On Submanifolds in a Riemannian Manifold with Golden Structure

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Abstract. A golden Riemannian structure \((J, g)\) on a Riemannian manifold is given by a tensor field \(J\) of type \((1, 1)\) satisfying the golden section relation \(J^2 = J + I\), and a pure Riemannian metric \(g\), that is a metric satisfying \(g(JX, Y) = g(X, JY)\). We investigate some fundamental properties of the induced structure on submanifolds immersed in golden Riemannian manifolds. We obtain effective relations for some induced structures on submanifolds of codimension 2. We also construct an example on submanifold of a golden Riemannian manifold.


Keywords: Golden structure, Riemannian manifold, totally geodesics, normal induced structure, Killing vector fields, invariant submanifolds.

1. Introduction

The theory of submanifolds is an interesting topic in the study of differential geometry. It has the origin in the study of geometry of plane curves and surfaces initiated by Fermat. Since then, it has been evolving in different directions of differential geometry and mechanics, especially. It is still an active research field playing an important role in the development of modern differential geometry. Ahmad M et. al. ([1–3, 8]) studied submanifolds of almost r-para-contact Riemannian manifold endowed with semi-symmetric and quater symmetric connections. Hretcanu [14] studied submanifolds of almost product Riemannian manifolds. CR-submanifolds of LP-Sasakian manifolds were studied by Ahmad, Ozgur and Haseeb [18].


Motivated by above studies in this paper, we study submanifold of a golden Riemannian manifold. The paper is organized as follows:

In section 2, we define golden Riemannian manifolds.

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In section 3, we establish several properties of induced structure \((P, g, \xi, u, a)\) on the submanifold immersed in golden Riemannian manifold. In last section, we construct an example of golden Riemannian structure on Euclidean space and its submanifolds.

2. GOLDEN RIEMANNIAN MANIFOLDS

In this section, we give a brief information of golden Riemannian manifolds.

**Definition 2.1.** [6] Let \((M, g)\) be a Riemannian manifold. A golden structure on \((\overline{M}, g)\) is a non-null tensor \(J\) of type \((1,1)\) which satisfies the equation

\[ J^2 = J + I, \]

where \(I\) is the identity transformation. We say that the metric \(g\) is \(J\)-compatible if

\[ g(JX, Y) = g(X, JY) \]

for all \(X, Y\) vector fields on \(\overline{M}\). If we substitute \(JX\) into \(X\) in (2.2), then we have

\[ g(JX, JY) = g(JX, Y) + g(X, Y). \]

The Riemannian metric (2.2) is called \(J\)-compatible and \((\overline{M}, J, g)\) is called a Golden Riemannian manifold.

**Proposition 2.2.** [6] A golden structure on the manifold \(M\) has the power

\[ J^n = F_n J + F_{n-1} I \]

for any integer \(n\), where \((F_n)\) is the Fibonacci sequence.

Using an explicit expression for the Fibonacci sequence namely the Binet’s formula

\[ F_n = \frac{\phi^n - (1 - \phi)^n}{\sqrt{5}}, \]

we obtain a new form for the equality (2.3) as

\[ J^n = \left( \frac{\phi^n - (1 - \phi)^n}{\sqrt{5}} \right) J + \left( \frac{\phi^{n-1} - (1 - \phi)^{n-1}}{\sqrt{5}} \right) I. \]

The straight forward computations yield:

**Proposition 2.3.** [6] (i) The eigen values of a golden structure \(J\) are the golden ratio \(\phi\) and \(1 - \phi\).

(ii) A golden structure \(J\) is an isomorphism on the tangent space \(T_x M\) of the manifold \(M\) for every \(x \in M\).

(iii) It follows that \(J\) is invertible and its inverse \(\tilde{J} = J^{-1}\) satisfies

\[ \tilde{\phi}^2 = -\phi + 1. \]

3. PROPERTIES OF INDUCED STRUCTURE ON SUBMANIFOLDS IN GOLDEN RIEMANNIAN MANIFOLDS

Let us consider that \(M\) is an \(n\)-dimensional submanifold of codimension \(r\), isometrically immersed in an \((n + r)\)-dimensional golden Riemannian manifold \((\overline{M}, \overline{g}, J)\) with \(n, r \in N\).

We denote by \(T_x M\) the tangent space of \(M\) in a point \(x \in M\) and by \(T^\perp_x M\) the normal space of \(M\) in \(x\). Let \(i\) be the differential of the immersion \(i : M \to \overline{M}\). The induced Riemannian metric \(g\) on \(M\) is given by \(g(X, Y) = g(iX, iY)\) for every \(X, Y \in \chi(M)\).

We consider a local orthonormal basis \(N_1, N_2, \ldots, N_r\) of the normal space \(T^\perp_x M\). We assume that the indices \(\alpha, \beta, \gamma\) run over the range \(1, 2, \ldots, r\).

For every \(X \in T_x M\) the vector fields \(J(iX)\) and \(J(N_\alpha)\) can be decomposed in tangential and normal components as follows:

\[ J(iX) = i(P(X)) + \sum_{\alpha=1}^r u_\alpha(X) N_\alpha, \]

\[ J(N_\alpha) = \xi_\alpha + \sum_{\beta=1}^r a_{\alpha\beta} N_\beta, \]

where \(P\) is a \((1, 1)\) tensor field on \(M\), \(\xi \in \xi(\overline{M})\), \(u_\alpha\) are 1-forms on \(M\) and \((a_{\alpha\beta})\) is an \(r \times r\) matrix of smooth real functions on \(M\).
Proposition 3.1. [7] The structure \( \Sigma = (P, g, u_\alpha, \xi_\alpha, (a_{\alpha\beta})) \) induced on the submanifold \( M \) by the golden Riemannian structure \( (g, J) \) on \( \overline{M} \) satisfies the following equalities:

\[
P^2(X) = P(X) + X - \sum_\alpha u_\alpha(X) \xi_\alpha,
\]

\[
u_\alpha(P(X)) = u_\alpha(X) - \sum_\beta a_{\alpha\beta} u_\beta(X),
\]

\[
a_{\alpha\beta} = a_{\beta\alpha},
\]

\[
u_\alpha(\xi_\alpha) = \delta_{\alpha\beta} + a_{\alpha\beta} - \sum_\gamma a_{\alpha\gamma} a_{\gamma\beta},
\]

\[
P(\xi_\alpha) = \xi_\alpha - \sum_\beta a_{\alpha\beta} \xi_\beta,
\]

\[
u_\alpha(X) = g(X, \xi_\alpha),
\]

\[
g(PX, Y) = g(X, PY).
\]

\[
g(PX, PY) = g(X, PY) + g(X, Y) + \sum_\alpha u_\alpha(X) u_\alpha(Y)
\]

for every \( X, Y \in \chi(M) \), where \( \delta_{\alpha\beta} \) is the Kronecker delta.

Definition 3.2. A submanifold \( M \) in a manifold \( \overline{M} \) endowed with structural tensor field \( J \) (i.e. \( J \) is a tensor field on \( \overline{M} \)) is called invariant with respect to \( J \) if \( J(T_x) \subset T_x(M) \) for every \( x \in M \).

Remark 3.3. The induced structure \( \Sigma = (P, g, u_\alpha, \xi_\alpha, (a_{\alpha\beta})) \) on the submanifold \( M \) by the golden Riemannian structure \( (g, J) \) is invariant if and only if \( u_\alpha = 0 \) (equivalently \( \xi_\alpha = 0 \)) for every \( \alpha \in (1, ..., r) \).

The Gauss and Weingarten formula are

\[
\overline{\nabla}_X Y = \nabla_X Y + \sum_{\alpha=0}^r h_\alpha(X, Y) N_\alpha,
\]

(3.3)

\[
\overline{\nabla}_X N_\alpha = -A_\alpha X + \nabla_X N_\alpha.
\]

(3.4)

If \( \{N_1, ..., N_r\} \) and \( \{N'_1, ..., N'_r\} \) are two local orthogonal basis on a normal space \( T_x \perp \), then the decomposition of \( N'_\alpha \) in the basis \( \{N_1, ..., N_r\} \) is the following

\[
N'_\alpha = \sum_{\gamma=1}^r k'_{\alpha\gamma} N_\gamma
\]

for any \( \alpha \in \{1, ..., r\} \), where \( (k'_{\alpha\gamma}) \) is an \( r \times r \) orthogonal matrix and we have

\[
u'_\alpha = \sum_{\gamma} k'_{\alpha\gamma} u_\gamma, \xi'_\alpha = \sum_{\gamma} k'_{\alpha\gamma} \xi_\gamma \text{ and } a'_{\alpha\beta} = \sum_{\gamma} k'_{\alpha\gamma} a_{\gamma\beta} k'_{\beta\gamma}.
\]

Thus, if \( \xi_1, \xi_2, ..., \xi_r \) are linearly independent vector fields, then \( \xi'_1, \xi'_2, ..., \xi'_r \) are also linearly independent.

