac{\phi_1}{\phi} - \psi_\lambda[(\alpha_1+\alpha_3)' + \tfrac{(\beta_1+\beta_3)}{2}] - \tfrac{1}{2}(2\alpha_2'-\beta_2)\psi_\theta;
\end{aligned}
\tag{3.8}
$$

$$
\begin{aligned}
&f_1^D = 0\,, && f_2^D = \tfrac{\alpha_1\alpha_2}{2\alpha},\\
&f_3^D = 0\,, && f_4^D = \tfrac{\alpha_1}{2}\psi_\theta,\\
&f_5^D = -\tfrac{\alpha_2(\beta_1+\beta_3)}{2\alpha}, && f_6^D = -(\alpha_1+\alpha_3)'\tfrac{\alpha_2}{\alpha} + \tfrac{(2\alpha_2'-\beta_2)(\alpha_1+\alpha_3)}{2\alpha},\\
&f_7^D = -\tfrac{(\beta_1+\beta_3)\phi_2}{2\phi} - \tfrac{1}{2}(2\alpha_2'-\beta_2)\tfrac{(\phi_1+\phi_3)}{\phi}, && f_8^D = -(\beta_1+\beta_3)'\tfrac{\phi_2}{\phi} - [(\alpha_1+\alpha_3)' + \tfrac{\beta_1+\beta_3}{2}]\psi_\theta - \tfrac{1}{2}(2\alpha_2'-\beta_2)\psi_\omega;
\end{aligned}
\tag{3.9}
$$

$$
\begin{aligned}
&f_1^E = f_2^E = f_3^E = f_4^E = 0, && f_5^E = f_6^E = (\alpha_2' + \tfrac{1}{2}\beta_2)\tfrac{\alpha_1}{\alpha} - \alpha_1'\tfrac{\alpha_2}{\alpha},\\
&f_7^E = \beta_2\tfrac{\phi_1}{\phi} - (\beta_1-\alpha_1')\tfrac{\phi_2}{\phi}, && f_8^E = 2\beta_2'\tfrac{\phi_1}{\phi} - \beta_1'\tfrac{\phi_2}{\phi} - (2\alpha_2'+\beta_2)\psi_\lambda - 2\alpha_1'\psi_\theta;
\end{aligned}
\tag{3.10}
$$

$$(3.11)\quad \begin{array}{ll} f_1^F = f_2^F = f_3^F = f_4^F = 0, & f_5^F = f_6^F = -(\alpha_2' + \frac{1}{2}\beta_2)\frac{\alpha_2}{\alpha} \\ & +\alpha_1' \frac{(\alpha_1+\alpha_3)}{\alpha}, \\ f_7^F = (\beta_1 - \alpha_1')\frac{(\phi_1+\phi_3)}{\phi} - \frac{\beta_2\phi_2}{\phi}, & f_8^F = \beta_1' \frac{(\phi_1+\phi_3)}{\phi} - 2\beta_2' \frac{\phi_2}{\phi} \\ & -(2\alpha_2' + \beta_2)\psi_\theta - 2\alpha_1'\psi_\omega. \end{array}$$

Proof. We prove only (3.4), the proof of the other identities being the same. Let us set

$$(3.12)\qquad X = \sum_{i=1}^{m} X^i \partial_{x_i}, \quad Y = \sum_{i=1}^{m} Y^i \partial_{x_i}, \quad u = \sum_{i=1}^{m} u^i \partial_{x_i},$$

$$(3.13)\qquad \bar{\nabla}_{X^v} Y^h = \sum_{i=1}^{m} d_i \partial_{x_i}^h + \sum_{i=1}^{m} d_{m+i} \partial_{x_i}^v,$$

$$(3.14)\qquad s_i = G\left(\bar{\nabla}_{X^v} Y^h, \partial_{x_i}^h\right) \quad \text{and}$$

$$(3.15)\qquad s_{m+i} = G\left(\bar{\nabla}_{X^v} Y^h, \partial_{x_i}^v\right).$$

Koszul's formula gives

$$(3.16)\quad s_i = \frac{1}{2}\left\{X^v.G\left(Y^h, \partial_{x_i}^h\right) + Y^h.G\left(\partial_{x_i}^h, X^v\right) - \partial_{x_i}^h.G\left(X^v, Y^h\right) \right. \\ \left. + G\left(\partial_{x_i}^h, \left[X^v, Y^h\right]\right) - G\left(Y^h, \left[X^v, \partial_{x_i}^h\right]\right) - G\left(X^v, \left[Y^h, \partial_{x_i}^h\right]\right)\right\},$$

