Generalized \( \left( \frac{G'}{G} \right) \) - Expansion Method for Some Soliton Wave Solution of the Coupled Potential Korteweg–de Vries (KdV) equation

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Abstract
In this article, some soliton wave solutions of the coupled potential KdV equation have been found using the generalized \( (G'/G) \) - expansion method. For this equation, hyperbolic function solutions, trigonometric function solutions and rational function solutions have been obtained. It was seen that the solutions provided the equation using Mathematica 11.2 In addition, the graphic performances of some solutions are given.

Keywords: The coupled potential KdV-equation, Soliton wave solutions, Generalized \( \left( \frac{G'}{G} \right) \) - expansion method.

Öz
Bu makalede, genelleştirilmiş \( (G'/G) \) – açılım metodu kullanılarak potansiyel KdV denklem çiftinin bazı soliton dalga çözümleri bulunmuştur. Bu denklem için hiperbolik fonksiyon çözümleri, trigonometrik fonksiyon çözümleri ve rasyonel fonksiyon çözümleri elde edilmiştir. Çözümlerin Mathematica 11.2 kullanılarak denklemi sağladığı görülmüştür. Ayrıca, bazı çözümlerin grafik performansları verilmiştir.

Anahtar Kelimeler: Potansiyel KdV- denklem çifti, Soliton dalga çözümleri, Genelleştirilmiş \( \left( \frac{G'}{G} \right) \) - açılım metodu.
1. Introduction

Nonlinear partial differential equations (NPDEs) have an important place in applied sciences. There are some analytical methods for solving these equations in the literature (Bock and Kruskal, 1979; Malfliet, 1992; Chuntao, 1996; Cariello and Tabor, 1989; Fan, 2000a; Clarkson, 1989). In addition to these methods, there are many methods of solving such equations by using an auxiliary equation. By using these methods, partial differential equations are converted to ordinary differential equations and the solutions of partial differential equations are found with the help of these ordinary differential equations. Some of these methods are given in (Fan, 2000b; Elwakil et al., 2002; Chen and Zhang, 2004; Fu et al., 2001; Shen and Pan, 2003; Chen and Hong-Qing, 2004; Chen et al., 2004; Chen and Yan, 2006; Wang et al., 2008; Guo and Zhou, 2010; Lü et al., 2010; Li et al., 2010; Manafian, 2016; Khater, 2015; Manafian et al., 2017; Yan, 2001).

We used the generalized $\left(\frac{G'}{G}\right)$-Expansion Method for finding the some soliton wave solution of the coupled potential KdV-equation. This method is given in the second chapter.

2. Analysis of Method

The method will be introduced briefly. Consider a general partial differential equation of two variables,

$$Q(u, u_t, u_x, u_{xx}, \ldots) = 0,$$  \hspace{1cm} (1)

Using the wave variable $u(x, t) = u(\xi), \quad \xi = x - \mu t$ the Eq.(1) turns into an ordinary differential equation,

$$Q'(u', u'', u''', \ldots) = 0$$  \hspace{1cm} (2)

here $\mu$ is constant. With this conversion, we obtain a nonlinear ordinary differential equation for $u(\xi)$. We can express the solution of Eq.(2) as below,

$$u(\xi) = \sum_{k=0}^{m} d_k \Phi(\xi)^k + \sum_{k=1}^{m} e_k \Phi(\xi)^{-k}$$  \hspace{1cm} (3)

where $m$ is a positive integer is found as the result of balancing the highest order linear term and the highest order nonlinear term found in the equation, the coefficients $d_k$ and $e_k$ are constants. $\Phi(\xi) = \left(\frac{G'}{G}\right)$ satisfies the following ordinary differential equation,

$$k_4 G G'' - k_2 G G' - k_3 (G')^2 - k_4 G^2 = 0.$$  \hspace{1cm} (4)
Substituting solution (3) into Eq. (2) yields a set of algebraic equation for $\left(\frac{\partial r}{\partial \xi}\right)$, $\left(\frac{\partial r}{\partial \xi}\right)^{-k}$, then, all coefficients of $\left(\frac{\partial r}{\partial \xi}\right)$, $\left(\frac{\partial r}{\partial \xi}\right)^{-k}$, have to vanish. Then, $d_k, e_k, k_1, k_2, k_3, k_4$ and $\mu$ constants are found. The special solutions of Eq. (4) are as follows, (Manafian et al., 2017).