We know that \( a_{\alpha\beta} \) is symmetric in \( \alpha \) and \( \beta \), under a suitable transformation, we can find that \( a_{\alpha\beta} \) can be reduced to \( a'_{\alpha\beta} = \lambda_\alpha \delta_{\alpha\beta} \), where \( \lambda_\alpha (\alpha \in \{1, ..., r\}) \) are eigen values of the matrix \( (a_{\alpha\beta}) \), and in this case we have \( u'_\beta(\xi_\alpha) = \delta_{\alpha\beta} (1 + \lambda_\alpha - \lambda_\beta) \) and from this we obtain \( u'_\alpha(\xi_\alpha) = (1 + \lambda_\alpha - \lambda_\beta) \).

Remark 3.4. If \( M \) is a non-invariant \( n \)-dimensional submanifold of codimension \( r \), immersed in a golden Riemannian manifold \( (\overline{M}, \overline{g}, J) \) so that the tangent vector fields \( \xi_1, \xi_2, ..., \xi_r \) are linearly independent, then from Proposition 3.1 we obtain

\[
\|\xi_\alpha\|^2 = 1 + a_{\alpha\alpha} - \sum_\gamma a_{\alpha\gamma}^2
\]

and, for \( \alpha \neq \beta \), we have

\[
\sum_\gamma a_{\alpha\gamma} a_{\gamma\beta} = a_{\alpha\beta}.
\]
For the normal connection $\nabla^\perp_X N_\alpha$, we have the decomposition

$$\nabla^\perp_X N = \sum_{\beta=1}^r l_{\alpha\beta}(X)N_\beta$$

(3.5)

for every $X \in \chi(M)$. Therefore, we obtain an $r \times r$ matrix $(l_{\alpha\beta}(X))$, of 1-form on $M$. From $\bar{g}(N_\alpha, N_\beta) = \delta_{\alpha\beta}$, we get

$$\bar{g}(\nabla^\perp_X N, N_\beta) + \bar{g}(N_\alpha, \nabla^\perp_X N_\beta) = 0$$

which is equivalent with

$$\bar{g}(\sum_{\gamma} l_{\alpha\gamma}(X)N_\gamma, N_\beta) + \bar{g}(N_\alpha, \sum_{\gamma} l_{\beta\gamma}(X)N_\gamma) = 0$$

for any $X \in \chi(M)$. Thus, we obtain

$$l_{\alpha\beta} = -l_{\beta\alpha}$$

for any $\alpha, \beta \in \{1, \ldots, r\}$.

**Theorem 3.5.** Let $M$ is an $n$-dimensional submanifold of codimension $r$ in a golden Riemannian manifold with structure $(\bar{M}, \bar{g}, J)$. If the structure $J$ is parallel with respect to the Levi-Civita connection $\nabla$ defined on $\bar{g}$, then the induced structure $(P, g, u_\alpha, \xi_\alpha, (a_{\alpha\beta}))$ induced on $M$ by the structure $J$ has the following properties:

$$(\nabla_X P)(Y) = \sum_\alpha [g(A_\alpha X, Y)\xi_\alpha + u_\alpha(Y)A_\alpha X],$$

(3.6)

$$(\nabla_X u_\alpha)(Y) = \sum_\beta [h_\beta(X, Y)u_{\beta\alpha} - u_{\beta\alpha}(Y)l_{\alpha\beta}(X)] - h_\alpha(X, PY),$$

(3.7)

$$\nabla_X \xi_\alpha = -P(A_\alpha X) + \sum_\beta a_{\alpha\beta} A_\beta X + \sum_\beta l_{\alpha\beta}(X)\xi_\beta,$$

(3.8)

$$X(a_{\alpha\beta}) = -u_\alpha(A_\beta X) - u_\beta(A_\alpha X) + \sum_{\gamma} [l_{\alpha\gamma}(X)a_{\beta\gamma} + l_{\beta\gamma}(X)a_{\alpha\gamma}],$$

(3.9)

for any $X \in \chi(M)$.

**Proof.** Using (3.1) and $\nabla J = 0$, we obtain

$$J(\nabla_X Y) = \nabla_X (\nabla Y) + \nabla_X \sum_\alpha (u_\alpha(Y))N_\alpha$$

and

$$J(\nabla_X Y) = \nabla_X PY + \sum_\alpha [u_\alpha(Y)\nabla_X N_\alpha + N_\alpha \nabla_X (u_\alpha(Y))].$$

Using (3.3) and (3.4), we obtain

$$J[\nabla_X Y + \sum_{\alpha=1}^r h_\alpha(X, Y)N_\alpha] = \nabla_X (PY) + \sum_{\alpha=1}^r h_\alpha(X, PY)N_\alpha +$$

$$\sum_\alpha [u_\alpha(Y)(-A_\alpha X + \nabla^\perp_X N_\alpha) + N_\alpha (\nabla_X u_\alpha(Y) + \sum_{\beta=1}^r h_\beta(X, u(Y))N_\beta].$$

Using (3.1), (3.2) and (3.5), we obtain

$$\sum_{\alpha=1}^r h_\alpha(X, Y)\xi_\alpha + \sum_{\alpha=1}^r h_\alpha(X, Y)\sum_{\beta=1}^r a_{\alpha\beta} N_\beta$$

$$= (\nabla_X P)(Y) + \sum_{\alpha=1}^r h_\alpha(X, PY)N_\alpha - \sum_\alpha u_\alpha(Y)A_\alpha X +$$

$$\sum_\alpha u_\alpha(Y)\sum_{\beta} l_{\alpha\beta}(X)N_\beta] + \sum_\alpha [\nabla_X u_\alpha(Y)N_\alpha].$$

Comparing tangential and normal components, we obtain (3.6) and (3.7).
Using (3.2) and $\nabla J = 0$, we obtain
\[
J(\nabla_X (N_\alpha)) = \nabla_X \xi_\alpha + \nabla_X \sum_\beta a_{\alpha\beta} N_\beta.
\]

Using (3.3), (3.4) and (3.5), we obtain
\[
-P(A_\alpha X) - \sum_\alpha u_\alpha (A_\alpha X) N_\alpha + \sum_\beta l_{\alpha\beta} (X) \xi_\beta + \sum_\beta l_{\alpha\beta} (X) \sum_\gamma a_{\beta\gamma} N_\gamma =
\]
\[
\nabla_X \xi_\alpha + \sum_\alpha h_\alpha (X, \xi_\alpha) N_\alpha + \sum_\beta X (a_{\alpha\beta}) N_\beta - \sum_\beta a_{\alpha\beta} A_\beta X + \sum_\beta a_{\alpha\beta} \sum_\gamma l_{\beta\gamma} (X) N_\gamma.
\]

Thus, identifying the tangential part and respectively the normal part in the last equality, we obtain (3.8) and (3.9).

**Definition 3.6.** If we have the equality $N_\rho (X, Y) - 2 \sum_\alpha d u_\alpha (X, Y) \xi_\alpha = 0$ for any $X, Y \in \chi (M)$, then the $(P, g, u_\alpha, \xi_\alpha, (a_{\alpha\beta})_\gamma)$ induced structure on submanifold $M$ in a golden Riemannian manifold $(\overline{M}, \overline{g}, J)$ is said to be normal.

