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Is There Any Metric Line Which Can Be Represented By A Single Fixed Point?

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Abstract

In this paper, determining metric lines in Levenberg plane, a special metric line example which can be represented by a single fixed point is presented. Moreover, we prove nonexistence of periodic lines in Levenberg plane and give a problem whether there exists a distance space in which periodic lines represented by a single fixed point?

Keywords: Metric spaces, functional equations of metric and periodic lines and their solutions, Levenberg plane

Tek Sabit Nokta Yardımıyla Temsil Edilebilen Metrik Doğru Var Mıdır?

Öz

Bu çalışmada Levenberg düzleminde metrik doğruların tanımlanmasıyla tek sabit nokta ile temsil edilebilen özel bir metrik doğru örneği verilmiştir. Ayrıca Levenberg düzleminde hiçbir periyodik doğrunun olmadığı gösterilmiş olup, tek sabit noktalı periyodik doğruların bulunduğu uzaklık uzaylarının var olup olmayacağına dair bir problem sunulmuştur.

Anahtar Kelimeler: Metrik uzaylar, metrik ve periyodik doğruların fonksiyonel denklemleri ve çözümleri, Levenberg düzlemi

Introduction

A real distance space $\Delta = (\mathbb{S}, d)$ is a non-empty set $\mathbb{S}$ together with a mapping $d: \mathbb{S} \times \mathbb{S} \to \mathbb{R}$. The elements of $\mathbb{S}$ are called points and $d(x, y)$ is said to be the distance of the (ordered pair of) points x, y. The subset k of $\mathbb{S}$ is called a metric line of $\Delta = (\mathbb{S}, d)$ if and only if there exists a bijection $f: k \to \mathbb{R}$ such that [1]

$$d(x, y) = |f(x) - f(y)|$$

for all $x, y \in k$.

In [2], W. Benz proved that the distance space (k, d) is a metric space for every metric line k of $\Delta = (\mathbb{S}, d)$. Moreover W. Benz characterized the lines of Euclidean and hyperbolic geometries as metric lines in the sense of Blumenthal-Menger as follows:

Let X be a real inner product space of dimension ≥ 2, that $\mathbb{S} = X$ and

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$d(x, y) = \|x - y\|$ for $x, y \in X$. W. Benz proved that all metric lines in Euclidean geometry $\Delta = (\mathbb{S}, d)$ are given by

$\{p + tq: \quad t \in \mathbb{R}\}$

with $p, q \in X$ such that $\|q\| = 1$. The metric lines in Weierstrass model of hyperbolic geometry $\Delta = (\mathbb{S}, d)$ are determined by W. Benz [2] as

$\{pcosht + qsinht: \quad t \in \mathbb{R}\}$

where $\mathbb{S}$ is a a real inner product space of dimension ≥ 2,

$coshd(x, y) = \sqrt{1 + x^2}\sqrt{1 + y^2} - xy,$

p, q are arbitrary points of X with $pq = \langle p, q\rangle = 0$ and $qq = 1$.

If X is a real inner product space of dimension ≥ 2, then the metric lines in the Poincaré ball model of hyperbolic geometry $\Delta = (\mathbb{S}, d)$ is defined by O. Demirel et.al. [3] as

$\{p \oplus q \otimes t: \ t \in \mathbb{R}\}$

with $\|q\| = tanh1$, where

$\mathbb{S} = \{p \in X: \ \|p\| < 1\}$

and

$tanhd(x, y) = \|x \ominus y\|$.

Here, the notations " $\oplus$ ", " $\ominus$ "and " $\otimes$ " denote the Möbius addition, Möbius subtraction and Möbius scalar product in $\mathbb{S}$, respectively. For more details, we refer [3]. Moreover, O. Demirel and E.S. Sey-rantepe [4] compared the cogyrolines of Möbius gyrovector spaces which are defined by

$x(t) = p \otimes t \oplus q, \quad t \in \mathbb{R}$

($\|p\| = tanh1$) to metric lines of itself, where

$\mathbb{S} = \{X \in V: \| X \| < 1\}$

and

$tanhd(X, Y) = \| X \boxminus Y \|$.