then by using Proposition 1.1, Lemma 1.1 and Lemma 1.2, we obtain

$$(3.17)\quad s_i = (\alpha_1 + \alpha_3)' g(X,u) g(Y, \partial_{x_i}) + (\beta_1 + \beta_3)' g(X,u) g(Y,u) g(\partial_{x_i}, u) \\ + \frac{\beta_1 + \beta_3}{2} g(X,Y) g(\partial_{x_i}, u) + \frac{\beta_1 + \beta_3}{2} g(Y,u) g(X, \partial_{x_i}) \\ + \frac{\alpha_1}{2} g(R(Y, \partial_{x_i})u, X),$$

and similarly

$$(3.18)\quad s_{m+i} = \frac{1}{2}(2\alpha_2' - \beta_2) g(X,u) g(Y, \partial_{x_i}) - \frac{1}{2}(2\alpha_2' - \beta_2) g(X,Y) g(u, \partial_{x_i}).$$

By setting $\mathbf{d} = (d_i)_{1 \le i \le 2m}$ and $\mathbf{s} = (s_i)_{1 \le i \le 2m}$, we have $\mathbf{d} = G^{-1}\mathbf{s}$ (Matrix-valued function of G^{-1} with the column vector $\mathbf{s}$ as argument).

Then by using the expression of G^{-1} in Proposition 2.2, we obtain

$$(3.19)\quad d_i = \frac{\alpha_1^2}{2\alpha}\{R(u,X)Y\}^i - \frac{\alpha_1 \psi_\lambda}{2} g(R(Y,u)u, X) u^i \\ + [(\alpha_1 + \alpha_3)' \frac{\alpha_1}{\alpha} - \frac{\alpha_2}{2\alpha}(2\alpha_2' - \beta_2)] g(X,u) Y^i \\ + \frac{\alpha_1(\beta_1 + \beta_3)}{2\alpha} g(Y,u) X^i \\ + [\frac{1}{2}(\beta_1 + \beta_3)\frac{\phi_1}{\phi} + \frac{1}{2}(2\alpha_2' - \beta_2)\frac{\phi_2}{\phi}] g(X,Y) u^i \\ + \{(\beta_1 + \beta_3)' \frac{\phi_1}{\phi} - [(\alpha_1 + \alpha_3)' + \frac{\beta_1 + \beta_3}{2}]\psi_\lambda \\ - \frac{1}{2}\psi_\theta (2\alpha_2' - \beta_2)\} g(X,u) g(Y,u) u^i,$$

and

$$
\begin{aligned}
(3.20) \qquad d_{m+i} \;=\; & \frac{\alpha_1\alpha_2}{2\alpha}\{R(X,u)Y\}^i + \frac{\alpha_1\psi_\theta}{2} g(R(X,u)Y,u)u^i \\
& +[-(\alpha_1+\alpha_3)'\frac{\alpha_2}{\alpha} + \frac{(2\alpha_2'-\beta_2)(\alpha_1+\alpha_3)}{2\alpha}]g(X,u)Y^i \\
& -\alpha_2\frac{(\beta_1+\beta_3)}{2\alpha}g(Y,u)X^i \\
& +[-\frac{(\beta_1+\beta_3)\phi_2}{2\phi} - \frac{1}{2}(2\alpha_2'-\beta_2)\frac{\phi_1+\phi_3}{\phi}]g(X,Y)u^i \\
& +\{-(\beta_1+\beta_3)'\frac{\phi_2}{\phi} - [(\alpha_1+\alpha_3)' + \frac{(\beta_1+\beta_3)}{2}]\psi_\theta \\
& -\frac{1}{2}(2\alpha_2'-\beta_2)\psi_\omega\}g(X,u)(Y,u)u^i,
\end{aligned}
$$

where, for all $W \in \mathfrak{X}(M)$, $\{W\}^i$ are the components of W in the coordinates system $(U; x_i,\ i = 1, \cdots, m)$. So according to (3.13), the proof of (3.4) is completed. □

4. g-Natural metrics with constant sectional curvature

4.1. Riemannian curvature of nondegenerate g-natural metrics.

Some notations and properties of F-tensor fields. Fix $(x, u) \in TM$ and a system of normal coordinates $S := (U\,;\, x_i\,, i = 1, \cdots, m)$ of (M, g) centred at x. Then we can define on U the vector field $\mathbf{U} := \sum_i u^i \frac{\partial}{\partial x_i}$, where $(u^1, \cdots, u^m)$ are the coordinates of $u \in T_xM$ with respect to its basis $(\left(\frac{\partial}{\partial x_i}\right)_x; i = 1, \cdots, m)$.

Let P be an F-tensor field of type (r, s) on M. Then, on U, we can define an (r, s)-tensor field P_u^S (or P_u if there is no risk of confusion), associated to u and S, by

$$
(4.1) \qquad P_u(X_1, \cdots, X_s) := P(\mathbf{U}_z; X_1, \cdots, X_s),
$$

for all $(X_1, \cdots, X_s) \in T_zM$, $\forall z \in U$.

