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A GENERALIZATION OF SIMPLE-INJECTIVE RINGS

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Abstract. A ring R is called right 2-simple J-injective if, for every 2-generated right ideal $I \subseteq J(R)$, every R-linear map from I to R with simple image extends to R. The class of right 2-simple J-injective rings is broader than that of right 2-simple injective rings and right simple J-injective rings. Right 2-simple J-injective right Kasch rings are studied, several conditions under which right 2-simple J-injective rings are QF-rings are given.

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1. Introduction

Throughout this paper, R is an associative ring with identity, m is a positive integer unless otherwise stated, and all modules are unitary. As usual, $J(R)$ or J for short, Z_l (Z_r) and S_l (S_r) denote respectively the Jacobson radical, the left (right) singular ideal and the left (right) socle of R. The left annihilator of a subset X of R is denoted by $l(X)$, and the right annihilator of X is denoted by $r(X)$. If M is an R-module, then the notation $N \subseteq^{max} M$ means that N is a maximal submodule of M, and the notation $N \trianglelefteq M$ means that N is an essential submodule of M.

Recall that a ring R is called right simple injective [5] if for every right ideal I of R, every R-linear map $\gamma : I \to R$ with $\gamma(I)$ simple extends to R. We recall also that a ring R is called *quasi-Frobenius*, briefly QF, if it is right (or left) artinian (or noetherian), and right (or left) self-injective. Simple injective rings and their relationship with QF-rings have been studied by many authors, for example, see [2, 8, 10, 11, 16]. And the concept of right simple injective rings have been generalized in two ways in [18] and [16], respectively. Following [18], a ring R is called right 2-simple injective if for every 2-generated right ideal I of R, every R-linear map

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$\gamma : I \to R$ with $\gamma(I)$ simple extends to R; and following [16], a ring R is called right simple J-injective if for every right ideal $I \subseteq J(R)$, every R-linear map $\gamma : I \to R$ with $\gamma(I)$ simple extends to R.

In this paper, we shall generalize the concept of right simple J-injective rings and right 2-simple injective rings to 2-simple J-injective rings, some properties of this class of rings are studied, and several conditions under which 2-simple J-injective rings are QF-rings are given, many of them extending known results.

We next recall some other known concepts of general injectivity of modules and rings and facts needed in the sequel.

A module M_R is called *FP-injective* (or *absolutely pure*) if, for any finitely generated submodule K of a free right R-module F, every R-homomorphism $K_R \to M_R$ extends to a homomorphism $F_R \to M_R$. A ring R is called right FP-injective if R_R is FP-injective.

Let m be a positive integer. A ring R is called *right m-injective* [7] if, for any m-generated right ideal I of R, every R-homomorphism from I to R extends to an endomorphism of R. Right 1-injective rings are also called *right P-injective* [7]. A ring R is called *right JP-injective* [15] if, for any principal right ideal $I \subseteq J(R)$, every R-homomorphism from I to R extends to an endomorphism of R.

A ring R is called *right general principally injective* (briefly *right GP-injective*) [3] if, for any $0 \neq a \in R$, there exists a positive integer n such that $a^n \neq 0$ and any right R-homomorphism from a^nR to R extends to an endomorphism of R. A ring R is called *right JGP-injective* [15] if for any $0 \neq a \in J(R)$, there exists a positive integer n such that $a^n \neq 0$ and any right R-homomorphism from a^nR to R extends to an endomorphism of R. A ring R is called *right MGP-injective* [19, 20] if, for any $0 \neq a \in R$, there exists a positive integer n such that $a^n \neq 0$ and any right R-monomorphism from a^nR to R extends to an endomorphism of R. A ring R is called right AGP-injective [12, 17] if for any $0 \neq a \in R$, there exists a positive integer n such that $a^n \neq 0$ and Ra^n is a direct summand of $l(r(a^n))$.

A ring R is called *right mininjective* [8] if for any minimal right ideal I of R, every R-homomorphism from I to R extends to an endomorphism of R.

Clearly, the following implications hold:

- right self-injective $\Rightarrow$ right simple injective and right FP-injective;
- right simple injective $\Rightarrow$ right 2-simple injective and right simple J-injective;
- right FP-injective $\Rightarrow$ right m-injective for $m \geq 2$ $\Rightarrow$ right 2-injective $\Rightarrow$ right P-injective $\Rightarrow$ right GP-injective $\Rightarrow$ right AGP-injective and right JGP-injective and right MGP-injective;

- right P-injective ⇒ right JP-injective ⇒ right JGP-injective ⇒ right mininjective;
- right MGP-injective ⇒ right mininjective.