" $\boxminus$ " denotes Möbius cosubtraction in $\mathbb{S}$. The metric lines in de Sitter's world is studied by R. Höfer in [5]. It is easy to see that the representation of the metric lines above are written by one parameter and one point (or two points). Clearly this representations can be written in different forms, in other words, the representative point p in the equation of a metric line is not uniquely determined. The main purpose of this paper is give a special metric line example which can be represented by one parameter and a single fixed point. The problem to find all metric lines of $\Delta = (\mathbb{S}, d)$ can be expressed as follows:

Problem 1. Determine all injective functions $x: \mathbb{R} \to \mathbb{S}$ such that

$$d(x(\xi), x(\eta)) = |\xi - \eta| \qquad (1)$$

holds true for all $\xi, \eta \in R$, see [1]. This is a functional equation, see [1], [2].

Suppose that ρ is a positive number. A subset k of $\mathbb{S}$ is a ρ −periodic line of the real distance space $\Delta = (\mathbb{S}, d)$ if, and only if, there exists a bijection

$f: k \to \ [0, \rho) \ = \{\varepsilon \in R: \ 0 \leq \varepsilon < \rho\}$

such that

$$d(x,y) = \begin{cases} |f(x)-f(y)|, & \text{if } |f(x)-f(y)| \leq \frac{\rho}{2} \\ \rho - |f(x)-f(y)|, & \text{if } |f(x)-f(y)| > \frac{\rho}{2} \end{cases}$$

holds true for all $x, y \in k$.

The problem to find all periodic lines of $\Delta = (\mathbb{S}, d)$ can be expressed as follows:

Problem 2. Determine all injective functions $x: [0,\rho) \to \mathbb{S}$ such that

$$d(x(\xi), x(\eta)) = \begin{cases} |\xi-\eta|, & \text{if } |\xi-\eta| \leq \frac{\rho}{2} \\ \rho - |\xi-\eta|, & \text{if } |\xi-\eta| > \frac{\rho}{2} \end{cases} \qquad (2)$$

holds true for all $\xi, \eta \in \mathbb{R}$, see [2].

Let X be a real inner product space of dimension ≥ 2. Define the real distance spaces $\Delta_i = (\mathbb{S}_i, d_i)$ for $i = 1,2$ as

$$\mathbb{S}_1 = \{x \in X : \langle x, x\rangle = 1\}$$

and

$$\mathbb{S}_2 = \{\{x, -x\} \subset X : \langle x, x\rangle = 1\}$$

with

$$\cos d_1(x,y) = \langle x, y\rangle$$

for $d_1(x,y) \in [0,2\pi)$, where $x, y \in \mathbb{S}_1$ and

$$\cos d_2(\{x,-x\},\{y,-y\}) = |\langle x, y\rangle|$$

for $d_2(\{x,-x\},\{y,-y\}) \in [0,\frac{\pi}{2})$, where $\{x,-x\},\{y,-y\} \in \mathbb{S}_2$. The distance spaces $\Delta_i = (\mathbb{S}_i, d_i)$ are metric spaces. Spherical and elliptic geometries over X are based on Δ_1 and Δ_2, respectively. W. Benz proved that the $2\pi-$ periodic lines in $\Delta_1 = (\mathbb{S}_1, d_1)$ are given by

$$x(\xi) = p\cos\xi + q\sin\xi, \qquad \xi \in [0,2\pi)$$

with $p, q \in S$ such that $\langle p, q\rangle = 0$, and $\pi-$ periodic lines in $\Delta_2 = (\mathbb{S}_2, d_2)$ are given by

$$x(\xi) = R(p\cos\xi + q\sin\xi), \qquad \xi \in [0,\pi)$$

with $p, q \in X$ such that $\langle p, p\rangle = 1$, $\langle q, q\rangle = 1$ and $\langle p, q\rangle = 0$. In this equation $R = \{1,-1\}$, $Rx = \{x, -x\}$.