**Remark 3.7.** The compatibility condition $\nabla J = 0$, where $\nabla$ is Levi-Civita connection with respect to the metric $\overline{g}$ implies the integrability of the structure $J$ which is equivalent with the vanishing of the Nijenhuis torsion tensor field of $J$:
\[
\]

For this assumption, we must have the next general lemma:

**Lemma 3.8.** We suppose that we have golden structure $J$ on a manifold $\overline{M}$ and linear connection $D$ with the torsion $T$. If $N_J$ is Nijenhuis torsion tensor field of $J$, then we obtain
\[
N_J (X, Y) = (D_J X) (Y) - (D_J Y) (X) - T [JX, JY] - JT (X, Y) - T (X, Y) + J (D_J Y) (X) + J (T (X, JY)) - J (D_J J) (Y) + T (X, JY).
\]

**Proof.** From the definition of the torsion $T$ follows that
\[
[X, Y] = D_X Y - D_Y X - T (X, Y)
\]
and from this we get
\[
[JX, JY] = D_J X JY - D_J Y JX - T (JX, JY),
\]
\[
[JX, Y] = D_J X Y - D_J Y JX - T (JX, Y)
\]
and
\[
\]

Using relations $(D_J X) (Y) = D_X JY - J (D_X Y)$ and (2.1) and replacing the relations (3.10), (3.11), (3.12) and (3.13) in the formula of Nijenhuis tensor field of $J$, we obtain
\[
N_J (X, Y) = D_J X JY - D_J Y JX - T (JX, JY) + (J + J) [X, Y] -
\]
\[
J [D_J X Y - D_J Y JX - T (JX, Y)] - J [D_J JY - D_J JY X - T (X, JY)]
\]
\[
N_J (X, Y) = (D_J X) (Y) + J (D_J X Y) - (D_J Y) (X) - J (D_J Y JX) - T (X, JY) +
\]
\[
J (D_J X Y) - J (D_J Y) X - J (T (X, Y) + D_J Y X - D_J Y JX - T (X, Y) - J (D_J JY) +
\]
\[
J (D_J J) (X) + J (D_J JX) + J (D_J (Y) JX) + J (T (X, JY)) - J (D_J J) JY + T (X, JY)
\]
\[
N_J (X, Y) = (D_J X) (Y) - (D_J Y) (X) - T (JX, JY) - JT (X, JY) -
\]
\[
T (X, Y) + J (D_J J) Y (X) + J (T (X, Y)) - J (D_J J) JY + T (X, JY).
\]

**Proposition 3.9.** Let $M$ be a submanifold of codimension $r$ in a golden Riemannian manifold $(\overline{M}, \overline{g}, J)$. If the induced structure $(P, g, u_\alpha, \xi_\alpha, (a_{\alpha\beta})_\gamma)$ on $M$ is normal and the normal connection $\nabla^\perp$ on $M$ vanishes identically (i.e. $l_{\alpha\beta} = 0$), then we obtain the equality
\[
\sum_\alpha g (X, \xi_\alpha) (P A_\alpha - A_\alpha P) (Y) = \sum_\alpha g (Y, \xi_\alpha) (P A_\alpha - A_\alpha P) (X)
\]
for any $X, Y \in \chi (M)$. 
Proof. From the definition 3.6, we have
\[ N_P(X, Y) - 2 \sum_a du_a(X, Y) \xi_a = 0 \]
for any \( X, Y \in \chi(M) \). Then we have
\[
\sum_a g(X, \xi_a)(PA_a - A_a P) - \sum_a g(Y, \xi_a)(PA_a - A_a P)(X)
+ \sum_{a, \beta} [g(X, \xi_\beta)l_{a\beta}(X) - g(Y, \xi_\beta)l_{a\beta}(Y)]\xi_a = 0
\]
for any \( X, Y \in \chi(M) \).

Also, given that normal connection \( \nabla^\perp \) of \( M \) vanishes identically (i.e. \( l_{a\beta} = 0 \)), we obtain
\[
\sum_a g(X, \xi_a)(PA_a - A_a P)(Y) = \sum_a g(Y, \alpha)(PA_\alpha - A_\alpha P)(X).
\]
\[ \square \]

**Proposition 3.10.** Under the assumption of last result, Proposition 3.9 does not depend on the choice of a basis in the normal space \( T^\perp_x(M) \) for any \( x \in M \).

Proof. If \( \{N'_\alpha\} \) is another basis in \( T^\perp_x(M) \), then we have
\[
N'_\alpha = \sum_\beta O_{a\beta} N_\alpha, \tag{3.14}
\]
where \( (O_{a\beta})_a \) is an orthogonal matrix.

From the condition \( \nabla_X N'_\alpha = 0 \), we obtain
\[
\nabla_X N'_\alpha = \sum_\beta O_{a\beta} \nabla_X N_\beta + \sum_\beta \nabla_X O_{a\beta} N_\beta
+ \sum_\beta X(O_{a\beta}) N_\alpha = 0 \tag{3.15}
\]
for any \( X \in M \).

\( \{N_\beta\} \) is linearly independent set, then
\[ O_{a\beta} = \text{constant}.\]

On the other hand,
\[ \nabla_X N'_\alpha = -A'_\alpha X \]
and
\[ \nabla_X N'_\alpha = \sum_\beta \nabla_X O_{a\beta} N_\beta - \sum_\beta O_{a\beta} A_\beta X. \tag{3.16} \]

Thus, from the relations (3.14), (3.15) and (3.16), we obtain
\[
-A'_\alpha X = \sum_\beta \nabla_X O_{a\beta} N_\beta - \sum_\beta O_{a\beta} O_{\beta\alpha} X,
A'_\alpha X = \sum_\beta O_{a\beta} A_\beta X.
\]

Therefore, we have
\[ J(N'_\alpha) = i_* \xi_\alpha + \sum_\beta \alpha'_{a\beta} N'_\beta. \]

Using (3.14), we obtain
\[ J(N'_\alpha) = i_* \xi_\alpha + \sum_\beta \sum_\gamma d_{a\beta} O_{\beta\gamma} N_\gamma. \tag{3.17} \]

Using equality (3.2) and (3.14), we get
\[ J(N'_\alpha) = \sum_\beta O_{a\beta} \xi_\beta + \sum_\beta \sum_\gamma O_{a\beta} a_{\beta\gamma} N_\gamma. \tag{3.18} \]
From (3.17) and (3.18), we obtain

\[ \xi'_{\alpha} = \sum_{\beta} O_{\alpha \beta} \xi_{\beta} \]  

(3.19)

and

\[ \sum_{\beta} \sum_{\gamma} d_{\alpha \beta} O_{\gamma \gamma} = \sum_{\beta} \sum_{\gamma} O_{\alpha \beta} a_{\gamma \gamma}. \]

On the basis \( \{ N_1, ..., N_r \} \), the condition of Proposition 3.10 becomes

\[ \sum_{\alpha} g(X, \xi'_{\alpha})(PA'_{\alpha} - A'_{\alpha} P)(Y) = \sum_{\alpha} g(Y, \xi'_{\alpha})(PA'_{\alpha} - A'_{\alpha} P)(X). \]

From (3.16) and (3.19), we get

\[ \sum_{\alpha} g(X, O_{\alpha \beta})(PO_{\alpha \gamma} A'_{\gamma} - O_{\alpha \gamma} A'_{\gamma} P)(Y) - \sum_{\alpha} g(Y, O_{\alpha \beta})(PO_{\alpha \gamma} A'_{\gamma} - O_{\alpha \gamma} A'_{\gamma} P)(X) \]

\[ = \sum_{\alpha} O_{\alpha \beta} O_{\alpha \gamma} g(X, \xi'_{\beta})(PA_{\gamma} - A_{\gamma} P)(Y) - g(Y, \xi'_{\beta})(PA_{\gamma} - A_{\gamma} P) = 0. \]

From the orthogonality of the matrix \( (O_{\alpha \beta}) \), it follows that

\[ \sum_{\alpha} [g(X, \xi'_{\alpha})(PA_{\alpha} - A_{\alpha} P)(Y) - g(Y, \xi'_{\alpha})(PA_{\alpha} - A_{\alpha} P)(X)] = 0. \]

Therefore, the Proposition 3.9 does not depend on the choice of a basis in the normal space \( T^\perp_x(M) \) for any \( x \in M \).  \( \square \)

**Lemma 3.11.** Let \( M \) be a submanifold in a golden Riemannian manifold \( (\overline{M}, \overline{g}, J) \). Let \( (P, g, u_{\alpha}, \xi_\alpha, (a_{\alpha \beta})_\alpha) \) be the induced structure on \( M \). Then

\[ g((PA_{\alpha} - A_{\alpha} P)(X), Y) \]

on \( M \) is skew-symmetric for any \( X, Y \in \chi(M) \).