2. 2-Simple J-injective rings

We start with the following definition.

Definition 2.1. Let m be a positive integer. A ring R is called right m-simple J-injective if, for every m-generated right ideal $I \subseteq J(R)$, every R-linear map $\gamma : I \to R$ with $\gamma(I)$ simple extends to an endomorphism of R.

Recall that a ring R is called right (J, S_r)-m-injective [16] if, for any m-generated right ideal $I \subseteq J(R)$, every R-linear map $\gamma : I \to R$ with $\gamma(I) \subseteq S_r$ extends to an endomorphism of R; a ring R is called right (R, S_r)-m-injective [16] if, for any m-generated right ideal I of R, every R-linear map $\gamma : I \to R$ with $\gamma(I) \subseteq S_r$ extends to an endomorphism of R. Clearly, a right (R, S_r)-m-injective ring is right (J, S_r)-m-injective.

Proposition 2.2. *A ring R is right m-simple J-injective if and only if R is right (J, S_r)-m-injective.*

Proof. Assume that R is right m-simple J-injective. Let I be an m-generated right ideal contained in $J(R)$ and γ a homomorphism from I to R with $\gamma(I)$ semisimple. If $\gamma(I) = 0$ then $\gamma = 0$. Otherwise, let $\gamma(I) = K_1 \oplus \cdots \oplus K_n$, where the K_i are simple right ideals. If $\pi_i : \gamma(I) \to K_i$ is the projection, then $\pi_i\gamma = c_i\cdot$ for some $c_i \in R$ by hypothesis. It is routine to verify that $\gamma = (c_1 + \cdots + c_n)\cdot$, as required. □

Clearly, right simple J-injective rings and right 2-simple injective rings are both right 2-simple J-injective, but right 2-simple J-injective rings need neither be right simple J-injective nor right 2-simple injective.

Example 2.3. Let

$$R = \left\{ \begin{bmatrix} n & x \\ 0 & n \end{bmatrix} \middle| n \in \mathbb{Z}, x \in \mathbb{Z}_2 \right\},$$

then, by [16, Example 1.6], R is right simple J-injective but not right (R, S_r)-1-injective. So R is right 2-simple J-injective but not right 2-simple injective.

Example 2.4. Let $R = \mathbb{Z}_2[x_1, x_2, \cdots]$, where the x_i are commuting indeterminates satisfying the relations $x_i^3 = 0$ for all i, $x_ix_j = 0$ for all $i \neq j$, and $x_i^2 = x_j^2$ for all i and j. Write $m = x_1^2 = x_2^2 = \cdots$. Then by [9, Example 2.6], R is a commutative

FP-injective ring. So R is a commutative 2-injective ring and whence 2-simple injective ring, but it is not simple J-injective by the argument in [9, Example 5.45] because, in the notation of that example, $\gamma(J) = \mathbb{Z}_2 m$ is simple. So, in general, 2-simple J-injective rings need not be simple J-injective.

Recall that a ring R is called *right Kasch* [9] if every simple right R-module embeds in R, equivalently if $l(T) \neq 0$ for every maximal right ideal T of R. Left Kasch rings can be defined similarly. R is called *Kasch* if it is left and right Kasch.

Proposition 2.5. *If R is right 1-simple J-injective, then*

(1) *R is right mininjective.*

(2) *If R is right Kasch, then $l(J(R)) \cap L \neq 0$ for any non-zero small left ideal L of R.*

Proof. (1) Let aR be simple. If $(aR)^2 \neq 0$, then $aR = eR$ for an idempotent $e \in R$. Thus, every R-homomorphism from aR to R extends to R. If $(aR)^2 = 0$, then $a \in J(R)$. Since R is right 1-simple J-injective, so every right R-homomorphism from aR to R extends to R.
(2) Let L be a non-zero small left ideal of R and $0 \neq a \in L$. Then $a \in J(R)$. Suppose that T is a maximal submodule of aR. By the right Kasch hypothesis, let $\sigma : aR/T \to R$ be monic, and define $f : aR \to R$ by $f(x) = \sigma(x + T)$, then $im(f) = im(\sigma)$ is simple. Since R is right 1-simple J-injective, $f = c\cdot$ for some $c \in R$, and then $ca = f(a) = \sigma(a + T) \neq 0$. But $caJ(R) = f(a)J(R) = \sigma(a + T)J(R) \subseteq S_r J(R) = 0$, so $0 \neq ca \in Ra \cap l(J(R))$. And hence $l(J(R)) \cap L \neq 0$. □

