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Research Article

Pair of generalized derivations acting on multilinear polynomials in prime rings

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Abstract

Let R be a noncommutative prime ring of characteristic different from 2 with Utumi quotient ring U and extended centroid C and $f(r_1,\dots,r_n)$ be a multilinear polynomial over C, which is not central valued on R. Suppose that F and G are two nonzero generalized derivations of R such that $G \neq Id$ (identity map) and

$$F(f(r)^2) = F(f(r))G(f(r)) + G(f(r))F(f(r))$$

for all $r = (r_1,\dots,r_n) \in R^n$. Then one of the following holds:

(1) there exist $\lambda \in C$ and $\mu \in C$ such that $F(x) = \lambda x$ and $G(x) = \mu x$ for all $x \in R$ with $2\mu = 1$;
(2) there exist $\lambda \in C$ and $p, q \in U$ such that $F(x) = \lambda x$ and $G(x) = px + xq$ for all $x \in R$ with $p + q \in C$, $2(p + q) = 1$ and $f(x_1,\dots,x_n)^2$ is central valued on R;
(3) there exist $\lambda \in C$ and $a \in U$ such that $F(x) = [a, x]$ and $G(x) = \lambda x$ for all $x \in R$ with $f(x_1,\dots,x_n)^2$ is central valued on R;
(4) there exist $\lambda \in C$ and $a, b \in U$ such that $F(x) = ax + xb$ and $G(x) = \lambda x$ for all $x \in R$ with $a + b \in C$, $2\lambda = 1$ and $f(x_1,\dots,x_n)^2$ is central valued on R;
(5) there exist $a, p \in U$ such that $F(x) = xa$ and $G(x) = px$ for all $x \in R$, with $(p - 1)a = -ap \in C$ and $f(x_1,\dots,x_n)^2$ is central valued on R;
(6) there exist $a, q \in U$ such that $F(x) = ax$ and $G(x) = xq$ for all $x \in R$ with $a(q - 1) = -qa \in C$ and $f(x_1,\dots,x_n)^2$ is central valued on R.

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1. Introduction

Throughout this paper R always denotes an associative prime ring with extended centroid C and U its Utumi ring of quotients. By a derivation, we mean an additive mapping $d : R \to R$ such that $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. By a generalized derivation, we mean an additive mapping $F : R \to R$ such that $F(xy) = F(x)y + xd(y)$

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holds for all $x, y \in R$, where d is a derivation of R. Thus any derivation is a generalized derivation.

A famous result proved by Posner [17, Theorem 2] states that if a prime ring R has a nonzero derivation d such that $[d(x), x] \in Z(R)$ for all $x \in R$, then R is commutative. Brešar [2] studied the case $d(x)x - x\delta(x) \in Z(R)$ for all $x \in R$, where d and δ are two derivations of a prime ring R and obtained that either $d = \delta = 0$ or R is commutative. After that in [13] Lee and Shiue extended the previous result considering multilinear polynomial. They proved that if $d(f(x_1,\dots,x_n))f(x_1,\dots,x_n) - f(x_1,\dots,x_n)\delta(f(x_1,\dots,x_n)) \in Z(R)$ for all $x_1,\dots,x_n \in I$, where I is a nonzero ideal of R and $f(x_1,\dots,x_n)$ is a non-central multilinear polynomial over C, then either $d = 0 = \delta$ or $d = -\delta$ and $f(x_1,\dots,x_n)^2$ is central valued on RC unless $char(R) = 2$ and $dim_C RC = 4$.

Recently in [4], De Filippis et al. showed that if d and δ are nonzero derivations of R and $f(x_1,\dots,x_n)$ is a multilinear polynomial over C, non-central valued on R, such that $[d(f(x_1,\dots,x_n)), \delta(f(x_1,\dots,x_n))] \in Z(R)$ for all $x_1,\dots,x_n \in R$, then $\{d, \delta\}$ are linear dependent over C unless when $char(R) = 2$ and $dim_C RC = 4$.

