n-ABSORBING MONOMIAL IDEALS IN POLYNOMIAL RINGS

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Abstract. In a commutative ring $R$ with unity, given an ideal $I$ of $R$, Anderson and Badawi in 2011 introduced the invariant $\omega(I)$, which is the minimal integer $n$ for which $I$ is an $n$-absorbing ideal of $R$. In the specific case that $R = k[x_1, \ldots, x_n]$ is a polynomial ring over a field $k$ in $n$ variables $x_1, \ldots, x_n$, we calculate $\omega(I)$ for certain monomial ideals $I$ of $R$.

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1. Introduction

Throughout this paper, we set $\mathbb{N} := \{1, 2, \ldots \}$, $\mathbb{N}_0 := \{0, 1, 2, \ldots \}$, and $R$ will denote a commutative ring with unity. Given a nonzero ideal $I$ of $R$, $\text{Ass}(R/I)$ will denote the set of associated primes of $I$ in $R$. The primary notion we are interested in this paper is the following:

Definition 1. Let $n \in \mathbb{N}$, $R$ a commutative ring with unity, and $I$ an ideal of $R$. $I$ is said to be an $n$-absorbing ideal of a ring $R$ if for any $x_1, \ldots, x_{n+1} \in R$ such that $x_1 \cdots x_{n+1} \in I$, there are $n$ of the $x_i$'s whose product is in $I$. $I$ is said to be a strongly $n$-absorbing ideal of a ring $R$ if for any ideals $I_1, \ldots, I_{n+1}$ of $R$ such that $I_1 \cdots I_{n+1} \subseteq I$, there are $n$ of the $I_i$'s whose product is in $I$.

(Strongly) 2-absorbing ideals were initially defined and investigated by Badawi in [3] as a generalization of prime ideals, which are precisely the proper 1-absorbing ideals. In 2011, Anderson and Badawi together generalized this further to the notion of a (strongly) $n$-absorbing ideal for any $n \in \mathbb{N}$ defined above in [1]. For an ideal $I$ in a ring $R$, we let $\omega(I)$ denote the minimal integer $n \in \mathbb{N}$ such that $I$ is $n$-absorbing. In a general ring, $I$ may not be $n$-absorbing for any $n \in \mathbb{N}$, in which case we set $\omega(I) = \infty$. Similarly, we can define the invariant $\omega^*(I)$ to be the smallest integer $n \in \mathbb{N}$ for which an ideal $I$ is strongly $n$-absorbing, and set $\omega^*(I) = \infty$ if no such integer exists. We set $\omega(R) = \omega^*(R) = 0$. It is easy to see that $\omega(I) \leq \omega^*(I)$ holds for each ideal $I$ of $R$. In fact, Anderson and Badawi in
Conjecture 1 of [1, page 1669] postulate that $\omega(I) = \omega^*(I)$ holds for any ideal $I$ in an arbitrary ring $R$; that is, they conjecture that the notion of an $n$-absorbing ideal and strongly $n$-absorbing ideal coincide. As of this writing, this problem remains open. However, it is known that the conjecture holds true for any $n \in \mathbb{N}$ if $R$ is a Pr"{u}fer domain, i.e., an integral domain such that the set of ideals of $R_M$ is totally ordered under set inclusion for each maximal ideal $M$ of $R$ ([1, Corollary 6.9]), or if $R$ is a commutative algebra over an infinite field ([7]), and for any ring $R$ if $n = 2$ ([3, Theorem 2.13]). The interested reader may refer to the survey article [4, Section 5] for further information on strongly $n$-absorbing ideals. Anderson and Badawi made two more conjectures in [1] which were investigated by several researchers, and affirmative answers were given, either partial or complete. For example, the second Anderson-Badawi conjecture states that given an ideal $I$ of a ring $R$ and an indeterminate $X$, $\omega_{R[X]}(I[X]) = \omega_R(I)$ ([1, page 1661]). That is, for each $n \in \mathbb{N}$, $I$ is an $n$-absorbing ideal of $R$ if and only if $I[X]$ is an $n$-absorbing ideal of $R[X]$. This conjecture originates from the well-known result that $I$ is a prime ideal (i.e., 1-absorbing ideal) of $R$ if and only if $I[X]$ is a prime ideal of $R[X]$. Anderson and Badawi themselves proved this conjecture for an arbitrary commutative ring when $n = 2$ ([1, Theorem 4.15]), and Nasehpour proves that the second conjecture holds for every $n \in \mathbb{N}$ when $R$ belongs to certain classes of rings, including the class of Pr"{u}fer domains ([15]). In [11] Laradji independently proved that the second conjecture holds when $R$ is an arithmetical ring, i.e., when the set of ideals of $R_M$ is totally ordered under set inclusion for each maximal ideal $M$ of $R$.

Recall that for an ideal $I$ in a ring $R$, the Noether exponent of $I$, denoted by $e(I)$, is the minimal integer $\mu \in \mathbb{N}$ such that $(\sqrt{I})^\mu \subseteq I$. If such an integer does not exist, we set $e(I) = \infty$. We also set $e(R) = 0$. In a Noetherian ring, since $\sqrt{I}$ is finitely generated for any ideal $I$, $e(I) < \infty$. Anderson and Badawi in [1] establish a connection between $\omega^*(I)$ and Noether exponents:

**Theorem 1.1.** [1, Remark 2.2, Theorem 5.3, Section 6, Paragraph 2 on page 1669] Let $I_1, \ldots, I_r$ be ideals of a ring $R$. Then $\omega(I_1 \cap \cdots \cap I_r) \leq \omega(I_1) + \cdots + \omega(I_r)$ and $\omega^*(I_1 \cap \cdots \cap I_r) \leq \omega^*(I_1) + \cdots + \omega^*(I_r)$. In particular, let $I$ be an ideal in a Noetherian ring $R$. If $I = Q_1 \cap \cdots \cap Q_n$, where the $Q_i$ are primary ideals, then $\omega(I) \leq \omega^*(I) \leq \sum_{i=1}^{n} e(Q_i)$. Thus every ideal in a Noetherian ring is $n$-absorbing for some $n \in \mathbb{N}$.

On the other hand, the third Anderson-Badawi conjecture claims that for each $n \in \mathbb{N}$ and an $n$-absorbing ideal $I$ of a ring $R$, $(\sqrt[\tau]{I})^n \subseteq I$ ([1, Conjecture 2, page...
1669]). This conjecture was proved for \( n = 2 \) by Badawi ([3]), for \( n = 3 \) by Laradji ([11]), for \( n = 3, 4, 5 \) by Sihem and Sana ([17]), and for arbitrary \( n \) and \( R \) by the authors ([6]) and Donadze ([8]), independently. We summarize this as the following theorem in terms of \( \omega(I) \) and \( e(I) \), along with the result concerning primary ideals ([1, Theorem 6.3(c), Theorem 6.6]).

**Theorem 1.2.** Given an ideal \( I \) of a ring \( R \), \( e(I) \leq \omega(I) \). If \( Q \) is a primary ideal of \( R \), then \( \omega(Q) = \omega^*(Q) = e(Q) \).

This raises the question then if for an arbitrary ideal \( I \) whether \( \omega(I) \) can be described purely in terms of Noether exponents or possibly other well-known ring-theoretic invariants. This has been investigated to some extent by others in at least one case. Namely, Moghimi and Naghani [13, Theorem 2.21(1)] show that in a discrete valuation ring \( R \), \( \omega(I) \) is precisely the length of the \( R \)-module \( R/I \).

In this spirit, we attempt to give in this paper a description of \( \omega(I) \) in terms of other ring-theoretic invariants in the special case that \( I \) is a monomial ideal of a polynomial ring over a field. In some cases, our arguments are general enough to also give the same results for \( \omega^*(I) \), and thus as a side-effect we can show that in some cases the notions of a \( n \)-absorbing ideal and a strongly \( n \)-absorbing ideal coincide as Anderson and Badawi conjecture.

The present paper is divided into two parts. In Section 2, we review some definitions and facts concerning \( n \)-absorbing ideals and monomial ideals. Using these, we calculate \( \omega(I) \) for primary monomial ideals by computing Noether exponents and the standard primary decomposition of monomial ideals. These results lead to the study of how \( \omega(I) \) can be explicitly computed from the generating set of \( I \) when \( I \) is a monomial ideal of \( R = k[x_1, \ldots, x_n] \) with \( n \leq 3 \) in the following section.