In [3], O. Demirel et al. proved that there do not exist periodic lines in Poincaré ball model of hyperbolic geometry.

Levenberg Plane

The Levenberg plane is generalization of the Moulton plane which is well known as an example of an affine plane whose completion to projective plane does not satisfy the Desargues theorem. The set of points of Levenberg plane is the same as Euclidean analytical plane $\mathbb{R}^2$. The set of lines consists of vertical and horizontal Euclidean lines, and set of the form

$$\left\{(x,y) \in \mathbb{R}^2 :\ y = \begin{cases} mx+b, \text{if } x \leq 0 \\ cmx+b, \text{if } x > 0 \end{cases},\ m \in R, c \in \mathbb{R}^+, c \text{ is constant} \right\}$$

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Notice that a Levenberg line has a slope m such that $m = 0$ or $m \to \infty$ is the same as the Euclidean line as shown in Figure 1.

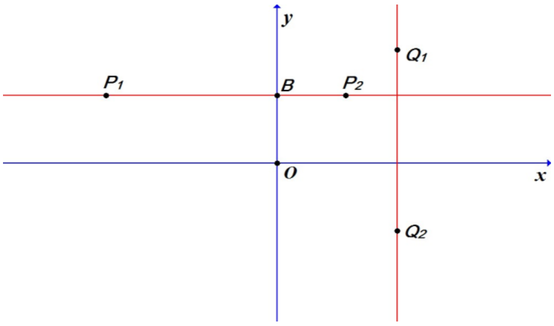

Figure1. *Levenberg lines have a slope m such that $m = 0$ or $m \to \infty$ are the same as the Euclidean lines.*

On the other hand, a Levenberg line gets bent as is passed across the y −axis as shown in Figure 2.

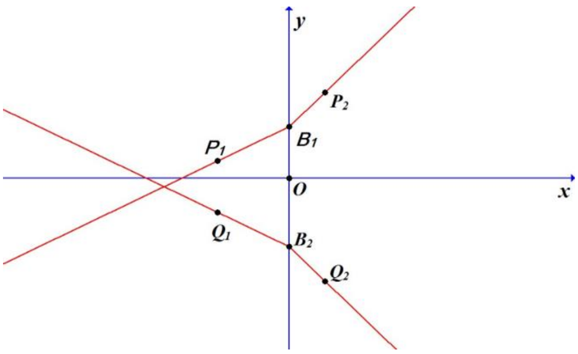

Figure 2. *Levenberg lines get bent as it passed across the y-axis.*

Therefore, the set of the Levenberg line L_L is the union of $L_{m,b}$ and L_a, where

$$L_{m,b} = \left\{(x,y) : \; y = \begin{cases} mx + b, x \leq 0 \\ cmx + b, x > 0 \end{cases}\right.$$

$$\left. m \in R, c \in \mathbb{R}^+, c \; is \; constant \right\}$$

and

$$L_a = \{(x,y) \in \mathbb{R}^2 : x = a, a \in \mathbb{R}\}.$$

Notice that the Levenberg lines are geodesics in this geometry. As in the representation of an Euclidean line in a parametric form, a Levenberg line $L_{m,b}$ can be expressed by a parametric form via

$$\alpha(t) = \begin{cases} B + tu, & \text{if } t \geq 0 \\ B - tv, & \text{if } t < 0 \end{cases}$$

where B is a point on y −axis, $\| u \| = \| v \| = 1$, and $d_L(u,v) = \| u \| + \| v \|$ $(u, v \in \mathbb{R}^2)$. Here, $\|\cdot\|$ denotes the Euclidean norm and d_L denotes the Levenberg distance function which is defined as follows:

The Levenberg distance between the points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ in $\mathbb{R}^2$ is defined by

$$d_L(P_1, P_2) = \begin{cases} d_E(P_1, B) + d_E(B, P_2), & \text{if } P_1, P_2, B \text{ lie on a line } l \in L_{m,b}, \; x_1 x_2 < 0 \\ d_E(P_1, P_2), & \text{otherwise} \end{cases} \quad (3)$$

where $B = (0, b) \in \mathbb{R}^2$ and d_E is Euclidean distance. Notice that the point B is defined uniquely for $P_1, P_2 \in \mathbb{R}^2$.