More recently, Fosner and Vukman [7] proved that if R is a prime ring of $char(R) \neq 2$, F_1 and F_2 are generalized derivations of R such that $F_1(x)F_2(x) + F_2(x)F_1(x) = 0$ for all $x \in R$ then either $F_1 = 0$ or $F_2 = 0$. In [18], Rania and Scudo extended this result to the case $G(f(x_1,\dots,x_n))d(f(x_1,\dots,x_n)) + d(f(x_1,\dots,x_n))G(f(x_1,\dots,x_n)) = 0$ for all $x_1,\dots,x_n \in R$, where G is a generalized derivation of R and d is any derivation of R, and proved that either $G = 0$ or $d = 0$, except when d is inner, there exists $\lambda \in C$ such that $G(x) = \lambda x, \forall x \in R$ and $f(x_1,\dots,x_n)^2$ is central valued on R. Recently, in [19] Yarbil and De Filippis studied the same situation, when G and d are two skew derivations of R associated to the same automorphism α and obtained that either $G = 0$ or $d = 0$. Here skew derivation means an additive mapping $d : R \to R$ such that $d(xy) = d(x)y + \alpha(x)d(y)$ for all $x, y \in R$, where α is an automorphism of R.

Recently, Dhara et al. [6] extended the above result by taking generalized derivation F in the place of derivation d, that is,

$$F(f(x_1,\dots,x_n))G(f(x_1,\dots,x_n)) + G(f(x_1,\dots,x_n))F(f(x_1,\dots,x_n)) = 0,$$

where F, G are two generalized derivations of R. In the present paper, we consider the case $F(f(r)^2) = F(f(r))G(f(r)) + G(f(r))F(f(r))$ for all $r = (r_1,\dots,r_n) \in R^n$, where F and G are two generalized derivations of R. If $G = Id$ (identity map), then F becomes a derivation of R. So our interest is to study the case when $G \neq Id$. More precisely, we prove the following theorem.

Main Theorem. *Let R be a noncommutative prime ring of characteristic different from 2 with Utumi quotient ring U and extended centroid C, and $f(r_1,\dots,r_n)$ be a multilinear polynomial over C, which is not central valued on R. Suppose that F and G are two nonzero generalized derivations of R such that $G \neq Id$ (identity map) and*

$$F(f(r)^2) = F(f(r))G(f(r)) + G(f(r))F(f(r))$$

for all $r = (r_1,\dots,r_n) \in R^n$. Then one of the following holds:

1. *there exist $\lambda \in C$ and $\mu \in C$ such that $F(x) = \lambda x$ and $G(x) = \mu x$ for all $x \in R$ with $2\mu = 1$;*
2. *there exist $\lambda \in C$ and $p, q \in U$ such that $F(x) = \lambda x$ and $G(x) = px + xq$ for all $x \in R$ with $p + q \in C$, $2(p + q) = 1$, and $f(x_1,\dots,x_n)^2$ is central valued on R;*
3. *there exist $\lambda \in C$ and $a \in U$ such that $F(x) = [a, x]$ and $G(x) = \lambda x$ for all $x \in R$ with $f(x_1,\dots,x_n)^2$ is central valued on R;*
4. *there exist $\lambda \in C$ and $a, b \in U$ such that $F(x) = ax + xb$ and $G(x) = \lambda x$ for all $x \in R$ with $a + b \in C$, $2\lambda = 1$ and $f(x_1,\dots,x_n)^2$ is central valued on R;*

(5) *there exist* $a, p \in U$ *such that* $F(x) = xa$ *and* $G(x) = px$ *for all* $x \in R$, *with* $(p-1)a = -ap \in C$ *and* $f(x_1, \ldots, x_n)^2$ *is central valued on* R;
(6) *there exist* $a, q \in U$ *such that* $F(x) = ax$ *and* $G(x) = xq$ *for all* $x \in R$ *with* $a(q-1) = -qa \in C$ *and* $f(x_1, \ldots, x_n)^2$ *is central valued on* R.

Following corollaries are straightforward.

Corollary 1.1. *Let* R *be a noncommutative prime ring of characteristic different from 2 with Utumi quotient ring* U *and extended centroid* C, *and* $f(r_1, \ldots, r_n)$ *be a multilinear polynomial over* C, *which is not central valued on* R. *Suppose that* F *is a nonzero generalized derivation of* R *and* d *is a nonzero derivation of* R *such that*

$$F(f(r)^2) = F(f(r))d(f(r) + d(f(r))F(f(r)$$

for all $r = (r_1, \ldots, r_n) \in R^n$. *Then there exist* $\lambda \in C$ *and* $p \in U$ *such that* $F(x) = \lambda x$ *and* $d(x) = [p, x]$ *for all* $x \in R$ *with* $f(x_1, \ldots, x_n)^2$ *is central valued on* R.