The second part is Section 4, where we define and investigate \( \omega \)-linear monomial ideals, i.e., monomial ideals \( I \) such that \( \omega(I^m) = m\omega(I) \) for each \( m \in \mathbb{N} \). We give a characterization theorem for primary \( \omega \)-linear monomial ideals, and in particular show that integrally closed monomial ideals in \( R = k[x, y] \) are \( w \)-linear, as well as the edge ideal of a cycle.

### 2. Some background

As a prerequisite of the main section of this paper, we briefly review some of the basic material excerpted from [10] regarding monomial ideals, and show that \( \omega(I) \) can be directly calculated from the generators of \( I \) when \( I \) is a primary monomial ideal.
Let $k$ be a field and $R = k[x_1,\ldots,x_n]$ be the polynomial ring with $n$ variables over $k$. An element of $R$ of the form $x_1^{a_1} \cdots x_n^{a_n}$ with $a_i \in \mathbb{N}_0$ is called a monomial, and an ideal of $R$ generated by monomials is called a monomial ideal. The degree of $f = x_1^{a_1} \cdots x_n^{a_n}$, denoted by $\deg(f)$, is defined to be $a_1 + \cdots + a_n$. $G(I)$ will denote the set of monomials in $I$ which are minimal with respect to divisibility. Any element of $R$ can be written uniquely as a $k$-linear combination of monomials; that is, given $f \in R$, we may write $f = \sum a_u u$ where the sum is taken over the monomial ideals of $R$ and $a_u \in k$ for each monomial $u$. Then the support of $f$, denoted by $\text{supp}(f)$, is the set of monomials $u$ such that $a_u \neq 0$. An ideal $I$ of a ring $R$ is irreducible if there are no ideals $I_1, I_2$ of $R$ such that $I = I_1 \cap I_2$ and $I \subsetneq I_1, I \subsetneq I_2$. We denote by $\mathfrak{m}$ the unique maximal homogeneous ideal of $R$.

**Lemma 2.1.** [10, Chapter 1] Let $R = k[x_1,\ldots,x_n]$ and $I$ a monomial ideal of $R$ generated by monomials $u_1,\ldots,u_r$ of $R$. Then the following hold:

(i) Given a monomial $f \in I$, there exists $i \in \{1,\ldots,r\}$ so $u_i | f$.

(ii) $G(I)$ is the unique minimal set of monomial generators of $I$.

(iii) $I$ can be written as a finite intersection of ideals of the form $(x_1^{d_1},\ldots,x_n^{d_n})$. An irredundant presentation of this form is unique ($I = Q_1 \cap \cdots \cap Q_r$ is irredundant if none of the ideals $Q_i$ can be omitted).

(iv) $I$ is irreducible if and only if $I$ is of the form $(x_1^{d_1},\ldots,x_n^{d_n})$. Moreover, every irreducible monomial ideal of the form $(x_1^{d_1},\ldots,x_n^{d_n})$ is $(x_1,\ldots,x_n)$-primary.

(v) If $J$ is another monomial ideal of $R$, then

$$I \cap J = (\{\text{lcm}(u,v) \mid u \in G(I), v \in G(J)\}).$$

In particular, if $a$ and $b$ are coprime monomials of $R$ and $I$ is a monomial ideal of $R$, then $(ab, I) = (a, I) \cap (b, I)$.

(vi) An ideal $I'$ of $R$ is monomial if and only if for each $f \in I'$, $\text{supp}(f) \subseteq I'$.

By Lemma 2.1(iv), the irredundant unique decomposition of Lemma 2.1(iii) is also a primary decomposition of $I$, which is known as the standard decomposition of $I$ (see [10, P. 12]). We will also need the following characterization of primary monomial ideals:

**Lemma 2.2.** [9, Exercise 3.6] Let $R = k[x_1,\ldots,x_n]$ and $P = (x_{i_1},\ldots,x_{i_r})$ a monomial prime ideal of $R$. Then given a $P$-primary monomial ideal $Q$, $G(Q)$ consists of monomials of the ring $k[x_{i_1},\ldots,x_{i_r}]$ and there exists $a_1,\ldots,a_r \in \mathbb{N}$
so \( \{x_{i_1}^{a_1}, \ldots, x_{i_r}^{a_r}\} \subseteq G(Q) \). Conversely, every monomial ideal of this form is a \( P \)-primary ideal.

**Proof.** Let \( f \in G(Q) \). If \( f \not\in k[x_{i_1}, \ldots, x_{i_r}] \), then \( x_j | f \) for some \( x_j \not\in P \) and \( g = \frac{f}{x_j} \in Q \) since \( Q \) is a \( P \)-primary ideal, but this contradicts the minimality of \( G(Q) \). Hence \( f \in k[x_{i_1}, \ldots, x_{i_r}] \). On the other hand, given \( j \in \{1, \ldots, r\} \) there exists \( a_j \in \mathbb{N} \) so \( x_{i_j}^{a_j} \in G(Q) \), since \( \sqrt{Q} = P \).

To prove the converse, let \( Q \) be a monomial ideal such that \( G(Q) \) consists of monomials of the ring \( k[x_{i_1}, \ldots, x_{i_r}] \) and there exists \( a_1, \ldots, a_r \in \mathbb{N} \) so \( \{x_{i_1}^{a_1}, \ldots, x_{i_r}^{a_r}\} \subseteq G(Q) \). Then \( \sqrt{Q} = P \) by [10, Proposition 1.2.4]. On the other hand, if \( P_1 \in \text{Ass}(R/Q) \setminus \{P\} \), then \( P_1 = Q : f \) for some monomial \( f \) of \( R \) ([10, Corollary 1.3.10]). Now choose \( d \in \{1, \ldots, n\} \) so \( x_d \in P_1 \setminus P \). Then \( x_d f \in Q \), and \( f \in Q \) by Lemma 2.1(i). But then \( P_1 = R \), a contradiction. Hence \( \text{Ass}(R/Q) = \{P\} \) and \( Q \) is a \( P \)-primary monomial ideal.

**Corollary 2.3.** Let \( P \) be a prime monomial ideal and \( I, J \) be \( P \)-primary monomial ideals of \( R \). Then both \( I \cap J \) and \( IJ \) are \( P \)-primary monomial ideals. Moreover, \( I : J \) is a \( P \)-primary monomial ideal provided \( J \not\subset I \).

**Proof.** This is an immediate consequence of Lemma 2.1(v) and Lemma 2.2. \( \square \)

We can now calculate \( \omega(I) \), where \( I \) is an irreducible monomial ideal.

**Lemma 2.4.** Let \( R = k[x_1, \ldots, x_n] \) denote a polynomial ring over a field \( k \). Let \( I = (x_{i_1}^{d_1}, \ldots, x_{i_m}^{d_m}) \), where \( d_1, \ldots, d_m \in \mathbb{N} \) and \( 1 \leq i_1 < i_2 < \cdots < i_m \leq n \). Then \( \omega(I) = \omega^*(I) = e(I) = d_1 + \cdots + d_m - m + 1 \).

**Proof.** Since \( I \) is an \( (x_{i_1}, x_{i_2}, \ldots, x_{i_m}) \)-primary ideal by Lemma 2.2, the first two equalities follow from Theorem 1.2. Thus it suffices to show that \( e(I) = r \), where \( r = d_1 + \cdots + d_m - m + 1 \). We have \( \sqrt{I} = (x_{i_1}, \ldots, x_{i_m}) \). For \( N \in \mathbb{N} \), \( (\sqrt{I})^N \subseteq I \) if and only if for every \( c_1, \ldots, c_m \in \mathbb{N}_0 \) with \( c_1 + \cdots + c_m = N \), we have \( x_{i_1}^{c_1} \cdots x_{i_m}^{c_m} \in I \). By Lemma 2.1(i), the latter happens precisely when \( c_i \geq d_i \) for some \( 1 \leq i \leq m \). Thus \( e(I) = r \). \( \square \)

Next, we produce a way to calculate \( \omega(I) \) when \( I \) is a monomial primary ideal not necessarily generated by pure powers.