On the other hand, if we fix $x \in M$ and s vectors $X_1, \cdots, X_s$ in T_xM, then we can define a C^∞-mapping $P_{(X_1,\cdots,X_s)} : T_xM \to \otimes^r T_xM$, associated to $(X_1, \cdots, X_s)$, by

$$
(4.2) \qquad P_{(X_1,\cdots,X_s)}(u) := P(u;\, X_1, \cdots, X_s),
$$

for all $u \in T_xM$.

Let $s > t$ be two non-negative integers, T be a $(1, s)$-tensor field on M and P^T be an F-tensor field, of type $(1, t)$, of the form

$$
(4.3) \qquad P^T(u; X_1, \cdots, X_t) = T(X_1, \cdots, u, \cdots, u, \cdots, X_t),
$$

for all $(u; X_1, \cdots, X_t) \in TM \oplus \cdots \oplus TM$, i.e., u appears $s - t$ times at positions $i_1, \cdots, i_{s-t}$ in the expression of T. Then

- P_u^T is a $(1, t)$-tensor field on a neighborhood U of x in M, for all $u \in T_xM$,
- $P^T_{(X_1,\cdots,X_t)}$ is a C^∞-mapping $T_xM \to T_xM$, for all $X_1, \cdots, X_t$ in T_xM.

Furthermore, we have

Lemma 4.1. [2]

1) *The covariant derivative of* P_u^T*, with respect to the Levi-Civita connection of* (M, g)*, is given by :*

$$(\nabla_X P_u^T)(X_1, \cdots, X_t) = (\nabla_X T)(X_1, \cdots, u, \cdots, u, X_t), \tag{4.4}$$

for all vectors $X, X_1, \cdots, X_t$ *in* T_xM*, where* u *appears at positions* $i_1, \cdots, i_{s-t}$ *in the right-hand side of the preceding formula.*

2) *The differential of* $P^T_{(X_1,\cdots,X_t)}$*, at* $u \in T_xM$*, is given by :*

$$\begin{aligned} d\left(P^T_{(X_1,\cdots,X_t)}\right)_u (X) &= T(X_1, \cdots, X, \cdots, u, \cdots, X_t) + \cdots \\ &\quad + T(X_1, \cdots, u, \cdots, X, \cdots, X_t), \end{aligned} \tag{4.5}$$

for all $X \in T_xM$*.*

Furthermore, in [2] the authors gave the expressions determining the Riemannian curvature $\bar{R}$ of any Riemannian g-natural metric G on TM (up to a misprint in the vertical component of the expression of $\bar{R}(X^h, Y^h)Z^h$, in which $(\nabla_Y A_u)(X, Z)$ should be written $(\nabla_Y B_u)(X, Z)$). Their formulas remain the same if we replace a Riemannian g-natural metric by a nondegenerate g-natural metric on TM. Indeed, a similar proof as that in [2] gives :

Proposition 4.1. *The Riemannian curvature* $\bar{R}$ *of a nondegenerate* g*-natural metric* G *is completely defined by*

$$\begin{aligned} \bar{R}(X^h, Y^h)Z^h &= [R(X,Y)Z]^h \\ &\quad + h\{(\nabla_X A_u)(Y,Z) - (\nabla_Y A_u)(X,Z) \\ &\quad + A(u; X, A(u; Y, Z)) - A(u; Y, A(u; X, Z)) \\ &\quad + C(u; X, B(u; Y, Z)) - C(u; Y, B(u; X, Z)) \\ &\quad + C(u; Z, R(X,Y)u)\} \\ &\quad + v\{(\nabla_X B_u)(Y,Z) - (\nabla_Y B_u)(X,Z) \\ &\quad + B(u; X, A(u; Y, Z)) - B(u; Y, A(u; X, Z)) \\ &\quad + D(u; X, B(u; Y, Z)) - D(u; Y, B(u; X, Z)) \\ &\quad + D(u; Z, R(X,Y)u)\}, \end{aligned} \tag{4.6}$$

$$\begin{aligned} \bar{R}(X^h, Y^h)Z^v &= h\{(\nabla_X C_u)(Y,Z) - (\nabla_Y C_u)(X,Z) \\ &\quad + A(u; X, C(u; Y, Z)) - A(u; Y, C(u; X, Z)) \\ &\quad + C(u; X, D(u; Y, Z)) - C(u; Y, D(u; X, Z)) \\ &\quad + E(u; R(X,Y)u, Z)\} \\ &\quad + v\{R(X,Y)Z + (\nabla_X D_u)(Y,Z) - (\nabla_Y D_u)(X,Z) \\ &\quad + B(u; X, C(u; Y, Z)) - B(u; Y, C(u; X, Z)) \\ &\quad + D(u; X, D(u; Y, Z)) - D(u; Y, D(u; X, Z)) \\ &\quad + F(u; R(X,Y)u, Z)\}, \end{aligned} \tag{4.7}$$