Corollary 1.2. *Let* R *be a noncommutative prime ring of characteristic different from 2 with Utumi quotient ring* U *and extended centroid* C, *and* $f(r_1, \ldots, r_n)$ *be a multilinear polynomial over* C, *which is not central valued on* R. *Suppose that* G *is a nonzero generalized derivation of* R *such that*

$$G(f(r))f(r) + f(r)G(f(r)) = f(r)^2$$

for all $r = (r_1, \ldots, r_n) \in R^n$, *then one of the following holds:*
(1) *there exists* $\mu \in C$ *such that* $G(x) = \mu x$ *for all* $x \in R$ *with* $2\mu = 1$;
(2) *there exist* $p, q \in U$ *such that* $G(x) = px + xq$ *for all* $x \in R$ *with* $p + q \in C$, $2(p+q) = 1$ *and* $f(x_1, \ldots, x_n)^2$ *is central valued on* R.

2. Main results

Lemma 2.1. [1, Lemma 3] *Let* R *be a noncommutative prime ring with Utumi quotient ring* U *and extended centroid* C, *and* $f(x_1, \ldots, x_n)$ *be a multilinear polynomial over* C, *which is not central valued on* R. *Suppose that there exist* $a, b, c, q \in U$ *such that* $(af(r) + f(r)b)f(r) - f(r)(cf(r) + f(r)q) = 0$ *for all* $r = (r_1, \ldots, r_n) \in R^n$. *Then one of the following holds:*
(1) $a, q \in C$ *and* $q - a = b - c \in C$;
(2) $f(x_1, \ldots, x_n)^2$ *is central valued on* R *and* $q - a = b - c \in C$;
(3) $char(R) = 2$ *and* R *satisfies* s_4.

In particular, from above Lemma, we have the followings:

Lemma 2.2. *Let* R *be a noncommutative prime ring of characteristic different from 2 with Utumi quotient ring* U *and extended centroid* C, *and* $f(x_1, \ldots, x_n)$ *be a multilinear polynomial over* C, *which is not central valued on* R. *Suppose that there exist* $a, b, q \in U$ *such that* $af(r)^2 + f(r)^2q + f(r)bf(r) = 0$ *for all* $r = (r_1, \ldots, r_n) \in R^n$. *Then one of the following holds:*
(1) $a, q \in C$ *and* $q + a = -b \in C$;
(2) $f(x_1, \ldots, x_n)^2$ *is central valued on* R *and* $q + a = -b \in C$;
(3) $char(R) = 2$ *and* R *satisfies* s_4.

Lemma 2.3. [6, Corollary 2.14] *Let* R *be a prime ring of characteristic different from 2, with Utumi quotient ring* U *and extended centroid* C, *and* $f(x_1, \ldots, x_n)$ *be a multilinear polynomial over* C. *Suppose that* d *and* δ *are two nonzero derivations of* R *such that*

$$d(f(r))\delta(f(r)) + \delta(f(r))d(f(r)) = 0$$

for all $r = (r_1, \ldots, r_n) \in R^n$. *Then* $f(x_1, \ldots, x_n)$ *is central valued on* R.

Lemma 2.4. [6, Lemma 2.10] *Let R be a prime ring of characteristic different from 2, U its Utumi quotient ring, and C its extended centroid, and $f(x_1,\ldots,x_n)$ a multilinear polynomial over C which is non-central valued on R. Suppose that $a,b,p\in U$ such that*

$$af(r)^2b+f(r)pf(r)=0$$

for all $r=(r_1,\ldots,r_n)\in R^n$. Then one of the following holds:

(1) *$a\in C$ and $ab=-p\in C$;*
(2) *$b\in C$ and $ab=-p\in C$;*
(3) *$f(x_1,\ldots,x_n)^2$ is central valued on R and $ab=-p\in C$.*