Theorem 2.6. *Let R be a right 2-simple J-injective, right Kasch ring. Then*

(1) *R is left JP-injective, and hence right and left mininjective.*

(2) *Ra is simple if and only if aR is simple. In particular, $S_r = S_l$.*

(3) *$J(R) = Z_l = r(S_r)$.*

(4) *If $e^2 = e$ is local then $Soc(Re)$ is simple.*

(5) *The map $\theta : T \mapsto l(T)$ gives a bijection from the set of maximal right ideals of R to the set of minimal left ideals of R, whose inverse map is given by $K \mapsto r(K)$.*

Proof. (1) Since R is right Kasch, by [9, Proposition 1.44], $rl(T) = T$ for every maximal right ideal T of R, and so $rl(J) \subseteq rl(T) = T$. It follows that $rl(J) \subseteq J$, and hence $rl(J) = J$. For every $a \in J(R)$, we always have $aR \subseteq rl(a)$. If $b \in rl(a) - aR$, then $b \in J$. Let $aR \subseteq T \subseteq^{max} (aR + bR)$. By the Kasch hypothesis, let $\sigma : (aR + bR)/T \to R$ be monic, and then define $\gamma : aR + bR \to R$ by

$\gamma(x) = \sigma(x+T)$. Since $im(\gamma) = im(\sigma)$ is simple and R is right 2-simple J-injective, $\gamma = c\cdot$ for some $c \in R$. So $ca = \gamma(a) = 0$. This gives $cb = 0$ because $b \in rl(a)$. But $cb = \sigma(b+T) \neq 0$ because $b \notin T$, which is a contradiction. Hence $rl(a) = aR$. This shows that R is left JP-injective by [15, Lemma 1.1].

(2) By (1), R is right and left mininjective, and so Ra is simple if and only if aR is simple by [8, Theorem 1.14 (1)]. Hence $S_r = S_l$.

(3) By (1), R is left JP-injective, so that R is left JGP-injective, and thus $J(R) \subseteq Z_l$ by [15, Theorem 3.6]. On the other hand, since R is right Kasch, by [9, Proposition 1.46], $Z_l \subseteq J(R)$, and hence $J(R) = Z_l$. For every maximal right ideal T of R, since R is right Kasch, R/T can be embedded in R_R, thus for each $x \in r(S_r)$, $(R/T)x = 0$, and then $x \in J(R)$. This implies that $r(S_r) \subseteq J(R)$. Noting that $J(R) \subseteq r(S_r)$ always holds, we have therefore that $J(R) = r(S_r)$.

(4) First we have $l(J)e \cong \mathrm{Hom}_R(eR/eJ, R)$ by [9, Lemma 3.1]. Since eR/eJ is simple (because e is local), and since R is right mininjective and right Kasch, by [9, Theorem 2.31], $l(J)e$ is a simple submodule of $Soc(Re)$. Hence (2) gives that $l(J)e \subseteq Soc(Re) = S_l \cap Re = S_l e = S_r e \subseteq l(J)e$. It follows that $Soc(Re) = l(J)e$ is simple.

(5) Let $K = Rk$ be any minimal left ideal. Then kR is a minimal right ideal by (2). Since R is right mininjective by Proposition 2.5 (1), we have that $lr(K) = K$ by [8, Lemma 1.1], and therefore (5) follows from [8, Theorem 2.3]. □

We call a ring R *left finite dimensional* in case ${}_RR$ is of finite Goldie dimension. We recall that a ring R is called *right* C_2 [9] if every right ideal of R that is isomorphic to a direct summand of R is itself a direct summand of R; a ring R is called *right* GC_2 [15] if every right ideal of R that is isomorphic to R is itself a direct summand of R.