The distance of two points in Levenberg plane may not be invariant under Euclidean translations. More precisely,

$$d_L(P_1, P_2) = d_L(T_w(P_1), T_w(P_2))$$

holds true for all $P_1, P_2 \in \mathbb{R}^2$ (where $T_w: \mathbb{R}^2 \to \mathbb{R}^2$ such that $T_w(u) = w + u$) if and only if w lies on the $y-$axis. There are even more interesting properties of Levenberg plane, for instance, triangle inequality and Pythagorean theorem are fail in Levenberg plane. For more details we refer [6].

Metric and Periodic Lines in Levenberg Plane

In this section we define the real distance space $\mathbb{S} = \mathbb{R}^2$ and $d(x, y) = d_L(x, y)$ as in the equation (3) for all $x, y \in \mathbb{S}$.

Theorem 3. The metric lines of $\Delta = (\mathbb{S}, \mathrm{d})$ are exactly the classical lines of Δ.

Proof. Let us consider the Levenberg line

$$\alpha(t) = \begin{cases} B + tu, & \text{if } t \geq 0 \\ B - tv, & \text{if } t < 0 \end{cases}$$

where B is a point on $y-$ axis satisfying $\| u \| = \| v \| = 1$, $d_L(u, v) = \| u \| + \| v \|$ for $u, v \in \mathbb{R}^2$. Clearly, for all $\xi > 0$ and $\eta < 0$, we get

$$\begin{aligned} d_L(\alpha(\xi), \alpha(\eta)) &= \|\xi u\| + \|\eta v\| \\ &= |\xi| + |\eta| \\ &= |\xi - \eta| \end{aligned}$$

For the all cases of $\xi, \eta \in \mathbb{R}$ satisfying $\xi\eta \geq 0$, we immediately get

$$d_L(\alpha(\xi), \alpha(\eta)) = |\xi - \eta|$$

The other type of Levenberg lines which have a slope m such that $m = 0$ or $m \to \infty$ are already metric lines, see [2]. Thus, $\alpha(t)$ is a metric line of Δ.

Suppose that the function $x: \mathbb{R} \to \mathbb{S}$ solves (1) for all $\xi, \eta \in \mathbb{R}$. If the metric line, defined by x, does not meet $y-$axis at any point, thus we immediately get that the metric line must be a classical line with slope $m \to \infty$, see [2].

Now assume that the metric line defined by x meets $y-$ axis. In this case, without loss of generality, we may assume $x(0) = 0$. Since x is a solution of (1), we have

$$\begin{aligned} d_L(x(\xi), x(0)) &= \| x(\xi) - x(0) \| \\ &= |\xi - 0| \\ &= |\xi| \end{aligned}$$

for all $\xi \in \mathbb{R}$. Let us assume $\xi > 0$. Clearly, there are two cases such as

$$d_L(x(\xi), x(1)) = \| x(\xi) - B \| + \| B - x(1) \| \qquad (4)$$

where B is a point on $y-$ axis, or

$$d_L(x(\xi), x(1)) = \| x(\xi) - x(1) \|. \qquad (5)$$

Now we consider the first case (4). It is not hard to see that the points $x(\xi)$ and $x(1)$ must be in different sides of $y-$ axis. There are two subcases:

Subcase 1: Firstly assume $B = x(0) = 0$, that is the points $x(\xi), x(1)$ and $x(0)$ lie on the same Levenberg line. Observe by (1), we get

$$\begin{aligned} |\xi - 1| &= \| x(\xi) - B \| + \| B - x(1) \| \\ &= \xi + 1 \end{aligned}$$

and clearly this is not possible for all $\xi > 0$.