Lemma 2.5. [5, Lemma 1.5] *Let C be an infinite field and $m\geq 2$. If $A_1,\ldots,A_k$ are not scalar matrices in $M_m(C)$ then there exists some invertible matrix $P\in M_m(C)$ such that any matrices $PA_1P^{-1},\ldots,PA_kP^{-1}$ have all nonzero entries.*

Proposition 2.6. *Let $R=M_m(C)$ be the ring of all $m\times m$ matrices over the infinite field C, $f(x_1,\ldots,x_n)$ a non-central multilinear polynomial over C and $a,b,p,q\in R$. If*

$$(af(r)^2+f(r)^2b)=(af(r)+f(r)b)(pf(r)+f(r)q)+(pf(r)+f(r)q)(af(r)+f(r)b)$$

for all $r=(r_1,\ldots,r_n)\in R^n$, then either a or p and either b or q are scalar matrices.

Proof. By our assumption, R satisfies the generalized polynomial identity

$$\begin{aligned}&(af(r_1,\ldots,r_n)^2+f(r_1,\ldots,r_n)^2b)\\&=(af(r_1,\ldots,r_n)+f(r_1,\ldots,r_n)b)(pf(r_1,\ldots,r_n)+f(r_1,\ldots,r_n)q)\\&+(pf(r_1,\ldots,r_n)+f(r_1,\ldots,r_n)q)(af(r_1,\ldots,r_n)+f(r_1,\ldots,r_n)b).\end{aligned}\tag{2.1}$$

We assume first that $a\notin Z(R)$ and $p\notin Z(R)$. Now we shall show that this case leads to a contradiction.
Since $a\notin Z(R)$ and $p\notin Z(R)$, by Lemma 2.5 there exists a C-automorphism ϕ of $M_m(C)$ such that $a_1=\phi(a)$, $p_1=\phi(p)$ have all nonzero entries. Clearly a_1, p_1, $b_1=\phi(b)$ and $q_1=\phi(q)$ must satisfy the condition (2.1). Without loss of generality we may replace a,b,p,q with a_1,b_1,p_1,q_1, respectively.
Here e_{kl} denotes the usual matrix unit with 1 in (k,l)-entry and zero elsewhere. Since $f(x_1,\ldots,x_n)$ is not central, by [14] (see also [15]), there exist $u_1,\ldots,u_n\in M_m(C)$ and $\gamma\in C-\{0\}$ such that $f(u_1,\ldots,u_n)=\gamma e_{kl}$, with $k\neq l$. Moreover, since the set $\{f(r_1,\ldots,r_n):r_1,\ldots,r_n\in M_m(C)\}$ is invariant under the action of all C-automorphisms of $M_m(C)$, then for any $i\neq j$ there exist $r_1,\ldots,r_n\in M_m(C)$ such that $f(r_1,\ldots,r_n)=e_{ij}$. Hence from (2.1) we have

$$0=(ae_{ij}+e_{ij}b)(pe_{ij}+e_{ij}q)+(pe_{ij}+e_{ij}q)(ae_{ij}+e_{ij}b)\tag{2.2}$$

and then left multiplying by e_{ij}, it follows $e_{ij}ae_{ij}pe_{ij}+e_{ij}pe_{ij}ae_{ij}=0$, which gives $2a_{ji}p_{ji}=0$, that is a contradiction, since a and p have all nonzero entries. Thus we conclude that either a or p is central.

Similarly, we can prove that b or q is central.
Therefore we conclude that either a or p and either b or q are scalar matrices. □

Proposition 2.7. *Let $R=M_m(C)$ be the ring of all matrices over the field C with $char(R)\neq 2$ and $f(x_1,\ldots,x_n)$ a non-central multilinear polynomial over C and $a,b,p,q\in R$. If*

$$(af(r)^2+f(r)^2b)=(af(r)+f(r)b)(pf(r)+f(r)q)+(pf(r)+f(r)q)(af(r)+f(r)b)$$

for all $r=(r_1,\ldots,r_n)\in R^n$, then either a or p and either b or q are scalar matrices.