**Proof.**

\[ g(PA_{\alpha}X - A_{\alpha} PX, Y) = g(PA_{\alpha}X, Y) - g(A_{\alpha} PX, Y) \]

\[ g(PA_{\alpha}X - A_{\alpha} PX, Y) = g(X, A_{\alpha} PY) - g(PA_{\alpha} Y, X) \]

\[ g(PA_{\alpha}X - A_{\alpha} PX, Y) = -g((PA_{\alpha} - A_{\alpha} P)(Y), X). \]

So, \( g((PA_{\alpha} - A_{\alpha} P)(X), Y) \) is skew-symmetric.  \( \square \)

**Proposition 3.12.** Let \( M \) be a submanifold of codimension \( r \) \( (r \geq 2) \) in a golden Riemannian manifold \( (\overline{M}, \overline{g}, J) \) and structure \( J \) is parallel to Levi-Civita connection \( \nabla \) defined on \( \overline{M} \) with \( (P, g, u_{\alpha}, \xi_\alpha, (a_{\alpha \beta})_\alpha) \) induced structure on \( M \) by \( J \). If the normal connection \( \nabla^\perp \) vanishes identically on the normal bundle \( T^\perp(M) \) (i.e. \( l_{\alpha \beta} = 0 \), then the tangent vector fields \( \{ \xi_1, \xi_2, ..., \xi_r \} \) are linearly independent if and only if the determinant of the matrix \( (I + A - A^2) \) does not vanish in any \( x \in M \), (where \( I \) is the \( r \times r \) identity matrix).

**Proof.** Let \( k_1, ..., k_r \) be the real number with the properties that

\[ k_1 \xi_1 + k_2 \xi_2 + .... + k_r \xi_r = 0 \]  

(3.20)

in any point \( x \in M \).

From the equality (3.6), we obtain

\[ g(\xi_\alpha, \xi_\beta) = u_\beta(\xi_\alpha) = \delta_{\alpha \beta} + a_{\alpha \beta} - \sum_{\gamma} a_{\alpha \gamma} a_{\gamma \beta}. \]  

(3.21)

Multiplying the equality (3.20) by \( \xi_\alpha \) (for any \( \alpha \in \{ 1, 2, ..., r \} \) and using the equality (3.21), we obtain

\[ \begin{cases}
  k_1 (1 + a_1) - \sum_j a_{1j} a_{j1} + k_2 (1 + a_2) - \sum_j a_{2j} a_{j2} + .... + k_r (1 + a_r) - \sum_j a_{rj} a_{jr} = 0 \\
  k_1 (a_{21} - \sum_j a_{2j} a_{j1}) + k_2 (a_{22} - \sum_j a_{2j} a_{j2}) + .... + k_r (a_{2r} - \sum_j a_{2j} a_{jr}) = 0 \\
  k_1 (a_{31} - \sum_j a_{3j} a_{j1}) + k_2 (a_{32} - \sum_j a_{3j} a_{j2}) + .... + k_r (a_{3r} - \sum_j a_{3j} a_{jr}) = 0 \\
  k_1 (a_{41} - \sum_j a_{4j} a_{j1}) + k_2 (a_{42} - \sum_j a_{4j} a_{j2}) + .... + k_r (1 + a_{rr} - \sum_j a_{jr} a_{jr}) = 0.
\end{cases} \]
This linear system of equations has the unique solution \( k_1 = k_2 = \ldots = k_r = 0 \) if and only if it does not have a vanishing determinant. Furthermore, the determinant of the linear system of equations is the determinant of the following matrix

\[
I_r + \begin{pmatrix}
    a_{11} & a_{12} & a_{13}\ldots & a_{1r} \\
    a_{21} & a_{22} & a_{23}\ldots & a_{2r} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{r1} & a_{r2} & a_{r3}\ldots & a_{rr}
\end{pmatrix} = \begin{pmatrix}
    a_{11} & a_{12} & a_{13}\ldots & a_{1r} \\
    a_{21} & a_{22} & a_{23}\ldots & a_{2r} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{r1} & a_{r2} & a_{r3}\ldots & a_{rr}
\end{pmatrix},
\]

that is determinant of the matrix

\[
I_r + A - A^2.
\]

**Lemma 3.13.** Let \( M \) be an \( n \)-dimensional submanifold of co-dimension 2 in a golden Riemannian manifold \((\overline{M}, \overline{g}, J)\), with the normal induced structure \((P, g, \alpha, \xi_0, (\alpha_{0\beta}))\) and structure \( J \) is parallel to Levi-Civita connection \( \nabla \). If the normal connection \( \nabla^\bot \) vanishes identically (i.e., \( I_{\alpha\beta} \)) then the following equation is hold good

\[
g(Y, \xi_1)(PA_1 - A_1 P)(X) + g(Y, \xi_2)(PA_2 - A_2 P)(X) + g((PA_1 - A_1 P)(X), Y)\xi_1
\]

\[
= g((PA_2 - A_2 P)(X), Y) = 0
\]

(3.22)

for any \( X, Y \in \chi(M) \).

**Proof.** By virtue of Lemma 3.11 we obtain

\[
g(X, \xi_1)(PA_1 - A_1 P)(Y) + g(X, \xi_2)(PA_2 - A_2 P)(Y)
\]

\[
= g(Y, \xi_1)(PA_1 - A_1 P)(X) + g(Y, \xi_2)(PA_2 - A_2 P)(X)
\]

for any \( X, Y \in \chi(M) \).

Multiplying by \( Z \in \chi(M) \) we have

\[
g(X, \xi_1)g((PA_1 - A_1 P)(Y), Z) + g(X, \xi_2)g((PA_2 - A_2 P)(Y), Z)
\]

\[
= g(Y, \xi_1)g((PA_1 - A_1 P)(X), Z) + g(Y, \xi_2)g((PA_2 - A_2 P)(X), Z)
\]

(3.23)

for any \( X, Y, Z \in \chi(M) \).

Inverting \( Y \) by \( Z \) in the last equality we obtain

\[
g(X, \xi_1)g((PA_1 - A_1 P)Z, Y) + g(X, \xi_2)g((PA_2 - A_2 P)Z, Y)
\]

\[
= g(Z, \xi_1)g((PA_1 - A_1 P)X, Y) + g(Z, \xi_2)g((PA_2 - A_2 P)X, Y).
\]

(3.24)

Adding equalities (3.23) and (3.24) we obtain

\[
g(X, \xi_1)g((PA_1 - A_1 P)Z, Y) + g(X, \xi_2)g((PA_2 - A_2 P)Z, Y)
\]

\[
+ g(Y, \xi_1)g((PA_1 - A_1 P)Y, Z) + g(Y, \xi_2)g((PA_2 - A_2 P)Y, Z)
\]

\[
= g(Y, \xi_1)g((PA_1 - A_1 P)X, Z) + g(Y, \xi_2)g((PA_2 - A_2 P)X, Z)
\]

\[
+ g(Z, \xi_1)g((PA_1 - A_1 P)X, Y) + g(Z, \xi_2)g((PA_2 - A_2 P)X, Y).
\]

By property of skew-symmetry, we obtain

\[
g((Y, \xi_1)((PA_1 - A_1 P)(X), Z) + (g(Y, \xi_2)(PA_2 - A_2 P)(X), Z)
\]

\[
+ g((PA_1 - PA_1)X, Y)\xi_1 + g((PA_2 - A_2 P)X, Y)\xi_2, Z) = 0
\]

for any \( Z \in \chi(M) \). Thus we obtain equality (3.22). \( \square \)
Lemma 3.14. Let $M$ be an $n$-dimensional submanifold of codimension 2 in a golden Riemannian manifold $(\bar{M}, \bar{g}, J)$, with the normal induced structure $(\bar{P}, \bar{g}, \xi_\alpha, (\alpha_{ab})_\alpha)$ and structure $J$ is parallel to Levi - Civita connection $\nabla^\perp$ vanishes identically (i.e., $l_{ab} = 0$) and $\sigma \neq 0$, then the following equations are good:

\begin{align*}
(\bar{P}A_1 - A_1 \bar{P})\xi_1 &= 0, \\
(\bar{P}A_2 - A_2 \bar{P})\xi_2 &= 0, \\
(\bar{P}A_1 - A_1 \bar{P})\xi_2 &= 0, \\
(\bar{P}A_2 - A_2 \bar{P})\xi_1 &= 0.
\end{align*}

