

New mixed-dark soliton solutions to the hyperbolic generalization of the Burgers equation

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Abstract

In this paper, we apply the exponential function method to find mixed-dark, exponential and singular soliton solutions in the hyperbolic generalization of the Burgers equation. We obtain some entirely new mixed singular and dark soliton solutions. Under the suitable values of parameters, various dimensional simulations of results are plotted. Finally, we present a conclusion by giving novelties of paper.

Keywords: Exponential function method, Hyperbolic Generalization of the Burgers equation, mixed-dark and singular soliton solutions.

Genelleştirilmiş hiperbolik Burgers denkleminin yeni mixed-dark soliton çözümleri

Özet

Bu yazıda, Burgers denkleminin hiperbolik genelleştirilmesinde mixed-dark, üstel ve tekil çözümleri bulmak için üstel fonksiyon yöntemini kullanıyoruz. Tamamen yeni karışık tekil (mixed singular) ve dark soliton çözümleri elde ediyoruz. Parametrelerin uygun değerleri altında, sonuçların çeşitli boyutsal simülasyonları çizilmiştir. Son olarak, makalemizde yeni bir sonuç sunuyoruz.

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Anahtar kelimeler: Üstel fonksiyon metodu, Burgers denkleminin hiperbolik genelleştirilmesi, mixed-dark ve tekil çözümleri.

1.Introduction

In recent several decades, nonlinear evolution equations (NLEEs) arising in optical fiber, applied sciences, plane of symmetry, nonsymmetry and antisymmetry, and nonlinear algebraic structures, supergravitational field have been used to symbolize real world problem in the point of mathematical views. For the last few years, many scientist have converted NLEEs, especially, real word problems, into nonlinear ordinary differential equations (NODEs) to find new soliton solutions [1-3]. Such as the improved Bernoulli sub-equation function method [4-7], the extended sinh-Gordon equation expansion method [8-11] the exponential function method[12], the modified $\exp(-\phi(\eta))$ -expansion function [13] the implicit finite difference scheme and the Dufort–Frankel finite difference scheme methods [14] and the difference schemes method [15]. Therefore, papers conducted on travelling wave solutions along with important partial differential equations have attracted attention of researchers from all over the world . Moreover, these models symbolize the surfaces of wave from time to time. One of these models is the hyperbolic generalized Burgers equation (GBE) defined as [16]

$$\tau u_{tt} + u_t + uu_x + Bu_x - \kappa u_{xx} - \lambda u(u-s)(u+q) = 0, \quad (1)$$

where $\tau, B, \kappa, \lambda, s, q$ are real numbers with non-zero. Eq.(1) is hyperbolic model under the terms of $\kappa > 0$. Eq.(1) is used to describe interaction of the wave fronts [16]. V. A. Vladimirov *et al* have applied Hirota method and found exact solutions to the model. Moreover, some significant physical properties of the Burgers and its generalized versions have been presented by A. S. Makarenko and et al [17,18].

2.General properties of EFM

Here, we shortly give the main steps of the EFM. Let us consider the nonlinear partial differential equation (NPDE)

$$P(u, u_t, u_x, u^2, u_u^3, \dots) = 0, \quad (2)$$

where $u = u(x, t)$ is the unknown function, and P is a polynomial in u .

Step 1: By taking the transformation as:

$$u(x, t) = U(\xi), \quad \xi = kx - ct, \quad (3)$$

from Eq.(2), we obtain the nonlinear ordinary differential equation (NODE)

$$N(U, U', U'', \dots) = 0, \quad (4)$$

which N is a polynomial of U .

Step 2: Let us now assume the solution of equation (4) to have the form:

$$U = U(\xi) = \sum_{i=0}^N A_i \left[e^{-\Omega(\xi)} \right]^i, \tag{5}$$

where $A_i, (0 \leq i \leq N)$ are constants to be obtained later, such that $A_N \neq 0$, and $\Omega = \Omega(\xi)$ solves the following ODE;

$$\Omega' = \exp(-\Omega) + \mu \exp(\Omega) + w. \tag{6}$$

Eq.(6) is of the following set of solutions;

Family-1: When $\mu \neq 0, w^2 - 4\mu > 0$,

$$\Omega(\xi) = \ln \left(\frac{-\sqrt{w^2 - 4\mu}}{2\mu} \tanh \left(\frac{\sqrt{w^2 - 4\mu}}{2} (\xi + E) \right) - \frac{w}{2\mu} \right), \tag{7}$$