Proof. If one assumes that C is infinite, then the conclusions follow by Proposition 2.6. Now let C be finite and K be an infinite field which is an extension of the field C. Let

$\overline{R} = M_m(K) \cong R \otimes_C K$. Notice that the multilinear polynomial $f(r_1, \ldots, r_n)$ is central valued on R if and only if it is central valued on $\overline{R}$. Consider the generalized polynomial

$$\begin{aligned} P(r_1, \ldots, r_n) &= (af(r_1, \ldots, r_n)^2 + f(r_1, \ldots, r_n)^2 b) \\ &-(af(r_1, \ldots, r_n) + f(r_1, \ldots, r_n)b)(pf(r_1, \ldots, r_n) + f(r_1, \ldots, r_n)q) \\ &-(pf(r_1, \ldots, r_n) + f(r_1, \ldots, r_n)q)(af(r_1, \ldots, r_n) + f(r_1, \ldots, r_n)b) \end{aligned} \quad (2.3)$$

which is a generalized polynomial identity for R.
Moreover, it is a multi-homogeneous of multi-degree $(2, \ldots, 2)$ in $r_1, \ldots, r_n$.
Hence the complete linearization of $P(r_1, \ldots, r_n)$ is a multilinear generalized polynomial $\Theta(r_1, \ldots, r_n, y_1, \ldots, y_n)$ in $2n$ indeterminates, moreover

$$\Theta(r_1, \ldots, r_n, r_1, \ldots, r_n) = 2^n P(r_1, \ldots, r_n).$$

Clearly the multilinear polynomial $\Theta(r_1, \ldots, r_n, y_1, \ldots, y_n)$ is a generalized polynomial identity for R and $\overline{R}$ too. Since $char(C) \neq 2$ we obtain $P(r_1, \ldots, r_n) = 0$ for all $r_1, \ldots, r_n \in \overline{R}$ and then conclusion follows from Proposition 2.6. □

In the above Proposition, replacing $bp = b'$ and $qa = q'$, it is straightforward to prove the following:

Corollary 2.8. *Let $R = M_m(C)$ be the ring of all matrices over the field C with $char(R) \neq 2$ and $f(x_1, \ldots, x_n)$ a non-central multilinear polynomial over C and $a, b, p, q, b', q' \in R$. If*

$$(af(r)^2 + f(r)^2 b) = af(r)(pf(r) + f(r)q) + f(r)b'f(r) + f(r)bf(r)q$$

$$+pf(r)(af(r) + f(r)b) + f(r)q'f(r) + f(r)qf(r)b$$

for all $r = (r_1, \ldots, r_n) \in R^n$, then either a or p and either b or q are scalar matrices.

Lemma 2.9. *Let R be a noncommutative prime ring of characteristic different from 2 with Utumi quotient ring U and extended centroid C, and $f(r_1, \ldots, r_n)$ be a multilinear polynomial over C, which is not central valued on R. Suppose that F and G ($\neq Id$, identity map) are two nonzero inner generalized derivations of R such that*

$$F(f(r)^2) = F(f(r))G(f(r)) + G(f(r))F(f(r))$$

for all $r = (r_1, \ldots, r_n) \in R^n$. Then one of the following holds:

1. *there exist $\lambda \in C$ and $\mu \in C$ such that $F(x) = \lambda x$ and $G(x) = \mu x$ for all $x \in R$ with $2\mu = 1$;*
2. *there exist $\lambda \in C$ and $p, q \in U$ such that $F(x) = \lambda x$ and $G(x) = px + xq$ for all $x \in R$ with $p + q \in C$, $2(p + q) = 1$ and $f(x_1, \ldots, x_n)^2$ is central valued on R;*
3. *there exist $\lambda \in C$ and $a \in U$ such that $F(x) = [a, x]$ and $G(x) = \lambda x$ for all $x \in R$ with $f(x_1, \ldots, x_n)^2$ is central valued on R;*
4. *there exist $\lambda \in C$ and $a, b \in U$ such that $F(x) = ax + xb$ and $G(x) = \lambda x$ for all $x \in R$ with $a + b \in C$, $2\lambda = 1$ and $f(x_1, \ldots, x_n)^2$ is central valued on R;*
5. *there exist $a, p \in U$ such that $F(x) = xa$ and $G(x) = px$ for all $x \in R$, with $(p - 1)a = -ap \in C$ and $f(x_1, \ldots, x_n)^2$ is central valued on R;*
6. *there exist $a, p \in U$ such that $F(x) = ax$ and $G(x) = xq$ for all $x \in R$ with $a(q - 1) = -qa \in C$ and $f(x_1, \ldots, x_n)^2$ is central valued on R.*