**Lemma 2.5.** Let \( I \) be an ideal of a ring \( R \). Suppose there is \( P \in \text{Spec}(R) \) such that \( I = J_1 \cap \cdots \cap J_r \), where \( J_i \) are ideals of \( R \) with \( \sqrt{J_i} = P \) for each \( i \in \{1, \ldots, r\} \). Then \( e(I) = \max_{1 \leq i \leq r} \{e(J_i)\} \).
Corollary 2.6. Let \( R = k[x_1, \ldots, x_n] \) denote a polynomial ring over a field \( k \). If \( Q \) is a monomial primary ideal of \( R \) and \( Q = \bigcap_{i=1}^{r} Q_i \) is its standard decomposition, then

\[
\omega(Q) = \omega^*(Q) = \max_{1 \leq i \leq r} \{ e(Q_i) \}.
\]

Example 2.7. Let \( R = k[x, y, z] \) with a field \( k \) and \( I = (x^4, y^3, z^2, xy, y^2z) \). Then repeatedly applying Lemma 2.1(v), we obtain the standard decomposition \( I = (x, y^2, z^2) \cap (x^4, y, z^2) \cap (x, y^3, z) \). Thus by Lemma 2.4 and Corollary 2.6,

\[
\omega(I) = \omega^*(I) = \max\{1 + 2 + 2 - 3 + 1, 4 + 1 + 2 - 3 + 1, 1 + 3 + 1 - 3 + 1\} = 5.
\]

3. When \( I \) is a monomial ideal of \( R = k[x_1, \ldots, x_n] \) with \( n \leq 3 \)

In this section we show that when \( I \) is a monomial ideal of \( R = k[x_1, \ldots, x_n] \) with \( n \leq 3 \), then \( \omega(I) \) can be explicitly calculated from \( G(I) \). We first prove a theorem analogous to [2, Theorem 2.5]. Note that by \( a_1 \cdots \hat{a}_i \cdots a_n \) we mean \( \prod_{1 \leq j \leq n, j \neq i} a_j \).

Lemma 3.1. Let \( R \) be a UFD and \( p \) an irreducible element of \( R \). Then given \( n \in \mathbb{N} \), \( I \) is an \( n \)-absorbing ideal of \( R \) if and only if \( pI \) is an \( (n+1) \)-absorbing ideal of \( R \). In particular, \( \omega(pI) = \omega(I) + 1 \).

Proof. Suppose that \( I \) is \( n \)-absorbing. Let \( f_1, \ldots, f_{n+2} \in R \) and \( f_1 \cdots f_{n+2} \in pI \). Then since \( p \) is irreducible, \( p \mid f_i \) for some \( i \). Without loss of generality, suppose that \( p \mid f_1 \). Then \( f_1/p \in R \), and so \( (f_1/p)f_2 \cdots f_{n+2} \in I \). Since \( I \) is \( n \)-absorbing, and hence \( (n+1) \)-absorbing as well, we have that either \( (f_1/p)f_2 \cdots \hat{f}_i \cdots f_{n+2} \in I \) for some \( i \in \{2, \ldots, n+2\} \), in which case \( f_1f_2 \cdots \hat{f}_i \cdots f_{n+2} \in pI \) and we are done, or \( f_2 \cdots f_{n+2} \in I \). This is a product of length \( n + 1 \), so that since \( I \) is \( n \)-absorbing, for some \( j \) with \( 2 \leq j \leq n + 2 \), we have \( f_2 \cdots \hat{f}_j \cdots f_{n+2} \in I \). Thus \( pf_2 \cdots \hat{f}_j \cdots f_{n+2} \in pI \), and so \( f_1f_2 \cdots \hat{f}_j \cdots f_{n+2} \in pI \). This shows that \( pI \) is then \( (n+1) \)-absorbing, and \( \omega(pI) \leq \omega(I) + 1 \).

To show the converse, suppose that \( pI \) is an \( (n+1) \)-absorbing ideal. If \( I \) is not an \( n \)-absorbing ideal, then there exists \( f_1, \ldots, f_{n+1} \in R \) such that \( f = f_1 \cdots f_{n+1} \in I \) but \( f_1 \cdots \hat{f}_i \cdots f_{n+1} \notin I \) for each \( i \). Since \( pI \) is \( (n+1) \)-absorbing and \( pf \in pI \), it follows that either \( pf_1 \cdots \hat{f}_i \cdots f_{n+1} \in pI \) for some \( i \) or \( f \in pI \). But the former is impossible by our choice of \( f_i \)'s, and without loss of generality we may assume that
and thereby 
given an equivalence relation on \( \{1, \ldots, t\} \) by defining \( i \sim j \) iff \( \sqrt{T_i} = \sqrt{T_j} \), and set \( \{S_i\}_{i=1}^t \) to be the corresponding equivalence classes. Then \( Q_i = \cap_{\ell \in S_i} T_\ell \) is a monomial primary ideal for each \( i \in \{1, \ldots, r\} \), and \( I = \cap_{i=1}^r Q_i \) is an irredundant primary decomposition of \( I \). We will call this decomposition the canonical primary decomposition of \( I \).

**Corollary 3.2.** Given a monomial \( f \) and an ideal \( I \) of \( R = k[x_1, \ldots, x_n] \), \( \omega(fI) = \text{deg}(f) + \omega(I) \). In particular, \( \omega(fR) = \text{deg}(f) \).

Given a monomial ideal \( I \) with the standard decomposition \( I = \cap_{t=1}^r T_t \), we can define an equivalence relation on \( \{1, \ldots, t\} \) by defining \( i \sim j \) iff \( \sqrt{T_i} = \sqrt{T_j} \), and set \( \{S_i\}_{i=1}^t \) to be the corresponding equivalence classes. Then \( Q_i = \cap_{\ell \in S_i} T_\ell \) is a monomial primary ideal for each \( i \in \{1, \ldots, r\} \), and \( I = \cap_{i=1}^r Q_i \) is an irredundant primary decomposition of \( I \).

**Theorem 3.3.** Let \( R = k[x_1, \ldots, x_n] \). Let \( I \) be a monomial ideal with canonical primary decomposition \( I = \cap_{t=1}^r T_t \). If there exists \( k \in \{1, \ldots, r\} \) such that \( \sqrt{Q_k} \subseteq \sqrt{Q_j} \) for all \( j \in \{1, \ldots, r\} \), then \( \omega(I) = \max \{e(Q_k), \omega(\cap_{1 \leq i \leq r, i \neq k} Q_i)\} \) and \( \omega(I) = \max \{e(Q_k), \omega(\cap_{1 \leq i \leq r, i \neq k} Q_i)\} \).

**Proof.** Let \( t = \max \{e(Q_k), \omega(\cap_{1 \leq i \leq r, i \neq k} Q_i)\} \). We will first show that \( I \) is \( t \)-absorbing. If not, then there are \( f_1, \ldots, f_{t+1} \in R \) such that \( f = \prod_{j=1}^{t+1} f_j \in I \) but \( g_j := f/f_j \notin I \) for each \( j \in \{1, \ldots, t+1\} \). Hence given any \( i \in \{1, \ldots, t+1\} \), there exists \( \ell \in \{1, \ldots, r\} \) such that \( g_j \notin Q_\ell \), and since \( f_1g_\ell = f \in I \subseteq Q_\ell \), we must have \( f_\ell \in \sqrt{Q_\ell} \subseteq \sqrt{T_\ell} \). Therefore, \( g_j \in (\sqrt{Q_\ell})^t \subseteq (\sqrt{Q_\ell})^{e(Q_k)} \subseteq Q_k \) for all \( j \in \{1, \ldots, t+1\} \). On the other hand, \( \cap_{1 \leq i \leq r, i \neq k} Q_i \) is \( t \)-absorbing and \( f \in \cap_{1 \leq i \leq r, i \neq k} Q_i \), so that we conclude \( g_j \in \cap_{1 \leq i \leq r, i \neq k} Q_i \) for some \( j \in \{1, \ldots, t+1\} \) and thereby \( g_j \in I \), a contradiction. Thus \( \omega(I) \leq t \). Next, we show that \( \omega(I) \geq t \); that is, \( I \) is not \( (t-1) \)-absorbing. We now consider two cases.

**Case 1:** \( t = \omega(\cap_{1 \leq i \leq r, i \neq k} Q_i) \). Since \( \cap_{1 \leq i \leq r, i \neq k} Q_i \) is not \( (t-1) \)-absorbing, there are \( h_1, \ldots, h_t \in R \) such that \( h = \prod_{i=1}^t h_i \notin \cap_{1 \leq i \leq r, i \neq k} Q_i \) and \( \ell_j := h/h_j \notin \cap_{1 \leq i \leq r, i \neq k} Q_i \) for each \( j \in \{1, \ldots, t\} \). By an argument similar to the first paragraph of this proof, \( h_i \in \sqrt{Q_k} \) for each \( i \in \{1, \ldots, t\} \), and so \( h \in Q_k \). Hence \( h \in I \) and \( \ell_j \notin I \) for each \( j \in \{1, \ldots, t\} \), so that \( I \) is not \( (t-1) \)-absorbing.