1. if $k_2 \neq 0, f = k_1 - k_3$ and $s = k_2^2 + 4k_4(k_1 - k_3) > 0$, then
   \[
   \phi(x) = \frac{k_2}{2f} + \frac{\sqrt{s}}{2f} C_1 \sinh \left(\frac{\sqrt{s}}{2k_1} \xi\right) + C_2 \cosh \left(\frac{\sqrt{s}}{2k_1} \xi\right)
   \]

2. if $k_2 \neq 0, f = k_1 - k_3$ and $s = k_2^2 + 4k_4(k_1 - k_3) < 0$, then
   \[
   \phi(x) = \frac{k_2}{2f} + \frac{\sqrt{-s}}{2f} C_1 \sin \left(\frac{\sqrt{-s}}{2k_1} \xi\right) + C_2 \cos \left(\frac{\sqrt{-s}}{2k_1} \xi\right)
   \]

3. if $k_2 \neq 0, f = k_1 - k_3$ and $s = k_2^2 + 4k_4(k_1 - k_3) = 0$, then
   \[
   \phi(x) = \frac{k_2}{2f} + \frac{C_2}{C_1 + C_2}
   \]

4. if $k_2 = 0, f = k_1 - k_3$ and $g = f k_4 > 0$, then
   \[
   \phi(x) = \frac{\sqrt{g}}{f} C_1 \sinh \left(\frac{\sqrt{g}}{k_1} \xi\right) + C_2 \cosh \left(\frac{\sqrt{g}}{k_1} \xi\right)
   \]

5. if $k_2 = 0, f = k_1 - k_3$ and $g = f k_4 < 0$, then
   \[
   \phi(x) = \frac{\sqrt{-g}}{f} C_1 \sin \left(\frac{\sqrt{-g}}{k_1} \xi\right) + C_2 \cos \left(\frac{\sqrt{-g}}{k_1} \xi\right)
   \]

6. if $k_4 = 0$ and $f = k_1 - k_3$, then
   \[
   \phi(x) = \frac{C_1 k_2^2 \exp \left(-\frac{k_2^2}{k_1}\right)}{f k_1 + C_1 k_1 k_2 \exp \left(-\frac{k_2^2}{k_1}\right)}
   \]

7. if $k_2 \neq 0$ and $f = k_1 - k_3 = 0$, then
   \[
   \phi(x) = -\frac{k_4}{k_2} + C_1 \exp \left(\frac{k_2^2}{k_1}\right)
   \]
8. if $k_1 = k_3$, $k_2 = 0$ and $f = k_1 - k_3 = 0$, then
$$\Phi(\xi) = C_1 + \frac{k_4}{k_1} \xi$$

9. if $k_3 = 2k_1$, $k_2 = 0$ and $k_4 = 0$, then
$$\Phi(\xi) = -\frac{1}{C_1 + \left(\frac{k_3}{k_1} - 1\right)} \xi$$

3. Application

The coupled potential KdV equation were considered as follows (Yan, 2001),

$$u_t - u_{xxx} - 3uu_{xx} + 3vu_{xx} - 3u^2u_x + 6uu_x - 3v^2u_x = 0$$
$$v_t - v_{xxx} + 3uv_{xx} - 3vv_{xx} - 3v^2v_x + 6uv_x - 3v^2v_x = 0$$

(5)

If $u(x,t) = u(\xi)$, $\xi = x - \mu t$ conversion is used, the (5) equation becomes the following ordinary differential equation,

$$-\mu u' - u''' - 3u''u' - 3(u')^2 - 3u^2u' + 6uu' - 3v^2u' = 0$$
$$-\mu v' - v''' + 3uv'' - 3vv'' - 3(v')^2 - 3u^2v' + 6uv' - 3v^2v' = 0$$

(6)

When balancing $u'''$ with $vu'''$ and $uvu'$, $v'''$ with $uv''$ and $uvv'$ then gives $m_1 = 1$ and $m_2 = 1$. The solutions are as follows,

$$u(\xi) = d_0 + d_1 \Phi(\xi) + e_1 \Phi(\xi)^{-1}$$

$$v(\xi) = f_0 + f_1 \Phi(\xi) + g_1 \Phi(\xi)^{-1}$$

(7)