Subcase 2: Now assume $B \neq x(0)$, i.e. the points $x(\xi), x(1)$ and $x(0)$ are not collinear in the Levenberg sense. Put $\| B \| = b$. Here, we need to apply Euclidean triangle inequality to triangles $\Delta 0 B x(\xi)$ and $\Delta 0 x(1) B$ for six different cases. Notice that in these six cases

$$d_L(x(\xi), x(1)) = \| x(\xi) - B \| + \| B - x(1) \|$$

(i) If $b \leq \xi \leq 1$ holds, then

$$\begin{aligned} d_L(x(\xi), x(1)) &> |\xi - b| + |1 - b| \\ &= \xi + 1 - 2b \end{aligned}$$

holds. By (1), we get

$|\xi - 1| > \xi + 1 - 2b$

which contradicts $b \leq \xi \leq 1$.

(ii) If $b \leq 1 \leq \xi$ holds, then the proof is clear by (i).

(iii) If $1 \leq b \leq \xi$ holds, then one can easily get

$$\begin{aligned} d_L(x(\xi), x(1)) &> |\xi - b| + |1 - b| \\ &= \xi - 1. \end{aligned}$$

By (1), we have $|\xi - 1| > \xi - 1$ which contradicts $1 \leq b \leq \xi$.

(ıv) If $1 \leq \xi \leq b$ holds, then we get

$$\begin{aligned} d_L(x(\xi), x(1)) &> |\xi - b| + |1 - b| \\ &= -\xi - 1 + 2b. \end{aligned}$$

By (1), $|\xi - 1| > -\xi - 1 + 2b$ which contradicts $1 \leq \xi \leq b$.

(v) If $\xi \leq b \leq 1$ holds, then obviously

$$\begin{aligned} d_L(x(\xi), x(1)) &> |\xi - b| + |1 - b| \\ &= -\xi + 1 \end{aligned}$$

and by (1), we get $|\xi - 1| > -\xi + 1$ which contradicts $\xi \leq b \leq 1$.

(vı) If $\xi \leq 1 \leq b$ holds, then

$$\begin{aligned} d_L(x(\xi), x(1)) &> |\xi - b| + |1 - b| \\ &= -\xi - 1 + 2b \end{aligned}$$

is satisfied. Clearly, by (1), this yields $|\xi - 1| > -\xi - 1 + 2b$ which contradicts $\xi \leq 1 \leq b$.

Therefore, (5) must be valid for $x(1)$ and $x(\xi)$ with $\xi > 0$. In this case, we get

$$x(\xi) = \varphi(\xi) \cdot x(1) \text{ for } \xi > 0 \tag{6}$$

with $\varphi(\xi) = \xi,$ see [2].

If $\xi < 0$, following the same way above, one can easily get

$$x(\xi) = \varphi(\xi) \cdot x(-1) \text{ for } \xi < 0 \tag{7}$$

with $\varphi(\xi) = -\xi$.

Clearly, the equations (6) and (7) define two different rays in $\mathbb{R}^2$ start at $x(0)$ and goes off in a certain direction forever, to infinity.

Naturally, one may wonder whether these rays lie on a common Levenberg line? In order to see this, we need to check that the collinearity property of the points $x(-1)$, $x(0)$ and $x(1)$ in the Levenberg sense. It is easy to see that the points $x(-1)$ and $x(1)$ must be lie on different sides of y −axis; otherwise, by using Euclidean triangle inequality to $\Delta x(-1)x(0)x(1)$ and by (1), we get

$$2 = |1-(-1)| = \| x(1) - x(-1) \| < \| x(1) \| + \| x(-1) \| = 2.$$

which is not correct. Now, assume that these rays do not lie on a common Levenberg line. Therefore, the line which passes through $x(-1)$ and $x(1)$ intersect y −axis at a point, say K, different from 0. By using Euclidean triangle inequality for the triangles $\Delta x(-1)Kx(0)$ and $\Delta x(1), x(0), K$, we immediately get

$$\begin{aligned} \| x(-1) - K \| &< \| K \| + \| x(-1) \| \\ &= \| K \| + 1 \end{aligned}$$

and

$$\begin{aligned} \| x(1) - K \| &< \| K \| + \| x(1) \| \\ &= \| K \| + 1 \end{aligned}$$

respectively. Thus, we have

$$\begin{aligned} \| x(-1) - x(1) \| &= \| x(-1) - K \| + \\ &\quad \| x(1) - K \| \\ &< 2 + 2 \| K \| \end{aligned}$$

i.e.