Theorem 2.7. *Let R be a right 2-simple J-injective and right Kasch ring with $S_r \trianglelefteq {}_RR$. Then the following conditions are equivalent:*

(1) *R is left finitely cogenerated;*
(2) *R is left finite dimensional;*
(3) *R is a semilocal ring;*
(4) *S_r is a finitely generated left ideal;*
(5) *R is left Kasch and right finitely cogenerated;*
(6) *R is left Kasch and right finite dimensional;*
(7) *R is right C_2 and right finite dimensional.*

In these cases, $dim({}_RR) = length[(R/J)_R]$.

Proof. (1) $\Rightarrow$ (2) and (5) $\Rightarrow$ (6) are obvious.
(2) $\Rightarrow$ (3) Since R is right Kasch, by [9, Proposition 1.46], it is left C_2, and hence left GC_2. Note that a left GC_2 left finite dimensional ring is semilocal by [15, Corollary 2.5], so R is semilocal.
(3) $\Rightarrow$ (4) Since R is a semilocal and right mininjective ring, by [9, Theorem 5.52], S_r is a finitely generated left ideal.
(4) $\Rightarrow$ (1) By (4) and Theorem 2.6 (2), S_l is a finitely cogenerated left ideal. But $S_l \trianglelefteq {}_RR$ by hypothesis and Theorem 2.6 (2), so R is left finitely cogenerated.
(3), (4) $\Rightarrow$ (5) Since a semilocal two-sided mininjective right Kasch ring is left Kasch by [9, Lemma 5.49], so R is left Kasch. Observing that R is left JP-injective by Theorem 2.6 (1), we have $S_r = S_l \trianglelefteq R_R$ by [15, Theorem 3.8]. Moreover, as R is a semilocal left mininjective ring, by [9, Theorem 5.52], S_l is a finitely generated semisimple right R-module, and so S_l is a finitely cogenerated right ideal, which in turn implies that S_r is finitely cogenerated for $S_r = S_l$. Therefore, R is right finitely cogenerated.
(6) $\Rightarrow$ (7) By [9, Proposition 1.46], a left Kasch ring is right C_2.
(7) $\Rightarrow$ (4) Since right C_2 is right GC_2, and a right GC_2 right finite dimensional ring is semilocal.
Finally, assume that these equivalent conditions hold. Then observe that $l(J) \cong \mathrm{Hom}(R/J, R)$ and $R/J = K_1 \oplus \cdots \oplus K_n$, where each K_i is a simple right R-module, so we have $S_l = S_r = l(J) \cong \mathrm{Hom}(R/J, R) = \mathrm{Hom}(K_1 \oplus \cdots \oplus K_n, R) \cong \mathrm{Hom}(K_1, R) \oplus \cdots \oplus \mathrm{Hom}(K_n, R)$. Since R is right mininjective and right Kasch, by [9, Theorem 2.31 (2)], each $\mathrm{Hom}(K_i, R)$ is simple. Noting that $S_l \trianglelefteq {}_RR$, we have $dim({}_RR) = dim({}_RS_l) = n = length((R/J)_R$. $\square$

Recall that a ring R is called *semiregular* [9] if $R/J(R)$ is regular and idempotents of $R/J(R)$ lift to idempotents of R.

The three results of the following Theorem 2.8 improve the results of [16, Lemma 2.3(1), Theorem 2.11(3),(4)] respectively.

Theorem 2.8. *Let R be a semiregular ring and m be a positive integer. Then*

(1) *R is right m-simple injective if and only if R is right m-simple J-injective.*
(2) *R is right simple-injective if and only if R is right simple J-injective.*
(3) *R is right self-injective if and only if every R-homomorphism from a small right ideal of R to R can be extended to an endomorphism of R.*

Proof. (1) We need only to prove the sufficiency. Let I be an m-generated right ideal and $f : I \to R$ be a homomorphism from I to R with simple image. Since

R is semiregular, by [9, Theorem B.51], $R = P \oplus K$ with $P \subseteq I$ and $I \cap K \ll K$. Hence $R = I + K, I = P \oplus I \cap K$ and so $I \cap K$ is an m-generated right ideal in $J(R)$. Clearly, $f(I \cap K)$ is simple or 0. Since R is right m-simple J-injective, there exists a homomorphism $g : R \to R$ such that $g(x) = f(x)$ for all $x \in I \cap K$. Now we define $h : R \to R$ by $h(y + k) = f(y) + g(k)$, where $y \in I, k \in K$. Then it is easy to see that h is a right R-homomorphism which extends f.
(2) and (3) have proofs similar to the proof of part (1) and so are omitted. □

As the end of this section, we give two properties of a class of special 2-simple injective rings.