**Case 2:** \( t = e(Q_k) \). Consider the standard decomposition of \( I \), and choose an irreducible component \( T \) of \( I \) such that \( e(T) = e(Q_k) \) and \( \sqrt{T} = \sqrt{Q_k} \). Since
we obtained the canonical primary decomposition \( I = \bigcap_{i=1}^r Q_i \) from the standard decomposition, we can choose a monomial \( g \in (\bigcap_{1 \leq i \leq r, i \neq k} Q_i) \setminus T \) by Lemma 2.1(vi). Now \( T = (x_{i_1}^{a_1}, \ldots, x_{i_l}^{a_l}) \) for some \( a_j \in \mathbb{N} \) and \( 1 \leq i_1 < \cdots < i_l \leq n \). Note that we may assume that \( g = \prod_{j=1}^l x_{i_j}^{e_j} \) for some \( e_j \in \mathbb{N}_0 \) such that \( e_j < a_j \) for each \( j \in \{1, \ldots, l\} \). Set

\[
f := x_{i_1}^{a_1 - 1} \cdots x_{i_l}^{a_l - 1} (x_{i_1} + \cdots + x_{i_l}).
\]

Then \( f \) is a product of \( e(T) \) elements of \( \sqrt{T} \) by Lemma 2.4, and so \( f \in (\sqrt{T})^{e(T)} = (\sqrt{Q_k})^{e(Q_k)} \subseteq Q_k \). Since \( g \mid f \) it also follows that \( f \in \bigcap_{1 \leq i \leq r, i \neq k} Q_i \). Hence \( f \in I \).

However, given \( j \in \{1, \ldots, l\} \), \( f \frac{x_{i_j}}{x_{i_j}} \not\in T \). Indeed, \( x_{i_1}^{a_1 - 1} \cdots x_{i_l}^{a_l - 1} \in \text{supp}(\frac{f}{x_{i_j}}) \setminus T \) by Lemma 2.1(i), and \( f \frac{x_{i_j}}{x_{i_j}} \not\in T \) by Lemma 2.1(vi). Similarly \( x_{i_1}^{a_1 - 1} \cdots x_{i_l}^{a_l - 1} = \frac{f}{x_{i_1} + \cdots + x_{i_l}} \notin T \). Therefore \( I \) is not \( (e(Q_k) - 1) \)-absorbing, and \( \omega(I) \geq e(Q_k) = t \). Hence we have shown that \( \omega(I) = \max\{e(Q_k), \omega(\bigcap_{1 \leq i \leq r, i \neq k} Q_i)\} \). The proof of \( \omega^*(I) = \max\{e(Q_k), \omega^*(\bigcap_{1 \leq i \leq r, i \neq k} Q_i)\} \) can be obtained in a similar manner, and is omitted.

The following corollary is immediate.

**Corollary 3.4.** Let \( R = k[x_1, \ldots, x_n] \) and \( I \) a monomial ideal of \( R \) with standard decomposition \( I = \bigcap_{i=1}^r T_i \). Then \( \omega(I) = \omega^*(I) = \max_{1 \leq i \leq r} \{e(T_i)\} \) if Ass\((R/I)\) is totally ordered under set inclusion.

In the next proposition, we give a characterization of when the upper bound of \( \omega(I) \) from Theorem 1.1 is sharp.

**Proposition 3.5.** Let \( I \) be a monomial ideal of \( R = k[x_1, \ldots, x_n] \) with an irredundant primary decomposition \( I = Q_1 \cap \cdots \cap Q_r \). Then \( \omega(I) = \omega^*(I) = \sum_{i=1}^r e(Q_i) \) if and only if \( I \) has no embedded associated primes.

**Proof.** Set \( P_i = \sqrt{Q_i} \) for each \( i = 1, \ldots, r \).

\( \Rightarrow \): We prove the contrapositive; assume that \( P_1, \ldots, P_r \) are not incomparable prime ideals. Then without loss of generality we may assume that \( P_1 \subseteq P_2 \), and we have \( \omega(Q_1 \cap Q_2) = \max\{e(Q_1), e(Q_2)\} \) by Corollary 3.4. Therefore by Theorem...
1.1 we have

\[
\omega(I) = \omega\left(Q_1 \cap Q_2 \cap \left( \bigcap_{i \neq 1,2} Q_i \right) \right)
\]

\[
\leq \omega\left(Q_1 \cap Q_2\right) + \omega\left( \bigcap_{i \neq 1,2} Q_i \right)
\]

\[
= \max\{e(Q_1), e(Q_2)\} + \omega\left( \bigcap_{i \neq 1,2} Q_i \right)
\]

\[
\leq \max\{e(Q_1), e(Q_2)\} + \sum_{i \neq 1,2} e(Q_i)
\]

\[
< \sum_{i=1}^r e(Q_i).
\]

\[\Rightarrow:\] Assume that \(P_1, \ldots, P_r\) are incomparable prime ideals. The case when \(r = 1\) follows from Theorem 1.2, so we may assume that \(r \geq 2\). Since \(\omega(I) \leq \omega^\bullet(I) \leq \sum_{i=1}^r e(Q_i)\) by Theorem 1.1, it suffices to show that \(I\) is not \((\sum_{i=1}^r e(Q_i) - 1)\)-absorbing.

Now given \(i \in \{1, \ldots, r\}\), choose \(T_i\) to be an irreducible component of \(I\) with \(\sqrt{T_i} = P_i\) and \(e(T_i) = e(Q_i)\). Write \(T_i = (x_{i_1}^{a_1}, \ldots, x_{i_{s_i}}^{a_{s_i}})\) with \(1 \leq i_1 < \cdots < i_{s_i} \leq n\) and \(a_1, \ldots, a_{s_i} \in \mathbb{N}\). For \(i \in \{1, \ldots, r\}\) and \(j \in \{1, \ldots, s_i\}\), set

\[
f_{i,j} = x_{i_j} + \sum_{t \neq j} x_{i_t}^2 \quad \text{and} \quad f_i = \left( \sum_{l=1}^{s_i} x_{i_l} \right) \left( \prod_{j=1}^{s_i} f_{i,j}^{a_{j-1}} \right).
\]

It follows that \(f_i \in P_i^{e(T_i)} = (\sqrt{Q_i})^{e(Q_i)} \subseteq Q_i\). Thus \(f := \prod_{i=1}^r f_i \in I\), and \(f\) is a product of \(\sum_{i=1}^r e(Q_i)\) elements of \(R\). We wish to show that \(\sum_{l=1}^{s_i} x_{i_l} \not\in I\) and \(\frac{f}{f_{i,j}} \not\in I\) for each \(i \in \{1, \ldots, r\}\) and \(j \in \{1, \ldots, s_i\}\). Without loss of generality, let \(i = 1\).

Note that \(\frac{f_1}{f_{1,j}} \not\in T_1\), since \(\prod_{t=1}^{s_1} x_{1_t}^{a_{t-1}} \in \text{supp}(\frac{f_1}{f_{1,j}}) \setminus T_1\). On the other hand, \(\sum_{i=1}^{s_1} x_{i_l} \not\in P_1\) and \(f_{i,j} \not\in P_1\) for each \(i \neq 1\) and \(l \in \{1, \ldots, s_i\}\). Therefore \(f_i \not\in P_1\) for each \(i \neq 1\), and \(\frac{f}{f_1} = \prod_{i=2}^r f_i \not\in P_1\). Hence \(\frac{f}{f_{1,j}} = \left( \frac{f}{f_1} \right) \left( \frac{f_1}{f_{1,j}} \right) \not\in Q_1\). The proof that \(\sum_{i=1}^r x_{1_l} \not\in Q_1\) follows similarly. Hence we have \(\omega(I) = \sum_{i=1}^r e(Q_i)\).

\[\square\]

Theorem 3.3 and Proposition 3.5 yield the following corollary.
Corollary 3.6. Let $I$ be a monomial ideal of $R = k[x_1, \ldots, x_n]$ with $\dim(R/I) = 1$. Let $I = \bigcap_{i=1}^r Q_i$ be the canonical primary decomposition of $I$. Then

$$\omega(I) = \omega^*(I) = \begin{cases} \max \{ e(Q_k), \sum_{i \neq k} e(Q_i) \} & \text{if } \sqrt{Q_k} = m \text{ for some } k \in \{1, \ldots, r\}, \\ \sum_{i=1}^r e(Q_i) & \text{otherwise.} \end{cases}$$

Corollary 3.7. Let $f$ be a monomial of $R$. Then $\omega^*(fR) = \deg(f)$. In particular, $\omega(fR) = \omega^*(fR)$.