Proof. With $X = Y = \xi_1$ in equality (3.22)

\[ g(\xi_1, \xi_1)(\bar{P}A_1 - A_1 \bar{P})(\xi_1) + g(\xi_1, \xi_2)(\bar{P}A_2 - A_2 \bar{P})(\xi_1) + g((\bar{P}A_1 - A_1 \bar{P})(\xi_1, \xi_1) + g((\bar{P}A_2 - A_2 \bar{P})(\xi_1, \xi_2)) = 0. \]

Using $g(\xi_1, \xi_1) = a + \sigma \neq 0$, $g(\xi_1, \xi_2) = 0$

\begin{align*}
&g((\bar{P}A_1 - A_1 \bar{P})\xi_1, \xi_1) = -g((\bar{P}A_1 - A_1 \bar{P})\xi_1, \xi_2) \\
&g((\bar{P}A_1 - A_1 \bar{P})\xi_1, \xi_2) = 0 \\
&(\bar{P}A_1 - A_1 \bar{P})\xi_1 = 0.
\end{align*}

With $X = Y = \xi_2$, in equality (3.22), we obtain

\[ g(\xi_2, \xi_2)(\bar{P}A_1 - A_1 \bar{P})\xi_2 + g(\xi_1, \xi_2)(\bar{P}A_2 - A_2 \bar{P})\xi_1 + g((\bar{P}A_1 - A_1 \bar{P})\xi_1, \xi_2) = 0. \]

Using that $g(\xi_2, \xi_2) = b + \sigma \neq 0$, $g(\xi_1, \xi_2) = 0$, $g(\xi_1, \xi_1) = 0$

\[ (\bar{P}A_2 - A_2 \bar{P})\xi_2 = 0. \]

If we put $X = \xi_1$ and $Y = \xi_2$ in equality (3.22), we obtain

\begin{align*}
g(\xi_2, \xi_1)(\bar{P}A_1 - A_1 \bar{P})\xi_1 + g(\xi_2, \xi_2)(\bar{P}A_2 - A_2 \bar{P})\xi_1 \\
+ g((\bar{P}A_1 - A_1 \bar{P})\xi_1, \xi_2)\xi_1 + g((\bar{P}A_2 - A_2 \bar{P})\xi_1, \xi_2)\xi_2 = 0.
\end{align*}

Using that $g(\xi_2, \xi_2) = b + \sigma \neq 0$, $g(\xi_1, \xi_2) = 0$, we obtain

\[ (\bar{P}A_2 - A_2 \bar{P})\xi_1 = 0. \]

Again $X = \xi_2$ and $Y = \xi_1$, we obtain

\begin{align*}
g(\xi_1, \xi_1)(\bar{P}A_1 - A_1 \bar{P})\xi_2 + g(\xi_2, \xi_2)(\bar{P}A_2 - A_2 \bar{P})\xi_2 \\
+ g((\bar{P}A_1 - A_1 \bar{P})\xi_2, \xi_1)\xi_1 + g((\bar{P}A_2 - A_2 \bar{P})\xi_2, \xi_1)\xi_2 = 0.
\end{align*}

Using that $g(\xi_1, \xi_1) = a + \sigma \neq 0$, $g(\xi_1, \xi_2) = 0$, we obtain

\[ (\bar{P}A_1 - A_1 \bar{P})\xi_2 = 0. \]

Proposition 3.15. Let $M$ be an $n$-dimensional submanifold of codimension 2 in a golden Riemannian manifold $(\bar{M}, \bar{g}, J)$, with the normal induced structure $(\bar{P}, \bar{g}, \xi_\alpha, (\alpha_{ab})_\alpha)$ and structure $J$ is parallel to Levi - Civita connection $\nabla^\perp$ vanishes identically (i.e., $l_{ab} = 0$) and $\sigma \neq 0$, and trace $\Lambda = 0$. Then $\bar{P}$ commutes with the Weingarten operator $A_\alpha$ ($\alpha \in \{1, 2\}$), thus the following relations take place

\begin{align*}
(i) (\bar{P}A_1 - A_1 \bar{P})(X) &= 0, \quad (3.25) \\
(ii) (\bar{P}A_2 - A_2 \bar{P})(X) &= 0, \quad (3.26)
\end{align*}

\[ \forall X \in \chi(M) \]
Proof.

\[ g((PA_\alpha - A_\alpha P), \xi_\beta) = g((PA_\alpha X, \xi_\beta) - g((A_\alpha P)X, \xi_\beta) \]
\[ g((PA_\alpha - A_\alpha P)X, \xi_\beta) = -[g(PA_\alpha \xi_\beta, X) - g((A_\alpha P)\xi_\beta, X) \]
\[ g((PA_\alpha - A_\alpha P)X, \xi_\beta) = -g((PA_\alpha - A_\alpha P)\xi_\beta, X), \]

where \( \alpha \beta \in \{1, 2\} \), from the last Lemma

\[ (PA_\alpha - A_\alpha P)\xi_\beta = 0 \]

for any \( \alpha \beta \in \{1, 2\} \).

Similarly

\[ (PA_2 - A_2 P)X = 0 \]

for any \( \alpha \beta \in \{1, 2\} \).

In the following we assume that \( M \) is an \( n \)-dimensional submanifold of codimension 2 in golden Riemannian manifold \( (\overline{M}, \overline{g}, J) \) with induced structure \( (P, g, u_\alpha, \xi_\beta, (a_{\alpha\beta})_2) \) on \( M \) \((\alpha, \beta \in \{1, 2\})\). We suppose that the normal connection vanishes identically, thus \( (l_{\alpha\beta} = 0) \). In these conditions, the relations of Proposition 3.1 have the following forms:

\[ P^2X = P(X) + X - u_1(X)\xi_1 - u_2(X)\xi_2, \quad (3.27) \]

and

\[ u_1(PX) = u_1(X) - a_{11}u_1(X) - a_{12}u_2(X), \]
\[ u_2(PX) = u_2(X) - a_{21}u_1(X) - a_{22}u_2(X), \]
\[ u_1(\xi_1) = 1 + a_{11} - a_{11}^2 - a_{12}^2, \]
\[ u_2(\xi_2) = 1 + a_{22} - a_{12}^2 - a_{22}^2, \]
\[ u_1(\xi_1) = u_2(\xi_1) = a_{21} - a_{21}(a_{11} + a_{22}), \]
\[ P(\xi_1) = \xi_1 - a_{11}\xi_1 - a_{12}\xi_2, \]
\[ P(\xi_2) = \xi_2 - a_{21}\xi_1 - a_{22}\xi_2, \]
\[ g(PX, PY) = g(X, PY) + g(X, Y) + u_1(X)u_1(Y) + u_2(X)u_2(Y) \]

for any \( X, Y \in \chi(M) \). We denote by \( A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \).

Furthermore, from Theorem 3.5 under the assumption that the normal connection \( \nabla^\perp \) vanishes identically \( (i.e. \ l_{\alpha\beta}) \), we obtain