If Eq. (7) is substituted in Eq. (6), we have a system of algebraic equations for $d_0, d_1, e_1, f_0, f_1, g_1, k_1, k_2, k_3, k_4$ and $\mu$. These algebraic equations system are as follows

$$-\mu e_1 - 3d_0^2 e_1 - 3e_1^2 - 3d_0^2 e_2^2 + 6d_0 e_1 f_0 - 3e_1 f_0^2 + 6e_1^2 f_1 - 6e_1 f_1 g_1 + 3d_1 g_1^2 + \frac{3d_0 e_1 k_2^2}{k_1} - \frac{3d_0 e_1 k_2}{k_1} -$$

$$\frac{6d_0 e_1 f_1 k_2}{k_1} + \frac{6d_0 d_1 g_1 k_2}{k_1} - \frac{6d_0 d_1 g_1 k_2}{k_1} - \frac{e_1 k_2^2}{k_1^2} + \frac{3e_1 f_1 k_2^2}{k_1^2} + \frac{2d_1 g_1 k_2^2}{k_1^2} + \frac{\mu e_1 k_3}{k_1} + \frac{3d_0 e_1 k_3}{k_1} + \frac{6e_1 k_3}{k_1} +$$

$$\frac{3d_1 k_3}{k_1} - \frac{6d_0 e_1 f_0 k_3}{k_1} + \frac{3e_1 f_1 k_3}{k_1} - \frac{6e_1^2 f_1 k_3}{k_1} + \frac{6e_1 g_1 f_1 k_3}{k_1} - \cdots$$

(8)

If the system is solved, the coefficients are found as

Case 1.
\( e_1 = 0, f_0 = d_0, k_4 = 0, k_2 \neq 0, k_1 \neq 0, k_3 \neq 0, g_1 = 0, f_1 = d_1, k_3 = \frac{1}{2} (2k_1 - d_1 k_1), \}
\[ \mu = -\frac{k_2^2}{k_1^2}, d_1 \neq 0 \]

**Solution 1.**

\[
u(x, t) = d_0 + \frac{k_2}{k_1} + \left( \begin{array}{c}
\text{Sinh} \left[ \frac{k_2 (k_1^2 x + k_2^2 t)}{2k_1^2} \right] C_1 + \text{Cosh} \left[ \frac{k_2 (k_1^2 x + k_2^2 t)}{2k_1^2} \right] C_2 \end{array} \right) \frac{k_2}{k_1} \]

\[ (9) \]

\[
u(x, t) = d_0 + \frac{k_2}{k_1} + \left( \begin{array}{c}
\text{Sinh} \left[ \frac{k_2 (k_1^2 x + k_2^2 t)}{2k_1^2} \right] C_1 + \text{Cosh} \left[ \frac{k_2 (k_1^2 x + k_2^2 t)}{2k_1^2} \right] C_2 \end{array} \right) \frac{k_2}{k_1} \]

\[ (10) \]

**Case 2.**

\( e_1 = 0, f_0 = d_0, f_1 = d_1, g_1 = 0, k_2 = 0, k_3 = \frac{1}{2} (2k_1 - d_1 k_1), k_1 (-k_1 + k_3) \neq 0, \mu = \frac{-2d_1 k_4}{k_1} \)

**Solution 2.**

\[
u(x, t) = d_0 + \sqrt{2} \left( -\text{Sin} \left[ \frac{\sqrt{-d_1 k_1 k_4 (xk_1 + 2td_1 k_4)}}{\sqrt{2} k_1^2} \right] C_1 + \text{Cos} \left[ \frac{\sqrt{-d_1 k_1 k_4 (xk_1 + 2td_1 k_4)}}{\sqrt{2} k_1^2} \right] C_2 \right) \frac{\sqrt{-d_1 k_1 k_4}}{\sqrt{2} k_1} \]

\[ (11) \]

\[
u(x, t) = d_0 + \sqrt{2} \left( -\text{Sin} \left[ \frac{\sqrt{-d_1 k_1 k_4 (xk_1 + 2td_1 k_4)}}{\sqrt{2} k_1^2} \right] C_1 + \text{Cos} \left[ \frac{\sqrt{-d_1 k_1 k_4 (xk_1 + 2td_1 k_4)}}{\sqrt{2} k_1^2} \right] C_2 \right) \frac{\sqrt{-d_1 k_1 k_4}}{\sqrt{2} k_1} \]

\[ (12) \]

\[
u(x, t) = d_0 + \sqrt{2} \left( \text{Sinh} \left[ \frac{\sqrt{d_1 k_1 k_4 (xk_1 + 2td_1 k_4)}}{\sqrt{2} k_1^2} \right] C_1 + \text{Cosh} \left[ \frac{\sqrt{d_1 k_1 k_4 (xk_1 + 2td_1 k_4)}}{\sqrt{2} k_1^2} \right] C_2 \right) \frac{\sqrt{d_1 k_1 k_4}}{\sqrt{2} k_1} \]