Proof. Let $f = \prod_{k=1}^r x_k^{a_k}$ for some $a_1, \ldots, a_r \in \mathbb{N}$ and $1 \leq i_1 < i_2 < \cdots < i_r \leq n$. Then $fR = x_{i_1}^{a_1}R \cap \cdots \cap x_{i_r}^{a_r}R$, and by Lemma 2.4 and Proposition 3.5 we have $\omega(fR) = \omega^*(fR) = \sum_{i=1}^r e(x_{i_k}^{a_k}) = \sum_{i=1}^r a_k = \deg(f)$. □

Given a monomial ideal $I$ of $R = k[x, y, z]$, we can produce an algorithm that can compute $\omega(I)$. If $I$ is principal, then Corollary 3.7 says that $\omega(I)$ is equal to the degree of a generator for $I$. Otherwise, $I = hJ$ for some monomial $h$ and a monomial ideal $J$ with $\dim(R/J) \leq 1$. Now, $\omega(J)$ can be calculated explicitly using Corollary 2.6 or Corollary 3.6 after obtaining a canonical primary decomposition of $J$, and we have $\omega(I) = \deg(h) + \omega(J)$ by Corollary 3.2.

Example 3.8. Let $R = k[x, y, z]$ and $I = (x^3y^4, x^2y^5, x^4y^3z^2, x^5y^3z, x^2y^4z^2)$. Then $I = x^2y^3J$ with canonical primary decomposition $J = (x^2, y) \cap (y, z) \cap (x^3, y^2, z^2, xy)$. By Lemma 2.4 and Corollary 2.6, the standard decomposition $(x^3, y^2, z^2, xy) = (x, y, z^2) \cap (x^3, y, z^2)$ yields that $e((x^3, y^2, z^2, xy)) = 4$. Thus by Corollary 3.6,

$$\omega(I) = \deg(x^2y^3) + \omega(J)$$

$$= 5 + \max \{ e((x^3, y^2, z^2, xy)), e((x^2, y)) + e((y, z)) \}$$

$$= 5 + \max \{ 4, 2 + 1 \}$$

$$= 9.$$ 

Another interesting result that follows from Lemma 3.1 and Theorem 3.3 is a formula of $\omega(I)$ and $\omega^*(I)$ for monomial ideals of $R = k[x, y]$ where $k$ is a field and $x, y$ are indeterminates over $k$.

Theorem 3.9. Let $R = k[x, y]$ and $J$ a monomial ideal of $R$. Write $J = (x^{a_1}y^{b_1}, \ldots, x^{a_r}y^{b_r})$, where $\{a_i\}$ is strictly decreasing and $\{b_i\}$ is strictly increasing. Then

$$\omega(J) = \omega^*(J) = \begin{cases} a_1 + b_1 & \text{if } r = 1, \\ \max_{1 \leq i \leq r-1} \{ a_i + b_{i+1} \} - 1 & \text{if } r > 1. \end{cases}$$
Proof. The case when \( r = 1 \) follows from Corollary 3.7. For \( r > 1 \), first observe the standard decomposition of \( J \) is 
\( J = x^{a_r} R \cap y^{b_i} R \cap (x^{a_1}, y^{b_2}) \cap (x^{a_2}, y^{b_3}) \cap \cdots \cap (x^{a_{r-1}}, y^{b_r}) \) ([12, Proposition 3.2]). The case \( b_1 = a_r = 0 \) follows from Corollary 2.6. Suppose that at least one of \( a_r \) and \( b_1 \) is nonzero. Then by Lemma 2.4 and Corollary 3.6,
\[
\omega(J) = \omega^*(J) = \max\{e((x^{a_1}, y^{b_2}) \cap (x^{a_2}, y^{b_3}) \cap \cdots \cap (x^{a_{r-1}}, y^{b_r})), e(x^{a_r} R) + e(y^{b_1} R)\}
\]
\[
= \max\{\max_{1 \leq i \leq r-1} \{e((x^{a_i}, y^{b_{i+1}}))\}, a_r + b_1\}
\]
\[
= \max\{\max_{1 \leq i \leq r-1} \{a_i + b_{i+1} - 1\}, a_r + b_1\}
\]
\[
= \max_{1 \leq i \leq r-1} \{a_i + b_{i+1}\} - 1.
\]
\[
\square
\]

Example 3.10. If \( R = k[x, y] \) and \( J = (x^{11}y^4, x^8y^5, x^7y^9, x^4y^{10}, x^2y^{16}) \), then by Theorem 3.9,
\[
\omega(J) = \omega^*(J) = \max\{11 + 5, 8 + 9, 7 + 10, 4 + 16\} - 1 = 19.
\]

4. \( \omega \)-Linear ideals

Given an ideal \( I \) of a ring \( R \), we will say that \( I \) is an \( \omega \)-linear ideal if \( \omega(I^m) = m\omega(I) \) for each \( m \in \mathbb{N} \). Perhaps the most common example of \( \omega \)-linear ideals can be found amongst those \( P \in \text{Spec}(R) \) for which \( P^n \) is \( P \)-primary for each \( n \in \mathbb{N} \) ([1, Theorem 3.1, Theorem 5.7]). For instance,

1. \( R \) is a Prüfer domain and \( P^2 \neq P \).
2. \( R \) is a Noetherian ring and \( P \) is a maximal ideal that contains a nonzerodivisor.
3. \( R = k[x_1, \ldots, x_n] \) and \( P \) is a monomial ideal.

In this section, we investigate the properties of \( \omega \)-linear ideals. Again, we will restrict our concern to monomial ideals of a polynomial ring \( R = k[x_1, \ldots, x_n] \) where \( k \) is a field.

We first consider a few useful inequalities regarding monomial ideals.

Lemma 4.1. Let \( I \) be a monomial ideal of \( R = k[x_1, \ldots, x_n] \). Then \( \omega(I) \geq \max\{\deg(f) \mid f \in G(I)\} \).

Proof. Let \( f \in G(I) \). Then \( f = \prod_{k=1}^{r} x_{i_k}^{a_k} \) for some \( a_1, \ldots, a_r \in \mathbb{N} \) and \( 1 \leq i_1 < i_2 < \cdots < i_r \leq n \). Since \( f \in I \) but \( \frac{f}{x_{i_k}^{a_k}} \not\in I \) for each \( k \in \{1, \ldots, r\} \) by minimality of \( G(I) \), we have that \( I \) is not \( (\deg(f) - 1) \)-absorbing. Hence \( \omega(I) \geq \deg(f) \), and since \( f \) was chosen arbitrarily, we have the desired conclusion. \( \square \)
**Lemma 4.2.** Let $I \subseteq J$ be ideals of a ring $R$. If $\sqrt{I} = \sqrt{J}$, then $e(J) \leq \omega(I)$. In particular, if $I$ and $J$ are both $P$-primary ideals of a prime ideal $P$ of $R$, then $\omega(J) \leq \omega(I)$.

**Proof.** Since $\sqrt{I} = \sqrt{J}$, $(\sqrt{J})^{\omega(I)} \subseteq (\sqrt{I})^{\omega(J)} \subseteq I \subseteq J$ by Theorem 1.2 and $e(J) \leq \omega(I)$. The second statement follows immediately since $e = \omega$ for primary ideals.

**Lemma 4.3.** Let $P$ be a prime monomial ideal of $R = k[x_1, \ldots, x_n]$. If $I, J$ are $P$-primary monomial ideals of $R$, then $\omega(I+J) \leq \min\{\omega(I), \omega(J)\} \leq \max\{\omega(I), \omega(J)\} = \omega(I \cap J) \leq \omega(IJ) \leq \omega(I) + \omega(J)$. Moreover, $\omega(I : J) \geq \omega(I) - \omega(J)$.