$$\begin{aligned}(4.8)\qquad \bar{R}\left(X^h,Y^v\right)Z^h &= h\{(\nabla_X C_u)(Z,Y)+A(u;X,C(u;Z,Y))\\ &\quad +C(u;X,D(u;Z,Y))-C(u;A(u;X,Z),Y)\\ &\quad -E(u;Y,B(u;X,Z))-d\left(A_{(X,Z)}\right)_u(Y)\}\\ &\quad +v\{(\nabla_X D_u)(Z,Y)+B(u;X,C(u;Z,Y))\\ &\quad +D(u;X,D(u;Z,Y))-D(u;A(u;X,Z),Y)\\ &\quad -F(u;Y,B(u;X,Z))-d\left(B_{(X,Z)}\right)_u(Y)\},\end{aligned}$$

$$\begin{aligned}(4.9)\qquad \bar{R}\left(X^h,Y^v\right)Z^v &= h\{(\nabla_X E_u)(Y,Z)+A(u;X,E(u;Y,Z))\\ &\quad +C(u;X,F(u;Y,Z))-C(u;C(u;X,Z),Y)\\ &\quad -E(u;Y,D(u;X,Z))-d\left(C_{(X,Z)}\right)_u(Y)\}\\ &\quad +v\{(\nabla_X F_u)(Y,Z)+B(u;X,E(u;Y,Z))\\ &\quad +D(u;X,F(u;Y,Z))-D(u;C(u;X,Z),Y)\\ &\quad -F(u;Y,D(u;X,Z))-d\left(D_{(X,Z)}\right)_u(Y)\},\end{aligned}$$

$$\begin{aligned}(4.10)\qquad \bar{R}(X^v,Y^v)Z^h &= h\{d\left(C_{(Z,Y)}\right)_u(X)-d\left(C_{(Z,X)}\right)_u(Y)\\ &\quad +C(u;C(u;Z,Y),X)-C(u;C(u;Z,X),Y)\\ &\quad +E(u;X,D(u;Z,Y))-E(u;Y,D(u;Z,X))\}\\ &\quad +v\{d\left(D_{(Z,Y)}\right)_u(X)-d\left(D_{(Z,X)}\right)_u(Y)\\ &\quad +D(u;C(u;Z,Y),X)-D(u;C(u;Z,X),Y)\\ &\quad +F(u;X,D(u;Z,Y))-F(u;Y,D(u;Z,X))\},\end{aligned}$$

$$\begin{aligned}(4.11)\qquad \bar{R}(X^v,Y^v)Z^v &= h\{d\left(E_{(Y,Z)}\right)_u(X)-d\left(E_{(X,Z)}\right)_u(Y)\\ &\quad +C(u;E(u;Y,Z),X)-C(u;E(u;X,Z),Y)\\ &\quad +E(u;X,F(u;Y,Z))-E(u;Y,F(u;X,Z))\}\\ &\quad +v\{d\left(F_{(Y,Z)}\right)_u(X)-d\left(F_{(X,Z)}\right)_u(Y)\\ &\quad +D(u;E(u;Y,Z),X)-D(u;E(u;X,Z),Y)\\ &\quad +F(u;X,F(u;Y,Z))-F(u;Y,F(u;X,Z))\},\end{aligned}$$

for all $x \in M$ and $X, Y, Z \in T_xM$, where the lifts are taken at $u \in T_xM$ and R is the Riemannian curvature of g.

Remark 4.1. *Let $P = \sum_{i=5}^{8} f_i^P T^i$ and $Q = \sum_{i=5}^{8} f_i^Q T^i$ be F-tensors, such that f_i^P, f_i^Q are differentiable functions on $\mathbb{R}^+$ and T^i are defined in Notation 3.1 . For $(x,u) \in TM$ and X , Y , $Z \in T_xM$, we have*

$$\begin{aligned}(4.12)\qquad P(u;X,Q(u;Y,Z)) &- P(u;Y,Q(u;X,Z))\\ &= \{a_1(P,Q)(|u|^2)g(Y,Z)\\ &\quad +a_2(P,Q)(|u|^2)g(Y,u)g(Z,u)\}X\\ &\quad -\{a_1(P,Q)(|u|^2)g(X,Z)\\ &\quad +a_2(P,Q)(|u|^2)g(X,u)g(Z,u)\}Y\\ &\quad +a_3(P,Q)(|u|^2)\{g(X,Z)g(Y,u)\\ &\quad -g(Y,Z)g(X,u)\}u,\end{aligned}$$

where $a_i(P,Q)$, $i=1,2,3$; *are the functions on* $\mathbb{R}^+$ *given by*

$$a_1(P,Q)(t) = tf_6^P f_7^Q, \tag{4.13}$$
$$a_2(P,Q)(t) = f_6^P(f_6^Q + tf_8^Q) - (f_5^P f_6^Q - f_6^P f_5^Q), \tag{4.14}$$
$$a_3(P,Q)(t) = f_7^P f_5^Q - (f_5^P + f_7^P + tf_8^P) f_7^Q, \tag{4.15}$$

for all $t \in \mathbb{R}^+$.