Family-2: When $\mu \neq 0, w^2 - 4\mu < 0$,

$$\Omega(\xi) = \ln \left(\frac{\sqrt{-w^2 + 4\mu}}{2\mu} \tan \left(\frac{\sqrt{-w^2 + 4\mu}}{2} (\xi + E) \right) - \frac{w}{2\mu} \right), \tag{8}$$

Family-3: When $\mu = 0, w \neq 0$, and $w^2 - 4\mu > 0$,

$$\Omega(\xi) = -\ln \left(\frac{w}{\exp(w(\xi + E)) - 1} \right), \tag{9}$$

Family-4: When $\mu \neq 0, \lambda \neq 0$, and $w^2 - 4\mu = 0$,

$$\Omega(\xi) = \ln \left(-\frac{2w(\xi + E) + 4}{w^2(\xi + E)} \right), \tag{10}$$

Family-5: When $\mu = 0, \lambda = 0$, and $w^2 - 4\mu = 0$,

$$\Omega(\xi) = \ln(\xi + E), \tag{11}$$

$A_i, (0 \leq i \leq N), E, w, \mu$ are coefficients to be obtained later.

Step 3: Putting Eq.(5) along with its derivatives together with the Eq.(6) and simplifying, we find a polynomial of $\exp(-\Omega(\xi))$. When we solve this, we find the

values of the coefficients. By putting the values of the coefficients with one of Eqs. (7-11) into Eq.(5), we can obtain the new solitons to the NPDE equation (2).

3. Implementation of the EFM

In here, we use the EFM to gain various new mixed-dark, exponential and singular soliton solutions of equation (1). Substituting $u(x,t) = U(\xi)$, $\xi = kx - ct$ into equation (1), produces the following NODE;

$$(\tau c^2 - \kappa k^2)U'' + (kB - c)U' + kUU' - \lambda U^3 + (s\lambda - q\lambda)U^2 + s\lambda qU = 0, \tag{12}$$

Considering the balance principle, the value of N can be found as

$$N = 1. \tag{13}$$

Using $N = 1$, along with Eq.(5), yields;

$$U = A_0 + A_1 \exp(-\Omega(\xi)), \tag{14}$$

$$U' = -A_1 \exp(-\Omega(\xi))\Omega', \tag{15}$$

$$U'' = \dots \tag{16}$$

where $A_1 \neq 0$. Using equations (14-16) into equation (12), produces a polynomial equation in $e^{-\Omega(\xi)}$. We collect an algebraic equations from this polynomials by equating to zero. To obtain the solutions of Eq.(1), we put the values of the coefficients into Eq. (14) along with *Family-1* condition.

Case 1

$$A_0 = \frac{1}{2}(-q + wA_1), \quad s = -\frac{2c - 2Bk + kq + q\lambda A_1}{2\lambda A_1}, \quad \tau = \frac{2k^2\kappa + (k + \lambda A_1)A_1}{2c^2}, \quad \mu = \frac{1}{4}\left(w^2 - \frac{q^2}{A_1^2}\right),$$

with these coefficients, when $\mu \neq 0$, $w^2 - 4\mu > 0$, Eq.(1.1) gives the following singular soliton solution

$$u_1(x,t) = \frac{-q + wA_1}{2} + \frac{(q^2 - w^2 A_1^2) \coth(k\varpi x - \varpi ct + \varpi E)}{2q + 2wA_1 \coth(k\varpi x - \varpi ct + \varpi E)}, \tag{17}$$

where $q^2/A_1^2 > 0$ and $\varpi = q/2A_1$.