**Proof.** Note that by Corollary 2.3, $IJ \subseteq I \cap J \subseteq I + J$ are all $P$-primary monomial ideals. Therefore $\omega(I+J) \leq \min\{\omega(I), \omega(J)\} \leq \max\{\omega(I), \omega(J)\} \leq \omega(I \cap J) \leq \omega(IJ)$ by Lemma 4.2. On the other hand, let $I = \bigcap_{i=1}^{r} Q_i$ and $J = \bigcap_{j=1}^{s} T_j$ be the standard decompositions of $I$ and $J$, respectively. Then $I \cap J = (\bigcap_{i=1}^{r} Q_i) \cap (\bigcap_{j=1}^{s} T_j)$ is an irreducible decomposition of $I \cap J$, and by throwing away any redundant components, there are $A \subseteq \{1, \ldots, r\}$ and $B \subseteq \{1, \ldots, s\}$ so that $I \cap J = (\bigcap_{i \in A} Q_i) \cap (\bigcap_{j \in B} T_j)$ is the standard decomposition of $I \cap J$. Thus by Corollary 2.6,

$$\omega(I \cap J) = \max\{\max_{i \in A}\{e(Q_i)\}, \max_{j \in B}\{e(T_j)\}\}$$

$$\leq \max\{\max_{1 \leq i \leq r}\{e(Q_i)\}, \max_{1 \leq j \leq s}\{e(T_j)\}\}$$

$$= \max\{\omega(I), \omega(J)\}.$$ 

Moreover, $(\sqrt{IJ})^{e(I)+e(J)} = P^{e(I)+e(J)} = P^{e(I)}P^{e(J)} = (\sqrt{I})^{e(I)}(\sqrt{J})^{e(J)} \subseteq IJ$, and so $e(IJ) \leq e(I)+e(J)$. Combined with Theorem 1.2, we have $\omega(IJ) \leq \omega(I) + \omega(J)$. It remains to show that $\omega(I : J) \geq \omega(I) - \omega(J)$. When $J \subseteq I$, then we have $I : J = R$ and $\omega(I : J) = 0 \geq \omega(I) - \omega(J)$ by Lemma 4.2. If $J \not\subseteq I$, then $I : J$ is $P$-primary by Corollary 2.3, and since $J(I : J) \subseteq I$, we have $\omega(I : J) + \omega(J) \geq \omega(I)$ by the first part of this lemma, hence the claim.

As Anderson and Badawi pointed out ([1, Example 2.7]), the conclusion of Lemma 4.3 does not hold in every ring $R$. We add, that even in a polynomial ring over a field, the conclusion of the above lemma may fail if we drop any part of the hypothesis.

**Example 4.4.** Let $R = k[x, y, z]$ and $I = (x^2, xy, y^2, xz^2)$ and $J = (x^2, xy, y^2, yz^3)$, so that neither $I$ nor $J$ are primary ideals. The standard decompositions of $I, J$
and $I \cap J$ are

\[
I = (x^2, y, z^2) \cap (x, y^2)
\]

\[
J = (x, y^2, z^3) \cap (x^2, y)
\]

\[
I \cap J = (x, y^2) \cap (x^2, y)
\]

\[
I + J = (x, y) \cap (x, y^2, z^3) \cap (x^2, y, z^2).
\]

Thus we have $\omega(I) = 3$, $\omega(J) = 4$, $\omega(I \cap J) = 2$ and $\omega(I + J) = 4$, so that $\omega(I \cap J) < \omega(I + J) = \max\{\omega(I), \omega(J)\}$.

**Example 4.5.** Let $R = k[x, y, z]$ and $I = (x, y)$ and $J = (y, z^2)$, so that $I$ and $J$ are both primary, but $\sqrt{I} \neq \sqrt{J}$. Then we have $\omega(I) = 1$, $\omega(J) = 2$ and $\omega(I \cap J) = 3$, so that $\omega(I \cap J) > \max\{\omega(I), \omega(J)\}$.

**Corollary 4.6.** Let $I$ be a primary monomial ideal of $R = k[x_1, \ldots, x_n]$. Then for each $m \in \mathbb{N}$ we have $\omega(I^m) \leq m\omega(I)$.

**Proof.** Follows immediately by induction on $m$ and Lemma 4.3. \qed

Next, we derive a characterization of primary monomial $\omega$-linear ideals.

**Lemma 4.7.** Let $R = k[x_1, \ldots, x_n]$ and $Q$ a primary monomial ideal of $R$, so that $G(Q)$ consists of monomials of the ring $k[x_{i_1}, \ldots, x_{i_r}]$ for some $1 \leq i_1 < i_2 < \cdots < i_r \leq n$ and there exists $a_1, \ldots, a_r \in \mathbb{N}$ so $\{x_{i_1}^{a_1}, \ldots, x_{i_r}^{a_r}\} \subseteq G(Q)$. Choose $s \in \{1, \ldots, r\}$ so $a_s = \max_{1 \leq j \leq r}\{a_j\}$.

1. If $G(Q) = \{x_{i_1}^{a_1}, \ldots, x_{i_r}^{a_r}\}$, then $\omega(Q^m) = (m - 1)a_s + \omega(Q)$ for each $m \in \mathbb{N}$.
2. $Q$ is $\omega$-linear if and only if $\omega(Q) = a_s$.

**Proof.** (1) Let $Q = (x_{i_1}^{a_1}, \ldots, x_{i_r}^{a_r})$. Then given $m \in \mathbb{N}$, set $S_m = \{(k_1, \ldots, k_r) \in \mathbb{N}^r | \sum_{j=1}^{r} k_j = m+r-1\}$ and $Q_k = (x_{i_1}^{k_1a_1}, \ldots, x_{i_r}^{k_ra_r})$ for each $k = (k_1, \ldots, k_r) \in S_m$.

Then $Q^m = \bigcap_{k \in S_m} Q_k$ (cf. [14, Theorem 6.2.4]). Now by Corollary 2.6 and Lemma 2.4,

\[
\omega(Q^m) = \max_{k \in S_m} \{e(Q_k)\} = \max_{k \in S_m} \left\{ \sum_{j=1}^{r} k_ja_j \right\} - r + 1 = (m - 1)a_s + \omega(Q).
\]

(2) Fix $m \in \mathbb{N}$ and set

\[
I_1 = (x_{i_1}^{a_1}, \ldots, x_{i_r}^{a_r})^m, I_2 = (x_{i_1}, \ldots, x_{i_{r-1}}, x_{i_r}^{ma_s}, x_{i_{r+1}}, \ldots, x_{i_r}).
\]

It follows that $I_1 \subseteq Q^m \subseteq I_2$ are $(x_{i_1}, \ldots, x_{i_r})$-primary ideals, so we have $ma_s = \omega(I_2) \leq \omega(Q^m) \leq \omega(I_1) = (m - 1)a_s + \sum_{j=1}^{r} a_j - r + 1$ by Corollary 4.2,
Lemma 2.4 and part 1 of this lemma. Therefore if $Q$ is $\omega$-linear, then $\omega(Q) = \lim_{m \to \infty} \frac{m\omega(Q)}{m} = \lim_{m \to \infty} \frac{\omega(Q^m)}{m} = a_s$. Conversely, suppose that $\omega(Q) = a_s$ and fix $m \in \mathbb{N}$. Then since $x_i^{m_{ia_s}} \in G(Q^m)$ we have $\omega(Q^m) \geq ma_s = m\omega(Q)$ by Lemma 4.1. Hence $\omega(Q^m) = m\omega(Q)$ by Corollary 4.6 and so $Q$ is $\omega$-linear. \[\square\]

**Corollary 4.8.** Let $I$ be an irreducible monomial ideal of $R = k[x_1, \ldots, x_n]$ so that $I = (x_1^{a_1}, \ldots, x_r^{a_r})$ for some $1 \leq i_1 < i_2 < \cdots < i_r \leq n$ and $a_1, \ldots, a_n \in \mathbb{N}$. Set $a_s = \max_{1 \leq j \leq r}\{a_j\}$. Then the following are equivalent.

1. $I$ is $\omega$-linear.
2. $\omega(I^m) = m\omega(I)$ for some $m > 1$.
3. $\omega(I) = a_s$.
4. $a_i = 1$ for each $i \neq s$.

**Proof.** (1) $\Rightarrow$ (2) Obvious.
(2) $\Rightarrow$ (3) Suppose that $\omega(I^m) = m\omega(I)$ for some $m > 1$. By Lemma 4.7(1) we have $\omega(I^m) = (m-1)a_s + \omega(I)$. Hence $\omega(I) = a_s$.
(3) $\Leftrightarrow$ (4) Immediate consequence of Lemma 2.4.
(3) $\Leftrightarrow$ (1) Follows from Lemma 4.7(2). \[\square\]

**Lemma 4.9.** Let $P$ be a monomial prime ideal of $R$. If $I, J$ are $P$-primary $\omega$-linear monomial ideals of $R$, then so is $I \cap J$.