In the sequel we shall consider only Riemannian g-natural metrics G on TM.

4.2. **On the hereditary property of constant sectional curvature.** We prove the following result that improves [2, theorem 0.3].

Proposition 4.2. *If* (TM,G) *has constant sectional curvature then* (M,g) *is a flat Riemannian manifold.*

Proof. If (TM,G) has constant sectional curvature K, then by [2, theorem 0.3] (M,g) has constant sectional curvature $k \in \mathbb{R}$. Furthermore, since (TM,G) has constant sectional curvature then its Riemannian curvature $\bar{R}$ satisfies $\bar{R}(X^h,Y^h)Z^v_{|(x,u)} \in H_{(x,u)}TM$ for any $(x,u)\in TM$, and $X,Y,Z \in \mathfrak{X}(M)$. Then by (4.7), we have

$$\begin{aligned} R(X,Y)Z_{|x} = & -[(\nabla_X D_u)(Y,Z) - (\nabla_Y D_u)(X,Z) \\ & + B(u;X,C(u;Y,Z)) - B(u;Y,C(u;X,Z)) \\ & + D(u;X,D(u;Y,Z)) - D(u;Y,D(u;X,Z)) \\ & + F(u;R(X,Y)u,Z)]\,, \end{aligned} \tag{4.16}$$
$$\forall (x,u) \in TM.$$

Thus $R(X,Y)Z_{|x} = 0\,,\ \forall x \in M$ (by taking $(x,u)=(x,0)\in TM$).
This means that $k=0$.

□

In the following proposition, we investigate the g-natural metrics of constant sectional curvature.

Proposition 4.3. *For dim* $M \geq 3$*, the flat Riemannian* g*-natural metrics are the only* g*-natural metrics on* TM *that have constant sectional curvature.*

Proof. If (TM,G) has constant sectional curvature K, then

$$\begin{aligned} \bar{R}\left(X^h,Y^h\right)Z^h &= K\left[G\left(Z^h,Y^h\right)X^h - G\left(X^h,Z^h\right)Y^h\right] \\ &= K[(\alpha_1+\alpha_3)g(Z,Y) + (\beta_1+\beta_3)g(Z,u)g(Y,u)]X^h \\ &\quad -K[(\alpha_1+\alpha_3)g(X,Z) + (\beta_1+\beta_3)g(X,u)g(Z,u)]Y^h. \end{aligned} \tag{4.17}$$

So by Proposition 4.2, we have $R \equiv 0$ and thus from the formulas (4.6) and (4.12), we obtain

$$\begin{aligned}(4.18)\quad \bar{R}\left(X^h, Y^h\right) Z^h &= h\{A(u;X,A(u;Y,Z)) - A(u;Y,A(u;X,Z)) \\ &\quad +C(u;X,B(u;Y,Z)) - C(u;Y,B(u;X,Z))\} \\ &= \{[a_1(A,A) + a_1(C,B)]g(Y,Z) \\ &\quad +[a_2(A,A) + a_2(C,B)]g(Y,u)g(Z,u)\}X^h \\ &\quad -\{[a_1(A,A) + a_1(C,B)]g(X,Z) \\ &\quad +[a_2(A,A) + a_2(C,B)]g(X,u)g(Z,u)\}Y^h \\ &\quad +\{[a_3(A,A) + a_3(C,B)][g(X,Z)g(Y,u) \\ &\quad -g(Y,Z)g(X,u)]\}u^h.\end{aligned}$$

Then, let $(x,u) \in TM$ with $u \neq 0$:

1) Since $\dim M \geq 3$, there exists two non-vanishing vectors $X, Y \in T_xM$ such that the system (u, X, Y) is orthogonal. So by (4.17) and (4.18), for $Z = Y$, we obtain respectively $\bar{R}\left(X^h, Y^h\right) Y^h = K(\alpha_1+\alpha_3)g(Y,Y)X^h$ and $\bar{R}\left(X^h, Y^h\right) Y^h = [a_1(A,A) + a_1(C,B)]g(Y,Y)X^h$. Since $g(Y,Y) \neq 0$ and $X \neq 0$, we have

$$(4.19)\qquad K(\alpha_1 + \alpha_3)(t) = [a_1(A,A) + a_1(C,B)](t),\ \forall\, t > 0.$$