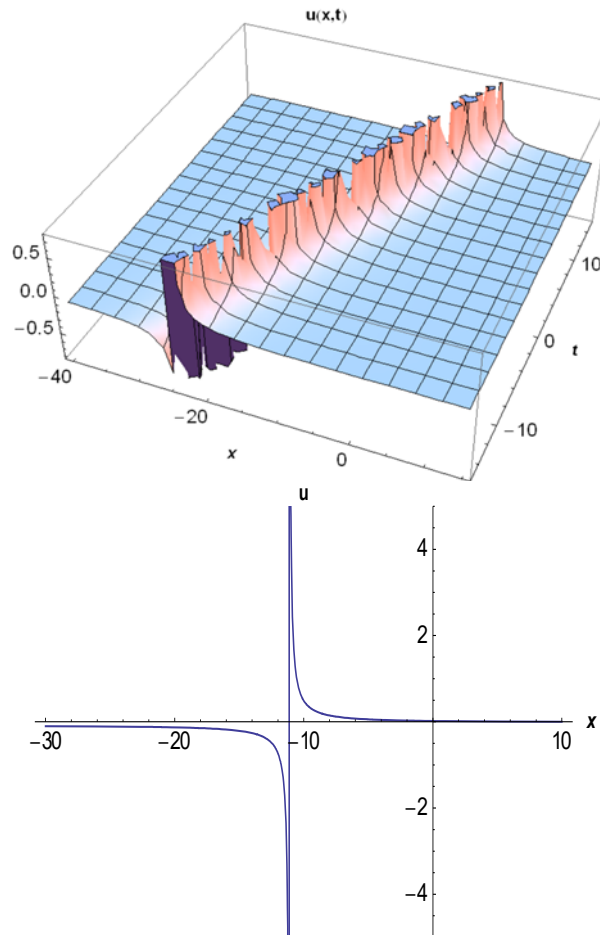


Figure 1. The 2D and 1D surfaces of Eq.(17) and $t = 1.5$ for 1D graphics.

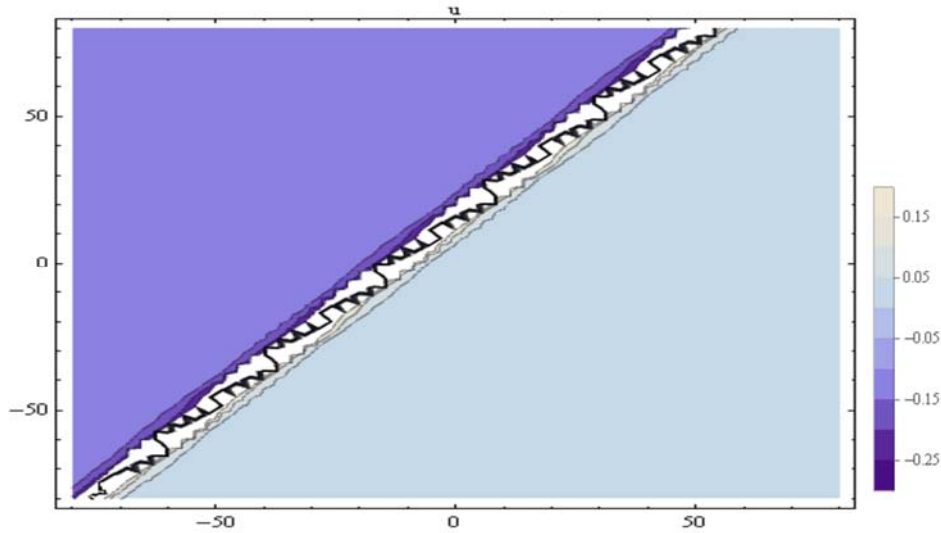


Figure 2. Contour graphs of Eq.(17).

Case 2 When

$$A_0 = \frac{1}{2}(-q - \sqrt{q^2 + 4\mu A_1^2}), \quad w = -\frac{\sqrt{q^2 + 4\mu A_1^2}}{A_1}, \quad s = -\frac{2c - 2Bk + kq + q\lambda A_1}{2\lambda A_1}, \quad \tau = \frac{2k^2\kappa + (k + \lambda A_1)A_1}{2c^2},$$

along with $\mu \neq 0$, $w^2 - 4\mu > 0$, Eq.(1.1) is of the following mixed dark soliton solution

$$u_2(x,t) = \frac{-\omega + \omega \tanh(k\varpi x - \varpi ct + \varpi E)}{\nu - 2q \tanh(k\varpi x - \varpi ct + \varpi E)}, \tag{18}$$

where $-4\mu + (q^2 + 4\mu A_1^2)/A_1^2 > 0$ for validity condition of Eq.(18) and $\omega = q^2 + q\sqrt{q^2 + 4\mu A_1^2}$, $\nu = 2\sqrt{q^2 + 4\mu A_1^2}$, $\varpi = q/2A_1$.