Proposition 2.9. *Assume that R is a semiperfect, right 2-simple injective ring in which $Soc(eR) \neq 0$ for every local idempotent e of R. Then the following hold:*

(1) *$S = S_r = S_l = r(J) = l(J)$ is essential in R_R and in ${}_RR$, and $Z_r = Z_l = J = r(S) = l(S)$.*
(2) *R is left and right finitely cogenerated.*

Proof. (1) By [18, Theorem 13], R is left P-injective and left Kasch. So, by [9, Proposition 5.19], S_r is essential in R_R. And thus (1) follows from [16, Proposition 2.5(2)].
(2) Since $S_r \trianglelefteq R_R$, by [16, Proposition 2.5 (3), (4)], R is left and right finitely cogenerated. □

3. Applications to quasi-Frobenius rings

Recall that a ring R is called *right CF* [9] if every cyclic right R-module embeds in a free R-module; a ring R is called left *pseudo-coherent* [1] if every left annihilator of a finite subset of R is a finitely generated left ideal; a ring R is called *right min-coherent* [6] if every minimal right ideal of R is finitely presented; a ring R is called a *left CS ring* [9] if every left ideal of R is essential in a summand of ${}_RR$; a ring R is called *right minsymmetric* [8] if kR is simple, $k \in R$, implies that Rk is simple; a ring R is called *right semiartinian* [9] if every nonzero right R-module has an essential socle. Next we give some applications of 2-simple J-injective rings to QF rings.

Theorem 3.1. *Let R be a right 2-simple J-injective ring. Then the following statements are equivalent:*

(1) *R is a QF-ring;*
(2) *R is right artinian;*

(3) *R is left artinian;*

(4) *R is left perfect and every cyclic right R-module is finite dimensional;*

(5) *R is left perfect, right min-coherent;*

(6) *R is left perfect, left pseudo-coherent;*

(7) *R is right perfect, left pseudo-coherent;*

(8) *R is a right noetherian ring with* $S_r \trianglelefteq R_R$*;*

(9) *R has ACC on right annihilators and* $S_r \trianglelefteq R_R$*;*

(10) *R is a right Kasch left noetherian ring;*

(11) *R is right Kasch and left* CF*;*

(12) *R is left* CS *and left* CF*;*

(13) *R is semilocal and right CF;*

(14) *R is right GP-injective with* ACC *on right annihilators;*

(15) *R is right* AGP*-injective with* ACC *on right annihilators;*

(16) *R is right MGP-injective with* ACC *on right annihilators;*

(17) *R is left GP-injective with* ACC *on left annihilators;*

(18) *R is left* AGP*-injective with* ACC *on left annihilators;*

(19) *R is left MGP-injective with* ACC *on left annihilators;*

(20) *R is semiprimary with* ACC *on left annihilators;*

(21) *R is semiprimary with* ACC *on right annihilators;*

(22) *R is left and right perfect with* ACC *on left annihilators;*

(23) *R is left perfect with* ACC *on left annihilators;*

(24) *R is left perfect with* ACC *on right annihilators;*

(25) *R is a right noetherian right and left Kasch ring.*

(26) *R is a semilocal right 2-*J*-injective ring with* ACC *on right annihilators.*

Proof. Since a semiperfect ring is semiregular, and by Theorem 2.8(1), every semiregular right 2-simple J-injective ring is right 2-simple injective. So the equivalences of (1), (2), (3), (4), (5), (20), (21), (22), (23) and (24) follow immediately from [18, Theorem 3.1].

(1) $\Rightarrow$ (2) $-$ (26), (8) $\Rightarrow$ (9) , (14) $\Rightarrow$ (15), (17) $\Rightarrow$ (18) are clear. (11) $\Rightarrow$ (3) by [4, Corollary 2.6]. (12) $\Rightarrow$ (3) by [4, Corollary 3.10]. (15) $\Rightarrow$ (21) and (18) $\Rightarrow$ (20) by [17, Corollary 1.6]. (16) $\Rightarrow$ (21) and (19) $\Rightarrow$ (20) by [19, Corollary 3.12 (1)].

(13) $\Rightarrow$ (2) Since R is right 2-simple J-injective, it is right mininjective, hence $S_r \subseteq S_l$ by [8, Theorem 1.14 (4)]. Therefore R is right artinian by [2, Theorem 2.10].