Proof. Since F and G are inner generalized derivations of R, there exist $a, b, p, q \in U$ such that $F(x) = ax + xb$ and $G(x) = px + xq$ for all $x \in R$. Then by hypothesis, we have

$$\begin{aligned} h(r_1, \ldots, r_n) &= (af(r_1, \ldots, r_n)^2 + f(r_1, \ldots, r_n)^2 b) \\ &-(af(r_1, \ldots, r_n) + f(r_1, \ldots, r_n)b)(pf(r_1, \ldots, r_n) + f(r_1, \ldots, r_n)q) \\ &-(pf(r_1, \ldots, r_n) + f(r_1, \ldots, r_n)q)(af(r_1, \ldots, r_n) + f(r_1, \ldots, r_n)b) = 0 \end{aligned} \quad (2.4)$$

for all $r_1, \ldots, r_n \in R$. Since R and U satisfy the same generalized polynomial identities (GPI) (see [3]), U satisfies $h(r_1, \ldots, r_n) = 0$ that is

$$\begin{aligned} h(r_1,\ldots,r_n) &= (af(r_1,\ldots,r_n)^2 + f(r_1,\ldots,r_n)^2 b) \\ &-(af(r_1,\ldots,r_n) + f(r_1,\ldots,r_n)b)(pf(r_1,\ldots,r_n) + f(r_1,\ldots,r_n)q) \\ &-(pf(r_1,\ldots,r_n) + f(r_1,\ldots,r_n)q)(af(r_1,\ldots,r_n) + f(r_1,\ldots,r_n)b) = 0 \end{aligned} \tag{2.5}$$

for all $r_1, \ldots, r_n \in U$. Suppose that $h(r_1, \ldots, r_n)$ is a trivial GPI for U and $C\{r_1, \ldots, r_n\}$, the free C-algebra in noncommuting indeterminates $r_1, \ldots, r_n$. Then, $h(r_1, \ldots, r_n)$ is zero element in $T = U *_C C\{r_1, \ldots, r_n\}$. This implies that $\{a, p, 1\}$ is linearly independent over C. Let $\alpha p + \beta a + \gamma = 0$, where $\alpha, \beta, \gamma \in C$. If $\alpha = 0$, then $\beta \neq 0$ and hence $a \in C$. If $\alpha \neq 0$, then $p = \lambda a + \mu$ for some $\lambda, \mu \in C$. In this case our identity reduces to

$$\begin{aligned} &(af(r_1,\ldots,r_n)^2 + f(r_1,\ldots,r_n)^2 b) \\ &-(af(r_1,\ldots,r_n) + f(r_1,\ldots,r_n)b)((\lambda a + \mu)f(r_1,\ldots,r_n) + f(r_1,\ldots,r_n)q) \\ &-((\lambda a + \mu)f(r_1,\ldots,r_n) + f(r_1,\ldots,r_n)q)(af(r_1,\ldots,r_n) \\ &+f(r_1,\ldots,r_n)b) = 0 \end{aligned} \tag{2.6}$$

in T. If a is not in C, then from above we have

$$\begin{aligned} &af(r_1,\ldots,r_n)((f(r_1,\ldots,r_n) - 2\lambda af(r_1,\ldots,r_n) - \mu f(r_1,\ldots,r_n) \\ &-f(r_1,\ldots,r_n)q - \lambda f(r_1,\ldots,r_n)b) = 0 \end{aligned} \tag{2.7}$$

in T, that is

$$af(r_1,\ldots,r_n)(2\lambda af(r_1,\ldots,r_n) + f(r_1,\ldots,r_n)(\mu + q + \lambda b - 1)) = 0. \tag{2.8}$$

This implies that $\lambda a \in C$ and hence $p = (\lambda a + \mu) \in C$. Thus we conclude that either $a \in C$ or $p \in C$. Similarly, we can prove that either $b \in C$ or $q \in C$.