\( (\nabla_X P)(Y) = g(A_1 X, Y)\xi_1 + g(A_2 X, Y)\xi_2 + g(Y, \xi_1)A_1 X + g(Y, \xi_2)A_2 X, \)
\( (\nabla_X u_1)(Y) = -g(A_1 X, PY) + a_{11}g(A_1 X, Y) + a_{21}g(A_2 X, Y), \)
\( (\nabla_X u_2)(Y) = -g(A_2 X, PY) + a_{12}g(A_1 X, Y) + a_{22}g(A_2 X, Y), \)
\[ \nabla_X \xi_1 = -P(A_1 X) + a_{11}A_1 X + a_{12}A_2 X, \]
\[ \nabla_X \xi_2 = -P(A_2 X) + a_{21}A_1 X + a_{22}A_2 X, \]
\[ X(a_{12}) = -2u_1(A_1 X), \]
\[ X(a_{22}) = -2u_2(A_2 X). \]
Remark 3.16. A simplifier assumption for these relations is $a_{11} + a_{22} = 0$. Thus, trace $A = 0$. Under this assumption, if we denote $a_{11} = -a_{22} = a$ $a_{12} = a_{21} = b$ and $1 - a^2 - b^2 = \sigma$, from the relations , we easily see that

$$u_1(\xi_1) = u_2(\xi_2) = a + \sigma \Leftrightarrow g(\xi_1, \xi_1) = g(\xi_2, \xi_2) = a + \sigma,$$

$$u_1(\xi_2) = u_2(\xi_1) = b,$$

$$u_1(PX) = (1 - a)u_1(X) - bu_2(X),$$

$$u_2(PX) = (1 - a)u_2(X) - bu_1(X),$$

and

$$P(\xi_1) = (1 - a)\xi_1 - b\xi_2,$$

$$P(\xi_2) = (1 - a)\xi_2 - b\xi_1.$$  

Proposition 3.17. Let $M$ ba a submanifold of codimension 2 in a golden Riemannian manifold $(\overline{M}, \overline{\nabla}, J)$ and structure $J$ is parallel to Levi - Civita connection $\overline{\nabla}$ defined on $\overline{M}$ with the normal induced structure $(P, g, u_\alpha, \xi_\alpha, (a_{ij})_2)$. If the normal connection $\nabla^\perp$ vanishes identically, that is $l_{ij} = 0$, trace $A = 0$ and $\sigma \neq 0$, then the following relations occurs :

$$(a + \sigma)A_1\xi_1 + bA_2\xi_2 = h_1(\xi_1, \xi_1)\xi_1 + h_1(\xi_1, \xi_2)\xi_2,$$  

$$(a + \sigma)A_1\xi_2 + bA_2\xi_1 = h_1(\xi_1, \xi_2)\xi_1 + h_1(\xi_2, \xi_2)\xi_2,$$  

$$(a + \sigma)A_2\xi_1 + bA_1\xi_1 = h_2(\xi_1, \xi_1)\xi_1 + h_2(\xi_1, \xi_2)\xi_2,$$  

$$(a + \sigma)A_2\xi_2 + bA_1\xi_2 = h_2(\xi_2, \xi_1)\xi_1 + h_2(\xi_2, \xi_2)\xi_2.$$  

Proof. Using (3.25) and applying $P$ it follows that

$$P^2A_1X = PA_1PX$$

for any $X \in \chi(M)$.

Using the equality (3.27) and if we put $X = \xi_1$ and $X = \xi_2$ respectively, we obtain

$$P(A_1\xi_1) + A_1\xi_1 - u_1(A_1\xi_1)\xi_1 - u_2(A_1\xi_1)\xi_2 = PA_1\xi_1.$$  

Using equality (3.28), we get

$$(2 - P)A_1\xi_1 + (P - 1)aA_1\xi_1 + (P - 1)bA_2\xi_2 = h_1(\xi_1, \xi_1)\xi_1 + h_1(\xi_1, \xi_2)\xi_2.$$  

Now,

$$P(A_1\xi_2) + A_1\xi_2 - u_1(A_1\xi_2)\xi_1 - u_2(A_1\xi_2)\xi_2 = PA_1\xi_2.$$  

Using (3.29), we obtain

$$A_1\xi_2 - A_1b\xi_1 - A_1a\xi_2 + A_1\xi_2 - PA_1(\xi_2 - b\xi_1 - a\xi_2) = h_1(\xi_1, \xi_2)\xi_1 + h_1(\xi_2, \xi_2)\xi_2.$$  

We replace $X \to PX$ in the equality (3.25), so

$$PA_1PX = A_1P^2X.$$  

Using equality (3.27) and if we put $X = \xi_1$ and $X = \xi_2$ respectively, we get

$$PA_1P\xi_1 = A_1P\xi_1 + A_1\xi_1 - u_1(\xi_1)A_1\xi_1 - u_2(\xi_1)A_1\xi_2.$$  

Using (3.28), we obtain

$$PA_1(\xi_1 - a\xi_1 - b\xi_2) = A_1(\xi_1 - a\xi_1 - b\xi_2) + A_1\xi_1 - u_1(\xi_1)A_1\xi_1 - u_2(\xi_1)A_1\xi_2,$$

$$(P - 2 + \sigma)A_1\xi_1 + (2 - P)aA_1\xi_1 + (2 - P)bA_1\xi_2 = 0$$  

and

$$PA_1PA\xi_2 = A_1P\xi_2 + A_1\xi_2 - u_1(\xi_2)A_1\xi_1 - u_2(\xi_2)A_1\xi_2.$$  

Using (3.29), we obtain

$$PA_1((1 - a)\xi_2 - b\xi_1) = A_1((1 - a)\xi_2 - b\xi_1) + A_1\xi_2 - bA_1\xi_1 - (a + \sigma)A_1\xi_2,$$

$$(P - 2 + \sigma)A_1\xi_2 + (2 - P)aA_1\xi_2 + (2 - P)bA_1\xi_1 = 0.$$  

Adding the relations (3.34) and (3.36), we obtain (3.30).

Adding (3.35) and (3.37), we obtain (3.31).
Applying $P$ in the equality (3.26), it follows that
\[ P^2A_2X = PA_2PX \]
for any $X \in \chi(M)$ and using in (3.27) and for $X = \xi_1$ and $X = \xi_2$ respectively we obtain
\[ (2 - P)A_2\xi_1 + (P - 1)aA_2\xi_1 + (P - 1)bA_2\xi_2 = h_2(\xi_1, \xi_1)\xi_1 + h_2(\xi_1, \xi_2)\xi_2 \]
(3.38)
and
\[ (2 - P)A_2\xi_2 + (P - 1)bA_2\xi_1 + (P - 1)aA_2\xi_2 = h_2(\xi_1, \xi_2) + h_2(\xi_2, \xi_2)\xi_2. \]
We replace $X = PX$ in the equality (3.26), so
\[ PA_2PX = A_2P^2X \]
and using equality (3.27) and if we put $X = \xi_1$ and $X = \xi_2$ we obtain
\[ (P - 2 + \sigma)A_2\xi_1 + (2 - P)aA_2\xi_1 + (2 - P)bA_2\xi_2 = 0 \]
(3.40)
and
\[ (P - 2 + \sigma)A_2\xi_2 + (2 - P)aA_2\xi_2 + (2 - P)bA_2\xi_1 = 0. \]
(3.41)
Adding (3.38) and (3.40), we obtain (3.32).
Adding the relation (3.39) and (3.41), we obtain (3.33).

**Theorem 3.18.** Let $M$ be a submanifold of a golden Riemannian manifold $\overline{M}$ and structure $J$ is parallel to Levi-Civita connection $\overline{\nabla}$ defined on $M$ (i.e $\overline{\nabla}J = 0$). If $\xi_\alpha$ ($\alpha = 1, 2, 3, \ldots, r$) are linearly independent, $T_\alpha(P) = \text{constant}$ and $M$ is totally umbilical, then $M$ is totally geodesic.

**Proof.** Since
\[ \nabla_X(a_{\alpha\beta}) = -u_\alpha(A_\beta X) - u_\beta(A_\alpha X) + \sum_\gamma [l_{\gamma\alpha}(X)a_{\gamma\beta} + l_{\gamma\beta}(X)a_{\gamma\alpha}]. \]
Putting $\alpha = \beta$, we have
\[ \nabla_X(a_{\alpha\alpha}) = -2u_\alpha(A_\alpha X) + \sum_\gamma [l_{\gamma\alpha}(X)a_{\gamma\alpha} + l_{\gamma\alpha}(X)a_{\gamma\alpha}]. \]
Since $a_{\alpha\beta}$ is symmetric and $l_{\alpha\beta}$ is skew-symmetric in $\alpha, \beta$, then $\sum_\alpha a_{\alpha\alpha}l_{\alpha\alpha}(X) = 0$.
Since, $T_\alpha(P) = \text{constant}$, we have $\sum_\alpha a_{\alpha\alpha} = \text{constant}.
Hence,
\[ \sum_\alpha u_\alpha(A_\alpha X) = 0 \]
\[ \sum_\alpha g(X, A_\alpha \xi_\alpha) = 0 \]
\[ \sum_\alpha A_\alpha \xi_\alpha = 0. \]
Since, $\xi_\alpha$ is linearly independent, then
\[ A_\alpha = 0. \]
Hence $M$ are totally geodesic.