2) Next, by choosing $Y = Z = u$ such that u is orthogonal to a vector $X \neq 0$ in T_xM, (4.17) gives

$$(4.20)\qquad \bar{R}\left(X^h, Y^h\right) Y^h = Kg(u,u)[(\alpha_1 + \alpha_3) + g(u,u)(\beta_1 + \beta_3)]X^h,$$

and (4.18) gives

$$\begin{aligned}(4.21)\qquad \bar{R}\left(X^h, Y^h\right) Y^h &= g(u,u)[a_1(A,A) + a_1(C,B) \\ &\quad +g(u,u)(a_2(A,A) + a_2(C,B))]X^h.\end{aligned}$$

Then, by (4.20) and (4.21), we have,

$$\begin{aligned}(4.22)\quad K[(\alpha_1 + \alpha_3) + g(u,u)(\beta_1 + \beta_3)] &= a_1(A,A) + a_1(C,B) \\ &\quad +g(u,u)[a_2(A,A) + a_2(C,B)].\end{aligned}$$

Thus, by (4.19), we obtain

$$(4.23)\qquad [a_2(A,A) + a_2(C,B)](t) = K(\beta_1 + \beta_3)(t)\ ,\forall\ t > 0.$$

3) Furthermore, by choosing $Y = u$ and $X = Z \neq 0$ such that X and u are orthogonal, (4.17) gives

$$\bar{R}\left(X^h, u^h\right) X^h = -K(\alpha_1 + \alpha_3)g(X,X)u^h,$$

and (4.18) gives

$$\begin{aligned}\bar{R}\left(X^h, u^h\right) X^h &= g(X,X)[-(a_1(A,A) + a_1(C,B)) \\ &\quad +g(u,u)(a_3(A,A) + a_3(C,B)]u^h.\end{aligned}$$

Then by (4.19), we obtain

$$(4.24)\qquad [a_3(A,A) + a_3(C,B)](t) = 0, \forall\, t > 0.$$

And we deduce that the identities (4.19), (4.23) and (4.24) are true for any $t \geq 0$, since the functions α_i , β_i , $i = 1, 2, 3$; are smooth on $\mathbb{R}^+$.
Hence we have

$$\left\{ \begin{array}{lcl} a_1(A,A) + a_1(C,B) & = & K(\alpha_1 + \alpha_3), \\ a_2(A,A) + a_2(C,B) & = & K(\beta_1 + \beta_3), \\ a_3(A,A) + a_3(C,B) & = & 0. \end{array} \right. \tag{4.25}$$

But (TM, G) is Riemannian, i.e.,

$$\left\{ \begin{array}{l} \alpha_1 > 0 \\ \alpha = \alpha_1(\alpha_1 + \alpha_3) - \alpha_2^2 > 0 \end{array} \right. , \tag{4.26}$$

and then

$$\left\{ \begin{array}{l} \alpha_1 > 0 \\ \alpha_1(\alpha_1 + \alpha_3) > \alpha_2^2 \end{array} \right. , \tag{4.27}$$

so $(\alpha_1 + \alpha_3) > 0$. Hence according to the first equation of (4.25) which means that

$$tf_6^A(t)f_7^A(t) + tf_6^C(t)f_7^B = K(\alpha_1 + \alpha_3)(t), \tag{4.28}$$

we obtain for $t = 0$, $0 = K(\alpha_1 + \alpha_3)(0)$, so $K = 0$.

□

If (M, g) is a flat Riemannian manifold and we choose

$$\left\{ \begin{array}{l} \alpha_1 \equiv 1, \\ \alpha_2 = \alpha_3 = \beta_1 = \beta_2 = \beta_3 \equiv 0 \end{array} \right. , \text{ (Sasaki's metric)} \tag{4.29}$$

we obtain that (TM, G) is a flat Riemannian manifold . But it is not the only way to choose the functions α_i, β_i, $i = 1, 2, 3$, for getting (TM, G) as a flat Riemannian manifold. Actually we estabish a characterization of flat Riemannian g-natural metrics in what follows.

4.3. **Flat Riemannian g-natural metrics.**

Lemma 4.2. *If (TM, G) is a flat Riemannian manifold with* $\dim M \geq 3$*, then*

a) $\beta_1 + \beta_3 = 0$,
b) $\alpha_1 + \alpha_3 = constant > 0$,
c) $2\alpha_2' = \beta_2$,
d) $f_6^F = f_7^F = f_8^F = 0$,

where α_2' denotes the first derivative of α_2.