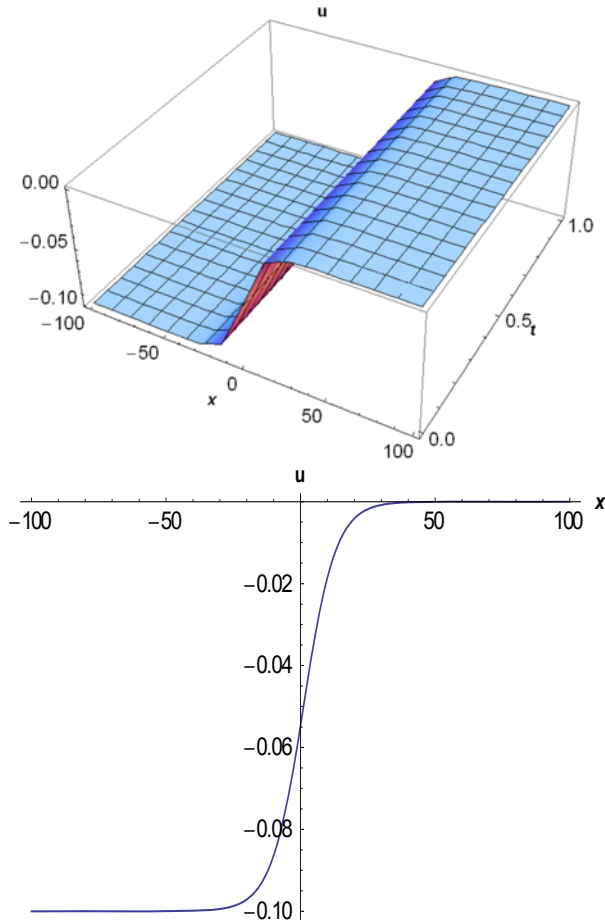


Figure 3. The 2D and 1D surfaces of Eq.(18) and $t = 0.85$ for 1D graphics.

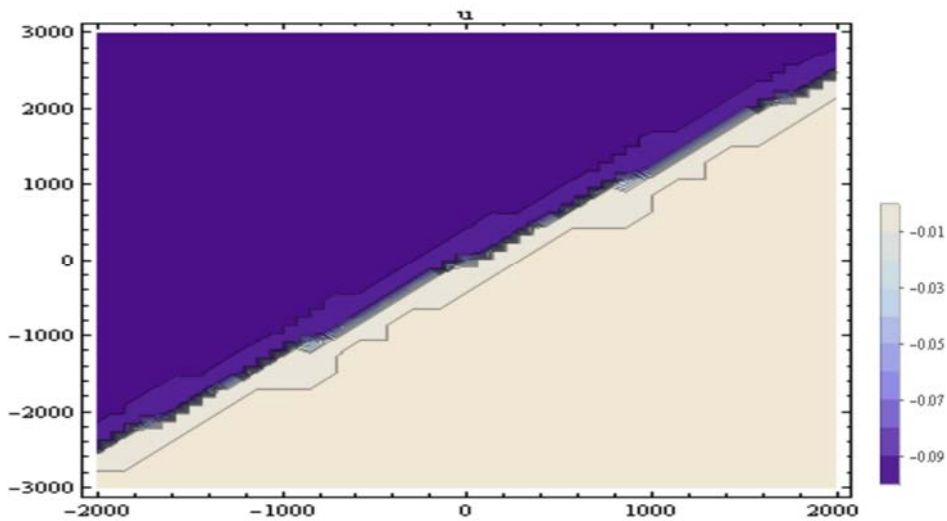


Figure 4. Contour graphs of Eq.(18).

Case 3 If

$$A_0 = \frac{1}{w}(q + 2A_0), s = -\frac{q}{2} - \frac{2cw - 2Bkw + kqw}{2q\lambda + 4\lambda A_0}, \kappa = -\frac{q(kw + q\lambda) - 2c^2w^2\tau + 2A_0(kw + 2q\lambda + 2\lambda A_0)}{2k^2w^2},$$

$\mu = \frac{w^2 A_0 (q + A_0)}{(q + 2A_0)^2}$, we find the following new exponential solution

$$u_3(x, t) = A_0 - \frac{2A_0(q + A_0)(e^{2(k\varpi x - \varpi ct + \varpi E)} + 1)}{2A_0 + (q + 2A_0)e^{2(k\varpi x - \varpi ct + \varpi E)} + qe^{2(k\varpi x - \varpi ct + \varpi E)}}, \tag{19}$$

where $w^2 - (4w^2 A_0 (q + A_0)) / (q + 2A_0)^2 > 0$ and $\varpi = (qw) / (2q + 2A_0)$.