**Theorem 3.19.** Let $M$ be a submanifold of a golden Riemannian manifold $\overline{M}$ and $J$ is parallel to Levi-Civita connection $\overline{\nabla}$ (i.e $\overline{\nabla}J = 0$). If $\xi_\alpha$ ($\alpha = 1, 2, \ldots, r$) are linearly independent, $\sum_\alpha(\nabla_\alpha P)e_\alpha = 0$ and $T_\alpha(P) = \text{constant}$, then $M$ is minimal.

**Proof.** Since,
\[ (\nabla_X P)(Y) = \sum_\alpha [g(A_\alpha X, Y)\xi_\alpha + u_\alpha(Y)A_\alpha X]. \]
Putting $X = Y = e_j$, we obtain
\[ \sum_j(\nabla_\alpha P)(e_j) = \sum_\alpha [A_\alpha \sum_j u_\alpha(e_j)e_j + \sum_j h_\alpha(e_j, e_j)\xi_\alpha]. \]
Using (3.8), we obtain
\[ \sum_j (\nabla_{e_j} P)(e_j) = \sum_{\alpha} [A_{\alpha} \xi_{\alpha} + \sum_j h_{\alpha}(e_j, e_j) \xi_{\alpha}] . \]

Since,
\[ T_r(P) = \text{constant}, \]
then from Theorem 3.18, we have
\[ \sum_{\alpha} h_{\alpha}(X, \xi_{\alpha}) = 0. \]

Therefore,
\[ \sum_{\alpha} g(A_{\alpha} X, \xi_{\alpha}) = 0 \]
then
\[ \sum_{\alpha} A_{\alpha} \xi_{\alpha} = 0. \]

Thus,
\[ \sum_{\alpha} \sum_j h_{\alpha}(e_j, e_j) \xi_{\alpha} = 0. \]

Since, \( \xi_{\alpha} \) are linearly independent, then
\[ h_{\alpha}(e_j, e_j) = 0. \]

Hence, \( M \) is minimal. \( \square \)

**Lemma 3.20.** Let \( M \) be a submanifold of a golden Riemannian manifold \( \overline{M} \). If \( \xi_\alpha \) (\( \alpha = 1, 2, ..., r \)) are linearly independent, then we have
\[ T_r(P) = -T_r(a_{\alpha\beta}), \]
where \( r = n \).

**Lemma 3.21.** Let \( M \) be a submanifold of a golden Riemannian manifold \( \overline{M} \). If \( \xi_\alpha \) (\( \alpha = 1, 2, ..., r \)) are linearly independent and \( \nabla_X P = 0 \), then \( T_r(a_{\alpha\beta}) = \text{constant} \).

**Proof.** Let \( \{e_1, e_2, ..., e_n\} \) be an orthogonal basis of \( T_P \) and extended \( e_j \) (\( j = 1, 2, ..n \)) to local vector field \( E_j \) which are covariantly constant at \( p \in M \).

Then at \( p \in M \),
\[ \nabla_X T_r(P) = \nabla_X \sum_j g(Pe_j, e_j) \]
\[ \nabla_X T_r(P) = \{ \sum_j g(\nabla_X (PE_j), E_j) \} P \]
\[ \nabla_X T_r(P) = \sum_j [g(\nabla_X P) E_j + P \nabla_X E_j, E_j] + g(PE_j, \nabla_X E_j) \]
\[ \nabla_X T_r(P) = \sum_j ((\nabla_X P) E_j, E_j) + \sum_j g(\nabla_X E_j, PE_j) + \sum_j g(\nabla_X E_j, PE_j) \]
\[ \nabla_X T_r(P) = 0. \]

Then,
\[ T_r(P) = \text{constant}. \]

From Lemma 3.13, we have
\[ T_r(a_{\alpha\beta}) = \text{constant}. \]
\( \square \)
Example 3.22. We consider that ambient space is a $(2a + b)$–dimensional Euclidean space $E^{2a+b}$ $(a, b \in \mathbb{N})$. Let $J : E^{2a+b} \rightarrow E^{2a+b}$ be an (1, 1) tensor field defined by

$$J(x^1, \ldots, x^a, y^1, \ldots, y^a, z^1, \ldots, z^b) = (\phi x^1, \ldots, \phi x^a, \phi y^1, \ldots, \phi y^a, (1 - \phi)z^1, \ldots, (1 - \phi)z^b)$$

for every point $(x^1, \ldots, x^a, y^1, \ldots, y^a, z^1, \ldots, z^b) \in E^{2a+b}$, where $\phi = \frac{1 + x^0}{2}$ and $1 - \phi = \frac{1 - x^0}{2}$ are roots of the equation $x^2 = x + 1$. On the other hand, for $(x^1, \ldots, x^a, y^1, \ldots, y^a, z^1, \ldots, z^b) \in E^{2a+b}$, we have

$$J^2(x^1, \ldots, x^a, y^1, \ldots, y^a, z^1, \ldots, z^b) = (\phi^2 x^1, \phi^2 x^2, \ldots, \phi^2 x^a, \phi^2 y^1, \ldots, \phi^2 y^a, (1 - \phi)^2 z^1, \ldots, (1 - \phi)^2 z^b)$$

and

$$J^2 = J + I.$$

For $(x^1, \ldots, x^a, y^1, \ldots, y^a, z^1, \ldots, z^b), (p^1, \ldots, p^a, q^1, \ldots, q^a, r^1, \ldots, r^b) \in E^{2a+b}$, we have

$$J((x^1, \ldots, x^a, y^1, \ldots, y^a, z^1, \ldots, z^b), (p^1, \ldots, p^a, q^1, \ldots, q^a, r^1, \ldots, r^b)) = ((\phi x^1, \ldots, \phi x^a, \phi y^1, \ldots, \phi y^a, (1 - \phi)z^1, \ldots, (1 - \phi)z^b), (p^1, \ldots, p^a, q^1, \ldots, q^a, r^1, \ldots, r^b))$$

for every $(x^1, \ldots, x^a, y^1, \ldots, y^a, z^1, \ldots, z^b), (p^1, \ldots, p^a, q^1, \ldots, q^a, r^1, \ldots, r^b) \in E^{2a+b}$. So, the product $\langle \rangle$ on $E^{2a+b}$ is $J$–compatible.

Therefore, $J$ is a golden structure defined on $(E^{2a+b}, \langle \rangle)$ and $(E^{2a+b}, \langle \rangle, J)$ is a golden Riemannian manifold.

In the following, we identify $iX$ with $X$ (where $X \in \mathfrak{X}(E^{2a+b}))$. It is obvious that $E^{2a+b} = E^a \times E^a \times E^b$ and in each of spaces $E^a, E^a$ and $E^b$ respectively, we can get a hypersphere

$$S^{a-1}(r_1) = \{(x^1, \ldots, x^a), \sum_{i=1}^{a}(x^i)^2 = r_1^2\},$$

$$S^{a-1}(r_2) = \{(y^1, \ldots, y^a), \sum_{i=1}^{a}(y^i)^2 = r_2^2\},$$

$$S^{b-1}(r_3) = \{(z^1, \ldots, z^b), \sum_{i=1}^{b}(z^i)^2 = r_3^2\}$$

respectively, where $r_1^2 + r_2^2 + r_3^2 = R^2$.