\[ (13) \]
\[ v(x, t) = d_0 + \frac{\sqrt{2} \sinh \left[ \frac{\sqrt{d_1 k_1 k_4 (x-k_1+2t d_1 k_4)}}{\sqrt{2 k_1^2}} \right]}{\cosh \left[ \frac{\sqrt{d_1 k_1 k_4 (x-k_1+2t d_1 k_4)}}{\sqrt{2 k_1^2}} \right]} C_1 + \frac{\cosh \left[ \frac{\sqrt{d_1 k_1 k_4 (x-k_1+2t d_1 k_4)}}{\sqrt{2 k_1^2}} \right]}{\sinh \left[ \frac{\sqrt{d_1 k_1 k_4 (x-k_1+2t d_1 k_4)}}{\sqrt{2 k_1^2}} \right]} C_2 \sqrt{d_1 k_1 k_4} k_1 \] (14)

**Case 3.**

\[ e_1 = g_1, \quad f_0 = d_0, \quad k_1 = 0, \quad e_1 \neq 0, \quad d_1 = 0, \quad k_4 = 0, \quad f_1 = 0, \quad k_1 (-k_1 + k_3) \neq 0, \mu = \frac{3(e_1 k_1 - e_1 k_3)}{k_1} \]

**Solution 3.**

\[ u(x, t) = d_0 + e_1 (-x - C_1 + 3e_1 t) \] (15)
\[ v(x, t) = d_0 + e_1 (-x - C_1 + 3e_1 t) \] (16)

**Case 4.**

\[ e_1 = 0, \quad f_0 = d_0, \quad f_1 = d_1, \quad k_2 = 0, \quad g_1 = 0, \quad k_1 = k_3, \quad k_1 k_4 \neq 0, \quad \mu = -\frac{3d_1 k_4}{k_1}, \quad d_1 \neq 0 \]

**Solution 4.**

\[ u(x, t) = d_0 + d_1 (C_1 + \frac{k_4 (k_1 x + 3d_1 k_4 t)}{k_1^2}) \] (17)
\[ v(x, t) = d_0 + d_1 (C_1 + \frac{k_4 (k_1 x + 3d_1 k_4 t)}{k_1^2}) \] (18)

**Case 5.**

\[ d_1 = 0, \quad f_1 = 0, \quad g_1 = e_1, \quad k_2 = 0, \quad k_4 = 0, \quad k_1 (-k_1 + k_3) \neq 0, \quad -d_0 e_1 + e_1 f_0 \neq 0, \quad \mu = \frac{3(d_0^2 k_1 + e_1 k_1 - 2d_0 f_0 k_1 + f_0^2 k_1 - e_1 k_3)}{k_1} \]

**Solution 5.**

\[ u(x, t) = d_0 + e_1 (-x - C_1 - 3(d_0^2 - e_1 - 2d_0 f_0 + f_0^2) t) \] (19)
\[ v(x, t) = 3e_1^2 t + f_0 - e_1 (x + C_1 + 3d_0^2 t - 6d_0 f_0 t + 3f_0^2 t) \] (20)

4. **Explanations and Graphical Representations of The Obtained Some Solutions**

The graphical performance of found some solutions are demonstrated Figs. 1-4. These figures have the following physical explanations.
Figure 1. The 3 Dimensional surfaces of Eq. (9) for $k_1 = 1, k_2 = 2, d_0 = 1, C_1 = 2, C_2 = 3$.

Figure 2. The 2 Dimensional surfaces of Eq. (9) for $k_1 = 1, k_2 = 2, d_0 = 1, C_1 = 2, C_2 = 3$ and $t = 1$.

Figure 3. The 3 Dimensional surfaces of Eq. (11) for $k_1 = 1, k_4 = 2, d_0 = 1, d_1 = 1, C_1 = 3, C_2 = 2$.

Figure 4. The 2 Dimensional surfaces of Eq. (11) for $k_1 = 1, k_4 = 2, d_0 = 1, d_1 = 1, C_1 = 3, C_2 = 2$ and $t = 1$. 
The coupled potential KdV-equation: The shapes of Eqs.(9)-(11) are represented in Figures 1-4 within the intervals $-20 \leq x \leq 20, -5 \leq t \leq 5$.

5. Conclusions

The Generalized $\left(\frac{\phi'}{\phi}\right)$-Expansion Method were used for some soliton wave solution the coupled potential KdV-equation. Some nonlinear partial differential equations were solved by this method. It can be solved similarly in a number of nonlinear partial differential equations.

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