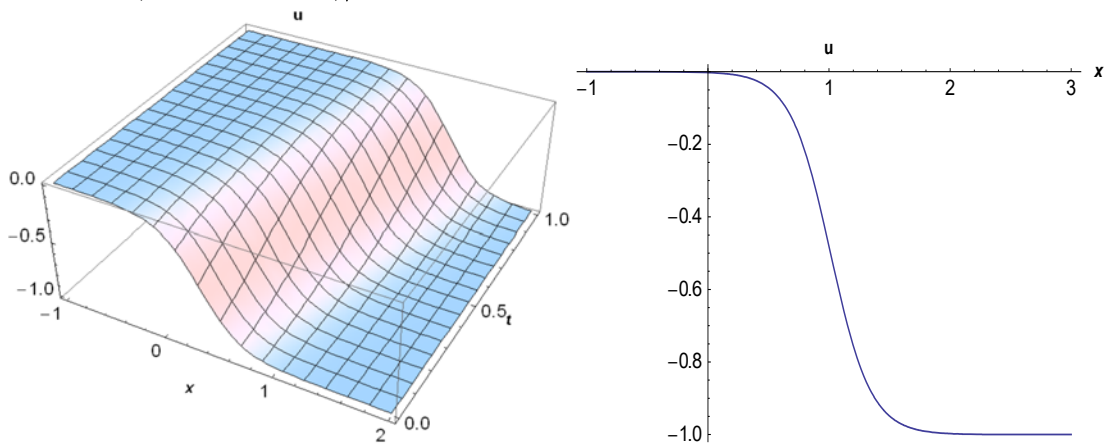


Figure 5. The 2D and 1D surfaces of Eq.(19) and $t = 0.85$ for 1D graphics.

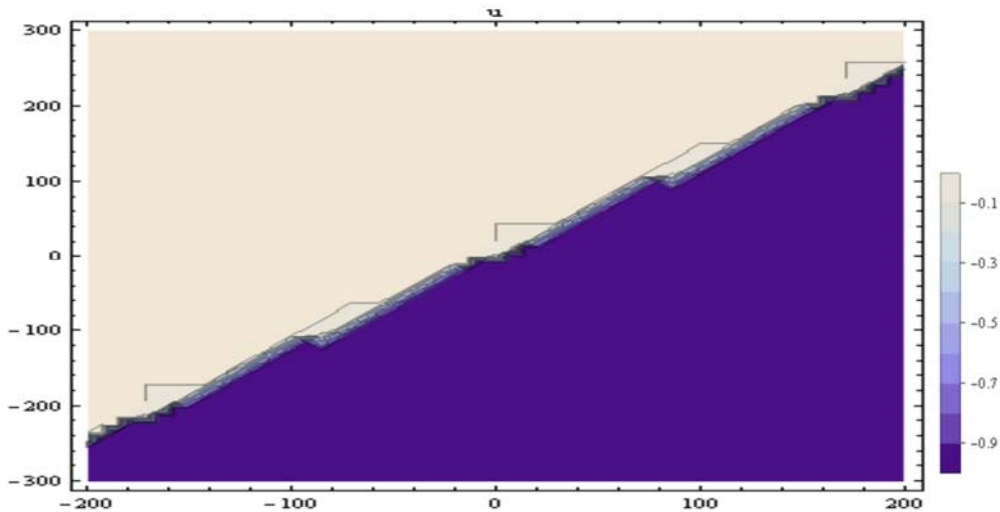


Figure 6. Contour graphs of Eq.(19).

4. Conclusion

In this paper, the EFM has been successfully used in finding mixed-dark, exponential and singular soliton results to the hyperbolic generalization of the Burgers equation (1). The constraint conditions for the existence of valid soliton solutions where necessary

are also given. The physical properties of the hyperbolic generalization of the Burgers model (1) are given as well. Solutions (17), (18) and (19) belong to mixed-dark, exponential and singular soliton solutions, respectively.

When it comes to the solution of Family-2 in Case-1 being $\mu \neq 0$, $-w^2 + 4\mu < 0$, is not satisfied the constraint conditions because it is always negative as $-w^2 + 4\mu = -q^2/A_1^2 < 0$.

The soliton solutions obtained in this manuscript might be physically beneficial in expressing how interaction of the wave fronts and also could be viewed from Figures (1-6). We observe that our result may be useful in detecting some soliton wave behaviors.

The results found in here are entirely new when comparing the results presented in [16]. Finally, one can be inferred from results that the $e^{-\Omega(\xi)}$ -exponential function method is a powerful and efficient mathematical tool that can be used to find many soliton solutions such as mixed-dark, exponential and singular soliton solutions to various nonlinear partial differential equations with high nonlinearity.

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