We construct the product manifold $S^{a-1}(r_1) \times S^{a-1}(r_2) \times S^{b-1}(r_3)$. Every point of $S^{a-1}(r_1) \times S^{a-1}(r_2) \times S^{b-1}(r_3)$ has the coordinate $(x^1, \ldots, x^a, y^1, \ldots, y^a, z^1, \ldots, z^b) = (x^i, y^j, z^k)$ $(i \in [1, a], j \in [1, b])$ such that:

$$\sum_{i=1}^{a}(x^i)^2 + \sum_{i=1}^{a}(y^i)^2 + \sum_{j=1}^{b}(z^j)^2 = R^2.$$

Thus, $S^{a-1}(r_1) \times S^{a-1}(r_2) \times S^{b-1}(r_3)$ is a submanifold of codimension $3$ in $E^{2a+b}$ and $S^{a-1}(r_1) \times S^{a-1}(r_2) \times S^{b-1}(r_3)$ is a submanifold of codimension $2$ in $S^{2a+b-1}(R)$ and $S^{a-1}(r_1) \times S^{a-1}(r_2) \times S^{b-1}(r_3)$ is a hypersurface in $S^{2a+b-1}(R)$. Therefore, we have

$$S^{a-1}(r_1) \times S^{a-1}(r_2) \times S^{b-1}(r_3) \hookrightarrow S^{2a+b-2}(r) \hookrightarrow S^{2a+b-1}(R) \hookrightarrow E^{2a+b}.$$

The tangent space in a point $(x^1, \ldots, x^a, y^1, \ldots, y^a, z^1, \ldots, z^b)$ at the product of spheres $S^{a-1}(r_1) \times S^{a-1}(r_2) \times S^{b-1}(r_3)$ is

$$T(x^1, \ldots, x^a, 0, 0, 0, 0) S^{a-1}(r_1) \bigoplus T(0, 0, 0) S^{a-1}(r_2) \bigoplus T(0, 0, 0) S^{b-1}(r_3).$$

A vector $(X^1, \ldots, X^a)$ from $T_{(x^1, \ldots, x^a)} E^a$ is tangent to $S^{a-1}(r_1)$ if and only if we have

$$\sum_{i=1}^{a} x^i X^i = 0$$

and it can be identified by $(X^1, \ldots, X^a, 0, 0, 0, 0, 0)$ from $E^{2a+b}$. 
A vector \((Y^1, \ldots, Y^a)\) from \(T_{(x^1, \ldots, x^a)}E^a\) is tangent to \(S^{a-1}(r_2)\) if and only if we have
\[
\sum_{i=1}^{a} y^i Y^i = 0
\]
and it can be identified by \((0, \ldots, Y^1, \ldots, 0)\) from \(E^{2a+b}\).

A vector \((Z^1, \ldots, Z^b)\) from \(T_{(z^1, \ldots, z^b)}E^b\) is tangent to \(S^{b-1}(r_3)\) if and only if we have
\[
\sum_{i=1}^{b} z^i Z^i = 0
\]
and it can be identified by \((0, \ldots, 0, \ldots, 0, Z^1, \ldots, Z^b)\) from \(E^{2a+b}\). Consequently, for every point \((x^i, y^j, z^k) \in S^{a-1}(r_1) \times S^{a-1}(r_2) \times S^{b-1}(r_3)\), we have
\[
(X^i, Y^j, Z^k) = (x^i, y^j, z^k) E \in T_{(x^i, y^j, z^k)}E^{2a+b}(r)
\]
for every point \((x^i, y^j, z^k) \in S^{a-1}(r_1) \times S^{a-1}(r_2) \times S^{b-1}(r_3)\). We consider a local orthonormal basis \((N_1, N_2, N_3)\) of \(T_{(x^i, y^j, z^k)}E^{2a+b}(r)\) in every point \((x^i, y^j, z^k) \in S^{a-1}(r_1) \times S^{a-1}(r_2) \times S^{b-1}(r_3)\) given by
\[
N_1 = \frac{1}{R} (x^i, y^j, z^k),
N_2 = \frac{1}{R} (x^i, y^j, -z^k),
N_3 = \frac{1}{r_3} \left( r_2 x^i, -r_1 y^j, 0 \right).
\]

From decomposition of \(J(N_\alpha) (\alpha \in \{1, 2, 3\})\) in tangential and normal components at \(S^{a-1}(r_1) \times S^{a-1}(r_2) \times S^{b-1}(r_3)\), we obtain
\[
J(N_\alpha) = \xi_\alpha + a_{\alpha 1} N_1 + a_{\alpha 2} N_2 + a_{\alpha 3} N_3,
\]
where \(\alpha \in \{1, 2, 3\}\).

(i) From \(a_{\alpha \beta} = (J(N_\alpha), N_\beta) (\alpha, \beta \in \{1, 2, 3\})\), we obtain
\[
a_{11} = a_{22} = \frac{1}{R^2} (\phi r_1^2 + \phi r_2^2 + (1 - \phi) r_3^2),
a_{12} = a_{21} = \frac{1}{R^2} (\phi r_1^2 + \phi r_2^2 - (1 - \phi) r_3^2),
a_{13} = a_{23} = 0 = a_{31} = a_{32},
a_{33} = \frac{\phi r_2^2 + \phi r_3^2}{r_3^2}.
\]

Thus, the matrix \(\Lambda = (a_{\alpha \beta})_{3}\) is given by
\[
\begin{pmatrix}
\frac{1}{R^2} (\phi r_1^2 + \phi r_2^2 + (1 - \phi) r_3^2) & \frac{1}{R^2} (\phi r_1^2 + \phi r_2^2 - (1 - \phi) r_3^2) & 0 \\
\frac{1}{R^2} (\phi r_1^2 + \phi r_2^2 - (1 - \phi) r_3^2) & \frac{1}{R^2} (\phi r_1^2 + \phi r_2^2 + (1 - \phi) r_3^2) & 0 \\
0 & 0 & \frac{\phi r_2^2 + \phi r_3^2}{r_3^2}
\end{pmatrix}
\] (3.42)

(ii)
\[
\xi_1 = \frac{(R - 2r_3)}{R^3} (\phi x^i, \phi y^j, (1 - \phi) z^k),
\xi_2 = \frac{(R - 2r_3)}{R^3} (\phi x^i, \phi y^j, -(1 - \phi) z^k),
\xi_3 = \frac{r_2 \phi (1 - \phi)}{r_1 r_3} x^i, \frac{r_1 \phi (1 - \phi)}{r_2 r_3} y^j, 0.
\] (3.43)
(iii) From $u_4(X) = u(X', Y', Z') = \langle (X', Y', Z'), \xi_4 \rangle$, we obtain

$$u_1 = \frac{1}{R}(\phi X'i^i + \phi Y'i^i + (1 - \phi)Z'j^j), \quad (3.46)$$

$$u_2 = \frac{1}{R}(\phi X'i^i + \phi Y'i^i - (1 - \phi)Z'j^j), \quad (3.47)$$

$$u_3(X) = \frac{1}{r_3} \left( \frac{r_2}{r_1} \phi X'i^i - \frac{r_1}{r_2} \phi Y'i^i + Z'j^j \right). \quad (3.48)$$

(iv) $P(X) = (\phi X'i^i - \frac{2\phi}{R^2}(X'i^i + Y'i^i) - \frac{r_1}{r_2} \left( \frac{r_2}{r_1} \phi X'i^i - \frac{r_1}{r_2} \phi Y'i^i + z'i^i \right) \phi Y'i^i + \frac{r_1}{r_2} \phi X'i^i - \frac{r_1}{r_2} \phi Y'i^i + z'i^i \phi Y'i^i - \frac{2\phi}{R^2}(X'i^i + Y'i^i) + r_1 \left( \frac{r_1}{r_2} \phi X'i^i - \frac{r_1}{r_2} \phi Y'i^i + z'i^i \right) \phi Y'i^i + \frac{r_1}{r_2} \phi X'i^i - \frac{r_1}{r_2} \phi Y'i^i + z'i^i \phi Y'i^i - \frac{2\phi}{R^2}(X'i^i + Y'i^i) + \frac{r_1}{r_2} \phi X'i^i - \frac{r_1}{r_2} \phi Y'i^i + z'i^i \phi Y'i^i - \frac{2\phi}{R^2}(X'i^i + Y'i^i) + \frac{r_1}{r_2} \phi X'i^i - \frac{r_1}{r_2} \phi Y'i^i + z'i^i \phi Y'i^i - \frac{2\phi}{R^2}(X'i^i + Y'i^i). \quad (3.49)$

Thus, we have $J(T_{(x', y')}(S^{a-1}(r_1) \times S^{a-1}(r_2) \times S^{b-1}(r_3))) \subseteq (T_{(x', y')}(S^{a-1}(r_1) \times S^{a-1}(r_2) \times S^{b-1}(r_3)))$ and we obtain $(P, \xi_4, u_4, (a_{ij}))$ induced structure on $(S^{a-1}(r_1) \times S^{a-1}(r_2) \times S^{b-1}(r_3))$ by the golden Riemannian structure $(J, \xi)$ on $E^{2a+b}$, which is effectively determined by the relations (3.41), (3.43), (3.44), (3.45), (3.46), (3.47), (3.48) and (3.49).

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**CONFLICTS OF INTEREST**

The authors declare that there are no conflicts of interest regarding the publication of this article.

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