Proof. If (TM, G) is flat Riemannian then, by [2, page 36], we have $\beta_1 + \beta_3 = 0$ and $\alpha_1 + \alpha_3 = constant$. We have also $\alpha_1 + \alpha_3 > 0$ since $\alpha > 0$. Therefore we have the parts *a)* and *b)* of Lemma 4.2. Furthermore,
by [3, Lemma 4.1], we have $2\alpha_2' - \beta_2 = 0$, and $A = B = C = D = 0$.
It remains to prove *d)*.
Since $D = 0$ then (4.11) gives,

$$\begin{array}{rcl} \bar{R}_v^6(X,Y)Z & = & \{[a_1(F,F) + f_7^F - f_6^F]g(Y,Z) \\ & & +[a_2(F,F) + f_8^F - 2f_6^{F'}]g(Y,u)g(Z,u)\}X \\ & & -\{[a_1(F,F) + f_7^F - f_6^F]g(X,Z) \\ & & +[a_2(F,F) + f_8^F - 2f_6^{F'}]g(X,u)g(Z,u)\}Y \\ & & +[a_3(F,F) + 2f_7^{F'} - f_8^F]\{g(X,Z)g(Y,u) \\ & & -g(Y,Z)g(X,u)\}u, \end{array} \tag{4.30}$$

where $\bar{R}_v^6(X,Y)Z$ is the vertical component of $\bar{R}\,(X^v, Y^v)\,Z^v$. Since $dim\ M \geq 3$, then similar arguments as in the proof of Proposition 4.3 applied to the identity $\bar{R}_v^6(X,Y)Z = 0$ for all $X, Y, Z \in T_xM$, yield

$$\left\{ \begin{array}{rcl} tf_6f_7 + f_7 - f_6 &=& 0, \\ f_6^2 + tf_6f_8 + f_8 &=& 2f_6', \\ f_7^2 + tf_8f_7 + f_8 &=& 2f_7', \end{array} \right. \tag{4.31}$$

where $f_i = f_i^F$, $i = 6, 7, 8$, and f_i' denotes the first derivative of f_i. Then the first equation of the system (4.31) gives

$$f_7(1 + tf_6) = f_6, \tag{4.32}$$

and so $1 + tf_6 \neq 0$, $\forall t \geq 0$, (otherwise $1 + tf_6 = 0$ would imply $f_6 = 0$ and $tf_6 = -1$, which is absurd). Hence (4.32) gives

$$f_7 = \frac{f_6}{1 + tf_6}. \tag{4.33}$$

Furthermore the second equation of (4.31) gives

$$f_8 = \frac{2f_6' - f_6^2}{1 + tf_6}. \tag{4.34}$$

Next by using (4.33), we obtain

$$f_7' = \frac{f_6' - f_6^2}{(1 + tf_6)^2}, \tag{4.35}$$

and

$$1 + tf_7 = \frac{1 + 2tf_6}{1 + tf_6}. \tag{4.36}$$

By replacing (4.33), (4.34), (4.36) and (4.35) into the 3^{rd} equation of the sytem (4.31), we obtain

$$4tf_6f_6' = -2f_6^2 + 2tf_6^3, \tag{4.37}$$

which implies

$$f_6(t) = 0\,, \quad \text{or} \tag{4.38}$$

$$f_6'(t) = -\frac{f_6(t)}{2t} + \frac{f_6^2(t)}{2}, \tag{4.39}$$

for $t > 0$.

So f_6 is a solution on the open set $I = \{t \in]0\,, +\infty[\ /\ f_6(t) \neq 0\}$ of the Bernouilli equation

$$y'(t) = -\frac{y(t)}{2t} + \frac{y^2(t)}{2}. \tag{4.40}$$

Besides, we have $f_6(0) = 0$. Indeed, if $0 \in Adh(I)$ the adherence of I in $\mathbb{R}^+$, then by equation (4.40), we have

$$\begin{aligned} f_6(0) &= \lim_{\substack{t\to 0 \\ t\in I}} f_6(t) \\ &= \lim_{\substack{t\to 0 \\ t\in I}} t[-2f_6'(t) + f_6^2(t)] = 0. \end{aligned}$$

But if $0 \notin Adh(I)$ then evidently, we have $f_6(0) = 0$.

Thus the frontier $Fr(I)$ of I is necessarily non empty, since $\mathbb{R}^+$ is connected and f_6 is smooth. In summary f_6 is a solution of the equation

$$\left\{ \begin{array}{l} y'(t) = -\frac{y(t)}{2t} + y^2(t), \ \forall t \in I, \\ y_{|_{Fr(I)}} \equiv 0 \, , \end{array} \right. \tag{4.41}$$

that has the unique solution $y \equiv 0$, so $f_6 \equiv 0$.
Next by using (4.33) and (4.34), we obtain $f_7 = f_8 = 0$, as stated.