**Proof.** Without loss of generality we may assume that $\omega(I) \geq \omega(J)$. By Lemma 4.7(2), there is $j \in \{1, \ldots, r\}$ so that $x_j^\omega(I) \in G(I)$. There exists $a \in \mathbb{N}$ so $x_j^a \in G(J)$. Then again, by Lemma 4.7(2), $a \leq \omega(J)$. Now, $x_j^\omega(I) = \text{lcm}(x_j^\omega(I), x_j^a) \in G(I \cap J)$. On the other hand, $\omega(I \cap J) = \omega(I)$ by Lemma 4.3. Hence $I \cap J$ is $\omega$-linear by Lemma 4.7(2). \[\square\]

Given a monomial ideal $I$ of $R = k[x, y]$ we will write $I = (x^{a_1}y^{b_1}, \ldots, x^{a_r}y^{b_r})$ where $\{a_i\}$ and $\{b_i\}$ are strictly decreasing and strictly increasing sequences of non-negative integers, respectively. Similarly, if $J$ is a monomial ideal of $R$ we write $J = (x^{c_1}y^{d_1}, \ldots, x^{c_s}y^{d_s})$ where $\{c_i\}$ and $\{d_i\}$ are strictly decreasing and strictly increasing sequence of non-negative integers, respectively. Hence $b_1 = a_r = 0$ iff $I$ is $(x, y)$-primary, and $d_1 = c_s = 0$ iff $J$ is $(x, y)$-primary.

**Lemma 4.10.** Let $R = k[x, y]$ and $I, J$ be $(x, y)$-primary monomial ideals with $\omega(I) \geq \omega(J)$. Then $\omega(IJ) \leq \omega(I) + \max\{c_1, d_s\}$.
Proof. We may assume that \( c_1 \geq d_s \). Then \( e(I) = \omega(I) \geq \omega(J) \geq c_1 \) by Lemma 4.1, so \((x, y)^{e(I)+c_1} = (x, y)^{e(I)}(x^{c_1}, y^{c_1}) = (\sqrt{I})^{e(I)}(x^{c_1}, y^{c_1}) \subseteq IJ \) are \((x, y)\)-primary ideals. Therefore \( \omega(IJ) \leq \omega((x, y)^{e(I)+c_1}) = e(I) + c_1 = \omega(I) + c_1 \) by Lemma 4.2.

We now classify \( \omega \)-linear monomial ideals \( I \) in \( R = k[x, y] \).

**Proposition 4.11.** Let \( R = k[x, y] \) and \( I = (x^{a_1}y^{b_1}, \ldots, x^{a_r}y^{b_r}) \) be a monomial ideal of \( R \). Then the following are equivalent.

1. \( I \) is \( \omega \)-linear.
2. \( \omega(I^m) = m\omega(I) \) for some \( m > 1 \).
3. \( \omega(I) = \max\{a_1 + b_1, a_r + b_r\} \).

Proof. Note that given \( m \in \mathbb{N} \) and a monomial \( f \) of \( R \), by Lemma 3.1 we have

\[
\omega(I^m) = m\omega(I)
\]

\[
\Leftrightarrow m(deg(f)) + \omega(I^m) = m(deg(f)) + m\omega(I)
\]

\[
\Leftrightarrow deg(f^m) + \omega(I^m) = m(deg(f) + \omega(I))
\]

\[
\Leftrightarrow \omega((fI)^m) = m\omega(fI).
\]

Moreover, if \( I \) is a principal ideal, then \( I \) satisfies all of 1, 2, and 3 by Corollary 3.2. Hence we may assume that \( I \) is a \((x, y)\)-primary monomial ideal of \( R \). That is, \( a_r = b_1 = 0 \).

1. \( \Rightarrow \) (2) is trivial.

2. \( \Rightarrow \) (3) Suppose that \( \omega(I^m) = m\omega(I) \) for some \( m > 1 \). Note that \( \omega(I^{m-1}) + \omega(I) \geq \omega(I^m) = m\omega(I) \) by Lemma 4.3 and \( \omega(I^{m-1}) \leq (m-1)\omega(I) \) by Corollary 4.6, and thereby \( \omega(I^{m-1}) = (m-1)\omega(I) \). Hence we must have \( \omega(I^2) = 2\omega(I) \). Since \( \omega(I^2) \leq \omega(I) + \max\{a_1, b_r\} \) by Lemma 4.10, \( \omega(I) = \omega(I^2) - \omega(I) \leq \max\{a_1, b_r\} \). On the other hand, \( \omega(I) \geq \max\{a_1, b_r\} \) by Lemma 4.1. Therefore \( \omega(I) = \max\{a_1, b_r\} \).

3. \( \Rightarrow \) (1) Follows from Lemma 4.7(2).

**Lemma 4.12.** The set of monomial \( \omega \)-linear ideals of \( R = k[x, y] \) is multiplicatively closed.

Proof. Let \( I \) and \( J \) be monomial \( \omega \)-linear ideals of \( R \). By Lemma 3.1 we may assume that \( I \) and \( J \) are \((x, y)\)-primary ideals of \( R \). Then \( \omega(I) = \max\{a_1, b_r\} \), \( \omega(J) = \max\{c_1, d_s\} \) by Proposition 4.11. Now, \( x^{a_1+c_1} \) and \( y^{b_r+d_r} \) are elements of \( G(IJ) \). Hence by Lemma 4.7(2) and Lemma 4.1, to show that \( IJ \) is \( \omega \)-linear it suffices to show that \( \omega(IJ) \leq \max\{a_1 + c_1, b_r + d_s\} \). Suppose that \( \omega(I) = a_1 \) and
\( \omega(J) = c_1 \). Then all we have to show is \( \omega(IJ) \leq a_1 + c_1 \), which follows from Lemma 4.3. The case when \( \omega(I) = b_r \) and \( \omega(J) = d_s \) can be derived in the exact same manner. Therefore, without loss of generality we may assume that \( \omega(I) = a_1 > b_r \) and \( \omega(J) = d_s > c_1 \). Observe now that \( Ix^{c_1} + Jy^{b_r} \) is an \((x,y)\)-primary ideal contained in \( IJ \). Thus by Lemma 4.2 and Theorem 3.9 we have
\[
\begin{align*}
\omega(IJ) & \leq \omega(Ix^{c_1} + Jy^{b_r}) \\
& = \max \{ \max_{1 \leq i \leq r-1} \{ a_i + b_{i+1} + c_1 \} - 1, \max_{1 \leq j \leq s-1} \{ c_j + d_{j+1} + b_r \} - 1 \} \\
& = \max \{ \omega(I) + c_1, \omega(J) + b_r \} \\
& = \max \{ a_1 + c_1, b_r + d_s \}.
\end{align*}
\]

\( \square \)

Recall that given an ideal \( I \) of a commutative ring \( R \), an element \( f \in R \) is said to be \textit{integral} over \( I \) if there is some \( k \in \mathbb{N} \) and \( c_i \in I \) for each \( i \in \{1, \ldots, k\} \) so that
\[
f^k + c_1 f^{k-1} + \cdots + c_{k-1} f + c_k = 0.
\]

The set of elements of \( R \) integral over \( I \) is called the \textit{integral closure} of \( I \) and is denoted by \( \overline{I} \). \( I \) is said to be \textit{integrally closed} if \( I = \overline{I} \).