(6) $\Rightarrow$ (3) Since R is left perfect, by [9, Theorem B.32], it is right semiartinian,

and so $S_r \trianglelefteq R_R$. Then by Theorem 2.8 (1), R is left Kasch, and thus $J = lr(J)$. Moreover, by Proposition 2.9, $r(J)$ is a finitely generated right ideal. But R is left pseudo-coherent, so J is a finitely generated left ideal, and hence J is nilpotent by [9, Lemma 5.64] since J is left T-nilpotent. Thus, R is semiprimary, and consequently right perfect. Since J/J^2 is a finitely generated left R-module, by Osofsky's Lemma [9, Lemma 6.50], R is left artinian.

(10) $\Rightarrow$ (3) Since R is left noetherian, it is left finite dimensional with ACC on left annihilators. Since R is right Kasch, it is left JP-injective by Theorem 2.6 (1), and so R is left JGP-injective. By [15, Theorem 3.6], $J \subseteq Z_l$. Since R has ACC on left annihilators, by Mewborn-Winton's Lemma [9, Lemma 3.29], Z_l is nilpotent, and thus J is nilpotent. Note that a right Kasch ring is left C_2 by [9, Proposition 1.46] and hence left GC_2. By [15, Corollary 2.5], R is semilocal. Thus, R is a left noetherian semiprimary ring, i.e., R is left artinian.

(7) $\Rightarrow$ (21) Since R is right perfect, R has DCC on finitely generated left ideals. Noting that R is left pseudo-coherent, every left annihilator of a finite subset of R is a finitely generated left ideals. So every left annihilator of a subset of R is a left annihilator of a finite subset of R, and hence every left annihilator in R is a finitely generated left ideal. It follows that R has DCC on left annihilators and thus R has ACC on right annihilators. This shows that R is semiprimary by [9, Lemma 4.20 (1)], and so (21) follows.

(9) $\Rightarrow$ (21) Since a right 2-simple J-injective ring is right mininjective and hence right minsymmetric by [8, Theorem 1.14 (1)]. So (21) follows from [14, Lemma 2.3].

(25) $\Rightarrow$ (2) Since R is right noetherian, it is right finite-dimensional and has ACC on right annihilators. Since R is right 2-simple J-injective and right Kasch, it is left JP-injective by Theorem 2.6 (1). Thus R is a left JP-injective right finite-dimensional ring, and so by [15, Theorem 3.8 (5)], R is semilocal. Since R is left Kasch and left JP-injective, by [15, Theorem 3.8 (4)], $J = Z_r$. Since R has ACC on right annihilators, by Mewborn-Winton's Lemma [9, Lemma 3.29], Z_r is nilpotent, and thus J is nilpotent. Therefore, R is a right noetherian semiprimary ring, i.e., R is right artinian.

(26) $\Rightarrow$ (21) Since R has the ascending chain condition on annihilator right ideals, by [9, Lemma 3.29], Z_r is nilpotent, and so $Z_r \subseteq J$. Since R is right JP-injective, by [15, Theorem 3.6], $J \subseteq Z_r$. Hence, $J = Z_r$ is nilpotent. Therefore, R is a semiprimary ring. □

Corollary 3.2. *The following statements are equivalent for a ring R:*

(1) *R is a QF-ring;*

(2) [13, Corollary 3] *R is right 2-injective with the ascending chain condition on annihilator right ideals;*

(3) [14, Theorem 2.8] *R is a right simple injective ring with ACC on right annihilators in which $S_r \trianglelefteq R_R$;*

(4) [14, Theorem 3.17 (4)] *R is a right small injective ring with ACC on right annihilators in which $S_r \trianglelefteq R_R$;*

(5) *R is a right simple injective right Kasch left noetherian ring;*

(6) *R is a right 2-injective right Kasch left noetherian ring.*

Proof. (1)⇔ (2) By Theorem 3.1 (14).
(1)⇔ (3) By Theorem 3.1 (9).
(1)⇔ (4) By Theorem 3.1 (9).
(1)⇔ (5)⇔ (6) By Theorem 3.1 (10). □

Corollary 3.3. *Let R be a right MGP-injective ring. Then the following statements are equivalent:*

(1) *R is a QF-ring.*

(2) *R is a right 2-simple injective ring with ACC on right annihilators.*

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