$$\| x(-1) - x(1) \| < 2 + 2 \| K \| \neq 2$$

which contradicts (1). Consequently, the points $x(-1), x(0)$ and $x(1)$ must be collinear, i.e. the rays mentioned above are lie on a common Levenberg line. Then by (6) and (7),

$$x(\xi) = \begin{cases} \xi x(1), & \text{if } \xi \geq 0 \\ -\xi x(-1), & \text{if } \xi < 0 \end{cases} \tag{8}$$

must be a classical line. Clearly, if

$$d_E(x(-1), x(1)) = 2$$

holds where d_E denotes Euclidean distance, then (8) is a Euclidean line with slope $m = 0$ which is also a Levenberg line.

Remark 4. In the proof of the theorem we have assumed $x(0) = 0$. If we remove this condition, one can easily find an appropriate translation g which sends $x(t_0)$ to 0, where $x(t_0)$ lies on the y −axis. Therefore, $x \circ g$ is a solution of (1).

Remark 5. As we mentioned before, the triangle inequality is fail in Levenberg plane, however this situation is not valid for some triangles. More precisely, if ΔABC is a triangle whose vertices are lie on the same side of y −axis (including being on), then triangle inequality is valid.

Theorem 6. For all $\rho > 0$, then there do not exist ρ −periodic lines in $\Delta = (\mathbb{S}, \mathrm{d})$.

Proof. Let us assume $x: [0, \rho) \to \mathbb{S}$ be a solution of (2) for a certain $\rho > 0$. Without loss of generality, we may assume $x(0) = 0$. By (2), we get

$$\begin{aligned} \| x(\frac{\rho}{2}) \| &= \| x(\frac{\rho}{2}) - x(0) \| \\ &= \left|\frac{\rho}{2} - 0\right| \\ &\left|\frac{\rho}{2}\right|. \end{aligned}$$

Clearly, for all $0 \leq \xi \leq \frac{\rho}{2}$, we have $\| x(\xi) \| = \xi$. If $0 \leq \xi, \eta \leq \frac{\rho}{2}$, following the way in the proof of the Theorem 3 and by (2), we immediately obtain

$$d_L\big(x(\xi), x(\eta)\big) = \| x(\xi) - x(\eta) \| \qquad (9)$$

Obviously, (9) implies that $x(\xi)$, $x(0)$ and $x(\eta)$ must be lie on the same Levenberg plane. Thus, we get

$$x(\xi) = \varphi(\xi) x\left(\frac{\rho}{2}\right), \quad \xi \in [0, \frac{\rho}{2})$$

with $\varphi(t) = \frac{2t}{\rho}$, see [2].

If $\frac{\rho}{2} < \zeta < \rho$, then, by (2),

$$\begin{aligned} \| x(\zeta) \| &= d\big(x(\zeta), x(0)\big) \\ &= \rho - |\zeta - 0| \\ &= \rho - \zeta. \end{aligned}$$

For $\zeta \in (\frac{\rho}{2}, \rho)$, there are two cases such as

$$d\left(x(\zeta), x\left(\frac{\rho}{2}\right)\right) = \| x(\zeta) - B \| + \| B - x\left(\frac{\rho}{2}\right) \| \qquad (10)$$

where B is a point on $y-$axis, or

$$d\left(x(\zeta), x\left(\frac{\rho}{2}\right)\right) = \| x(\zeta) - x\left(\frac{\rho}{2}\right) \| \qquad (11)$$

First, let us assume (10). It is not hard to see that the points $x(\zeta)$ and $x\left(\frac{\rho}{2}\right)$ must be on different sides of $y-$ axis. There are two subcases:

Subcase 1: Firstly assume $B = x(0) = 0$, i.e. the points $x(\zeta), x\left(\frac{\rho}{2}\right)$ and $x(0)$ are collinear in the Levenberg sense. Observe by (2), we get

$$\begin{aligned} |\zeta - \frac{\rho}{2}| &= d\left(x(\zeta), x\left(\frac{\rho}{2}\right)\right) \\ &= \| x(\zeta) \| + \| x\left(\frac{\rho}{2}\right) \| \\ &= (\rho - \zeta) + \frac{\rho}{2} \end{aligned}$$

which is not possible for all $\zeta \in (\frac{\rho}{2}, \rho)$.

Subcase 2: Now assume $B \neq x(0)$, i.e. the points $x(\zeta), x\left(\frac{\rho}{2}\right)$ and $x(0)$ are not collinear in the Levenberg sense. Put $\| B \| = b$. As in the proof of Theorem 3, we need to apply Euclidean triangle inequality for the triangles $\Delta 0 B x(\zeta)$ and $\Delta 0 x\left(\frac{\rho}{2}\right) B$ in three different cases. Notice that in these cases

$$d_L(x(\zeta), x(\frac{\rho}{2})) = \| x(\zeta) - B \| + \| B - x(\frac{\rho}{2}) \|.$$

(i) If $b \leq \rho - \zeta < \frac{\rho}{2}$, then

$$\begin{aligned} d_L(x(\zeta), x(\tfrac{\rho}{2})) &> |b - \rho + \zeta| + |\frac{\rho}{2} - b| \\ &= \frac{3\rho}{2} - \zeta - 2b \end{aligned}$$

and by (2), we have $|\zeta - \frac{\rho}{2}| > \frac{3\rho}{2} - \zeta - 2b$ which contradicts $b \leq \rho - \zeta < \frac{\rho}{2}$.

(ii) If $\rho - \zeta \leq b < \frac{\rho}{2}$, then

$$\begin{aligned} d_L(x(\zeta), x(\tfrac{\rho}{2})) &> |b - \rho + \zeta| + |\frac{\rho}{2} - b| \\ &= \zeta - \frac{\rho}{2} \end{aligned}$$

is satisfied. By (2), we get $|\zeta - \frac{\rho}{2}| > \zeta - \frac{\rho}{2}$ which contradicts $\rho - \zeta \leq b < \frac{\rho}{2}$.

(iii) If $\rho - \zeta < \frac{\rho}{2} \leq b$, then one get

$$\begin{aligned} d_L(x(\zeta), x(\tfrac{\rho}{2})) &> |b - \rho + \zeta| + |\frac{\rho}{2} - b| \\ &= \zeta + 2b - \frac{3\rho}{2} \end{aligned}$$

By (2), this yields $|\zeta - \frac{\rho}{2}| > \zeta + 2b - \frac{3\rho}{2}$ which contradicts $\rho - \zeta < \frac{\rho}{2} \leq b$. Hence, (11) must be valid for all $\zeta \in (\frac{\rho}{2}, \rho)$ and this implies $x(\zeta)$, $x\left(\frac{\rho}{2}\right)$ and $x(0)$ are collinear points in the Levenberg sense. Moreover,

$$x(\zeta) = \phi(\zeta) x\left(\frac{\rho}{2}\right), \quad \zeta \in (\frac{\rho}{2}, \rho)$$

with $\phi(t) = \frac{2}{\rho}(\rho - t)$, see [2]. Notice that $x\left(\frac{\rho}{4}\right) = x\left(\frac{3\rho}{4}\right)$, but this contradicts

$$|\frac{3\rho}{4} - \frac{\rho}{4}| = \| x\left(\frac{\rho}{4}\right) - x\left(\frac{3\rho}{4}\right) \|.$$

Hence, there do not exist ρ −periodic lines in $\Delta = (\mathbb{S}, d)$.

Problem 3 Is there any distance space $\Delta = (\mathbb{S}, \mathrm{d})$ in which ρ −periodic lines can be represented by a single fixed point?

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