\textbf{Corollary 4.13.} \textit{Every integrally closed monomial ideal of \( R = k[x, y] \) is \( \omega \)-linear.}

\textbf{Proof.} Let \( I \) be an integrally closed monomial ideal of \( R \). It is well known that \( R \) is an \textit{integrally closed domain} (i.e., \( R \) is an integral domain that contains every nonzero element of the quotient field of \( R \) that is integral over \( R \)), and that each principal ideal of \( R \) is integrally closed, and the product of an integrally closed ideal of \( R \) and a nonzero element of \( R \) yields another integrally closed ideal of \( R \). Hence by Lemma 3.1 we may assume that \( I \) is \((x,y)\)-primary. Now by [16, Proposition 2.6] there are monomial ideals \( I_1 = \{ x^{r-i} y^{b_i} \}_{i=0}^{r} \) and \( I_2 = \{ x^{a_i} y^{b_i} \}_{i=0}^{r} \) of \( R \) with \( 0 = b_0 < b_1 < \cdots < b_r \) and \( a_0 > a_1 > \cdots > a_r = 0 \) so \( I = I_1 I_2 \). Thus by Lemma 4.12, it suffices to show that \( I_1 \) and \( I_2 \) are \( \omega \)-linear. By Theorem 3.9, \( \omega(I_1) = \max_{0 \leq i \leq r} \{ c_i \} \), where \( c_i = r - i + b_{i+1} - 1 \) for each \( i \in \{0, 1, \ldots, r-1\} \). Since \( c_{i+1} - c_i = b_{i+1} - (b_i + 1) \geq 0 \) for each \( i \in \{0, 1, \ldots, r-1\} \), we have \( \omega(I_1) = c_{r-1} = b_r = \max \{ r, b_r \} \) and \( I_1 \) is \( \omega \)-linear by Proposition 4.11. The proof that \( I_2 \) is \( \omega \)-linear follows similarly. \( \square \)

\textbf{Remark 4.14.} (1) Even if \( I \) and \( J \) are \( \omega \)-linear monomial primary ideals such that \( \sqrt{I} = \sqrt{J} \), we may have \( \omega(I \cap J) < \omega(IJ) < \omega(I) + \omega(J) \). Indeed, set \( R = k[x, y] \),
Proposition 1.4.4. Thus ideals of \( R \) if it has no cycle of odd length as its subgraph. \( P \) corresponds to \( I = (x^2, xy, y^3) \).

Recall that a graph \( G \) is said to be a squarefree monomial ideal if \( \omega(G) = 4 \) by Theorem 3.9, and \( I \) is \( \omega \)-linear by Proposition 4.11. However, \((x^3, y^2) \) is not a squarefree monomial ideal, so \( \omega(I) = 4 \). Thus \( I \) is not integrally closed ([10, Theorem 1.4.2]).

So far, we have considered only \( \omega \)-linear monomial ideals of the form \( fI \) where \( f \) is a monomial and \( I \) is a primary ideal, and most of the proof is solely based on the fact that \( \epsilon(I) = \omega(I) \) when \( I \) is a primary ideal. We now show that there exists a class of (integrally closed) nonprimary \( \omega \)-linear monomial ideals. In fact, some of the squarefree monomial ideals are \( \omega \)-linear. Recall that a monomial \( f = x_1^{a_1} \cdots x_r^{a_r} \) is said to be squarefree if \( a_1 = \cdots = a_r = 1 \). A monomial ideal generated by squarefree monomials is said to be a squarefree monomial ideal.

Lemma 4.15. Let \( I \) be a squarefree monomial ideal. Then \( \omega(I^m) \geq m\omega(I) \) for each \( m \in \mathbb{N} \).

Proof. Let \( P_1, \ldots, P_r \) be minimal prime ideals of \( I \). Then \( I = \bigcap_{i=1}^r P_i \) and \( \omega(I) = r \) by Proposition 3.5. Set \( f_i = \sum_{x_j \in G(P_i)} x_j \) for each \( i \in \{1, \ldots, r\} \). Then \( f := \prod_{i=1}^r f_i \in \prod_{i=1}^r P_i \subseteq I \), so \( f^m \in I^m \). However, \( \frac{f^m}{f_i} \notin P_i^m \), so \( \frac{f^m}{f_i} \notin I^m([10, Proposition 1.4.4]) \). Thus \( I^m \) is not \((mr - 1)\)-absorbing and \( \omega(I^m) \geq m\omega(I) \). \( \square \)

Recall that a graph \( G \) consists of a set of vertices \( V = \{v_1, \ldots, v_n\} \) and a set of edges \( E \subseteq \{v_i v_j \mid v_i, v_j \in V\} \), and is called bipartite if there exists two disjoint subsets \( U_1, U_2 \) of \( V \) such that \( E \subseteq \{v_i v_j \mid v_i \in U_1, v_j \in U_2\} \). The edge ideal of \( G \) is defined to be the ideal \( I = (\{x_i x_j \mid v_i, v_j \in E\}) \) of \( R = k[x_1, \ldots, x_d] \), where \( k \) be a field and \( d \) is the number of vertices of \( G \). Given a graph \( G = (V, E) \), a subset \( W \) of \( V \) is said to be a vertex cover if given \( v_i v_j \in E \), either \( v_i \in W \) or \( v_j \in W \). A vertex cover \( W \) of \( G \) is said to be a minimal vertex cover if each proper subset of \( W \) is not a vertex cover of \( G \).

If \( I \) is an edge ideal of a graph, then it is a squarefree monomial ideal and a monomial prime ideal \( P \) is a minimal ideal of \( I \) if and only if the set of vertices that corresponds to \( P \) is a minimal vertex cover. Also, a graph is bipartite if and only if it has no cycle of odd length as its subgraph.
Our first example of a nonprimary $\omega$-linear ideal is the edge ideal of a bipartite graph.

**Lemma 4.16.** Let $R = k[x_1, \ldots, x_n]$. If $I$ is an ideal of $R$ that is also the edge ideal of a bipartite graph $G$, then $I$ is $\omega$-linear.

**Proof.** Let $I$ be an edge ideal of a graph $G$ and let $P_1, \ldots, P_r$ be the set of (incomparable) minimal prime ideals of $I$. Recall that a graph $G$ is bipartite if and only if

$$I^m = \bigcap_{P \text{ is a minimal prime of } I} P^m$$

for each $m \in \mathbb{N}$ ([18, Theorem 5.9]). Hence if $G$ is bipartite, then by Proposition 3.5, $\omega(I^m) = \sum_{i=1}^r e(P_i^m) = \sum_{i=1}^r m = mr$ for each $m \in \mathbb{N}$. Therefore the conclusion follows. □

There are nonbipartite graphs whose edge ideals are $\omega$-linear.

**Theorem 4.17.** Let $R = k[x_1, \ldots, x_n]$. Let $I = (x_1x_2, x_2x_3, \ldots, x_{n-1}x_n, x_nx_1)$ (that is, $I$ is the edge ideal of a cycle graph of length $n$). Then $I$ is $\omega$-linear.

**Proof.** Since a cycle of even length is bipartite, by Lemma 4.16 we may assume that $n = 2l + 1$ for some $l \in \mathbb{N}$. Fix $m \in \mathbb{N}$. $I$ is a squarefree monomial ideal, so $I = P_1 \cap \cdots \cap P_r$ where $P_1, \ldots, P_r$ are the minimal prime ideals of $I$ ([10, Lemma 1.3.5]). Thus by Proposition 3.5 we have $\omega(I) = \sum_{i=1}^r e(P_i) = r$, and we only need to show that $\omega(I^m) = mr$. Note that since $I$ is an edge ideal of a cycle of length $2l + 1$, $Ass(R/I^m) = \{P_1, \ldots, P_r\}$ if $m \leq l$ and $Ass(R/I^m) = \{P_1, \ldots, P_r, m\}$ if $m > l$ ([5, Lemma 3.1]). Hence if $m \leq l$, then $I^m = \bigcap_{i=1}^r P_i^m$ and $\omega(I^m) = \sum_{i=1}^r e(P_i^m) = mr$ by Proposition 3.5, so we are done. Assume that $m > l$. Then $I^m = (\bigcap_{i=1}^r P_i^m) \cap Q$ is the canonical primary decomposition of $I^m$, where $Q$ is an $m$-primary monomial ideal of $R$ ([10, Proposition 1.4.4]). Now, $Q = (x_1^{a_1}, \ldots, x_n^{a_n}, f_1, \ldots, f_t)$ for some $a_i \in \mathbb{N}$ and monomials $f_i$. Since $I$ is a squarefree monomial ideal and $Q$ is a primary component of $I^m$, we must have $a_i \leq m$ for each $i \in \{1, \ldots, n\}$, and thus $e(Q) \leq e((x_1^{a_1}, \ldots, x_n^{a_n})) \leq mn - n + 1 \leq mr$ by Lemma 2.4 and since $n \leq r$. It follows that $\omega(I^m) = \max\{\sum_{i=1}^r e(P_i^m), e(Q)\} = \max\{mr, e(Q)\} = mr$ by Theorem 3.3 and Proposition 3.5. □

We close the section with the following question: Is every integrally closed monomial ideal $\omega$-linear? Integrally closed monomial ideals considered in this note (certain monomial ideals in $R = k[x, y]$, irreducible monomial ideals, or edge ideal of
bipartite graphs) were all $\omega$-linear. Note also that if this question has an affirmative answer, then it follows that every edge ideal is $\omega$-linear.

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