□

Theorem 4.1. *Let (M, g) be a Riemannian manifold and (TM, G) its tangent bundle equipped with a g-natural metric G. Then (TM, G) is flat Riemannian if and only if*

i) *(M, g) is flat,*
ii) *$\alpha_1(t) > 0, \quad \phi_1(t) > 0, \quad \alpha(t) > 0, \quad \phi(t) > 0, \quad$ for all $t \in \mathbb{R}^+$,*
iii) *$\alpha_1 + \alpha_3 = constant > 0, \quad \beta_1 + \beta_3 = 0, \quad 2\alpha_2' = \beta_2$,*
iv) *$\alpha_1' = \frac{\alpha_2\beta_2}{\alpha_1+\alpha_3} \quad$ and $\quad \beta_1 = \frac{\beta_2(2\alpha_2+t\beta 2)}{\alpha_1+\alpha_3}$,*

where α_1' and α_2' are respectively the first derivatives of the functions α_1 and α_2.

Proof. Let us assume that (TM, G) is flat Riemannian. By Proposition 1.2 and Proposition 4.2, we obtain the parts *i)* and *ii)* of Theorem 4.1.
Next we obtain *iii)* from Lemma 4.2.
It remains to prove *iv)*. But according to Lemma 4.2 we have

$$2\alpha_2' = \beta_2 \quad \text{and} \tag{4.42}$$

$$f_6 = -\frac{\alpha_2}{\alpha}(\alpha_2' + \frac{1}{2}\beta_2) + \alpha_1' \frac{(\alpha_1+\alpha_3)}{\alpha} = 0. \tag{4.43}$$

Then by combining these identities, we obtain

$$\alpha_1' = \frac{\alpha_2\beta_2}{\alpha_1+\alpha_3}. \tag{4.44}$$

Lemma 4.2 gives again

$$\begin{aligned} f_7 &= (\beta_1 - \alpha_1')(\phi_1 + \phi_3) - \beta_2\phi_2 = 0, \quad \text{and} \\ \beta_1 + \beta_3 &= 0, \end{aligned} \tag{4.45}$$

then

$$\begin{aligned} \beta_1 &= \alpha_1' + \frac{\beta_2\phi_2}{\alpha_1+\alpha_3} \\ &= \frac{\alpha_2\beta_2}{\alpha_1+\alpha_3} + \frac{\beta_2(\alpha_2+t\beta_2)}{\alpha_1+\alpha_3} \quad \text{by} \quad (4.44) \\ \beta_1 &= \frac{\beta_2(2\alpha_2+t\beta_2)}{\alpha_1+\alpha_3}. \end{aligned} \tag{4.46}$$

So we prove *iv)*.
Conversely:
The part *ii)* shows that G is Riemannian. Next by combining the parts *i)* and *iii)* we obtain

$$A = B = C = D = 0. \tag{4.47}$$

Furthermore by combining the parts *iii*) and *iv*) we obtain

$$f_6^F = f_7^F = f_8^F = 0, \tag{4.48}$$

$$f_6^E = f_7^E = \frac{\beta_2}{\alpha_1+\alpha_3},\ f_8^E = \frac{2\beta_2'}{\alpha_1+\alpha_3}. \tag{4.49}$$

So (4.48) implies that $F = 0$, and by considering (4.47) we obtain: $\forall\ (x,u) \in TM$ and $\forall\, X, Y, Z \in T_xM$,

$$\begin{aligned} \bar{R}\left(X^h, Y^h\right) Z^h &= \bar{R}\left(X^h, Y^h\right) Z^v = 0, \\ \bar{R}\left(X^h, Y^v\right) Z^h &= \bar{R}\left(X^h, Y^v\right) Z^v = \bar{R}\left(X^v, Y^v\right) Z^h = 0, \end{aligned} \tag{4.50}$$

where the lifts are taken at (x,u). Next (4.49) implies

$$\begin{aligned} \bar{R}\left(X^v, Y^v\right) Z^v &= h\{d\left(E_{(Y,Z)}\right)_u(X) - d\left(E_{(X,Z)}\right)_u(Y)\} \\ &= \{(f_7^E - f_6^E)g(Y,Z) + (f_8^E - 2f_6^{E'})g(Y,u)g(Z,u)\}X \\ &\quad -\{(f_7^E - f_6^E)g(X,Z) + (f_8^E - 2f_6^{E'})g(X,u)g(Z,u)\}Z \\ &\quad +(2f_7^{E'} - f_8^E)\{g(Y,Z)g(X,u) - g(X,Z)g(Y,u)\}u \\ &= 0. \end{aligned} \tag{4.51}$$

Finally $\bar{R} \equiv 0$.

□

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