Next suppose that $h(r_1, \ldots, r_n)$ is a non-trivial GPI for U. In case C is infinite, we have $h(r_1, \ldots, r_n) = 0$ for all $r_1, \ldots, r_n \in U \otimes_C \overline{C}$, where $\overline{C}$ is the algebraic closure of C. Since both U and $U \otimes_C \overline{C}$ are prime and centrally closed [8, Theorems 2.5 and 3.5], we may replace R by U or $U \otimes_C \overline{C}$ according to C finite or infinite. Then R is centrally closed over C and $h(r_1, \ldots, r_n) = 0$ for all $r_1, \ldots, r_n \in R$. By Martindale's theorem [16], R is then a primitive ring with nonzero socle $soc(R)$ and with C as its associated division ring. Then, by Jacobson's theorem [10, p.75], R is isomorphic to a dense ring of linear transformations of a vector space V over C. Assume first that V is finite dimensional over C, that is, $\dim_C V = m$. By density of R, we have $R \cong M_m(C)$. Since $f(r_1, \ldots, r_n)$ is not central valued on R, R must be noncommutative and so $m \geq 2$. In this case, by Proposition 2.7, we get that either a or p and either b or q are in C. If V is infinite dimensional over C, then for any $e^2 = e \in soc(R)$ we have $eRe \cong M_t(C)$ with $t = \dim_C Ve$. In this case we prove that either a or p are in C. To prove this, assume that $a \notin C$ and $p \notin C$. Then there exist $h_1, h_2 \in soc(R)$ such that $[a, h_1] \neq 0$ and $[p, h_2] \neq 0$. By Litoff's Theorem [9], there exists idempotent $e \in soc(R)$ such that $ah_1, h_1 a, ph_2, h_2 p, h_1, h_2 \in eRe$. We have $eRe \cong M_k(C)$ with $k = \dim_C Ve$. Since R satisfies generalized identity

$$\begin{aligned} &e\{af(er_1e,\ldots,er_ne)^2 + f(er_1e,\ldots,er_ne)^2 b\}e \\ &= e\{(af(er_1e,\ldots,er_ne) + f(er_1e,\ldots,er_ne)b) \qquad (2.9) \\ &.(pf(er_1e,\ldots,er_ne) + f(er_1e,\ldots,er_ne)q) \\ &+(pf(er_1e,\ldots,er_ne) + f(er_1e,\ldots,er_ne)q) \\ &.(af(er_1e,\ldots,er_ne) + f(er_1e,\ldots,er_ne)b)\}e, \qquad (2.10) \end{aligned}$$

the subring eRe satisfies

$$\begin{aligned}
&eaef(r_1,\dots,r_n)^2+f(r_1,\dots,r_n)^2ebe\\
&=eaef(r_1,\dots,r_n)(epef(r_1,\dots,r_n)+f(r_1,\dots,r_n)eqe)\\
&+f(r_1,\dots,r_n)ebpef(r_1,\dots,r_n)+f(r_1,\dots,r_n)ebef(r_1,\dots,r_n)eqe\\
&+epef(r_1,\dots,r_n)(eaef(r_1,\dots,r_n)+f(r_1,\dots,r_n)ebe)\\
&+f(r_1,\dots,r_n)eqaef(r_1,\dots,r_n)+f(r_1,\dots,r_n)eqef(r_1,\dots,r_n)ebe. \qquad (2.11)
\end{aligned}$$

Then by Corollary 2.8, either eae or epe are central elements of eRe. Thus either $ah_1=(eae)h_1=h_1eae=h_1a$ or $ph_2=(epe)h_2=h_2(epe)=h_2p$, a contradiction. Hence either a or p are in C.

Similarly, we can prove that either b or q are in C.
Thus we have the following cases:

Case 1: Let $a,b\in C$.

In this case, by (2.5) U satisfies

$$\begin{aligned}
(a+b)f(r_1,\dots,r_n)^2-(a+b)(f(r_1,\dots,r_n)(pf(r_1,\dots,r_n)+f(r_1,\dots,r_n)q)-\\
(pf(r_1,\dots,r_n)+f(r_1,\dots,r_n)q)(a+b)f(r_1,\dots,r_n)=0. \qquad (2.12)
\end{aligned}$$

Since $F\neq 0$, $a+b\neq 0$. Hence from above

$$\begin{aligned}
&f(r_1,\dots,r_n)^2-(f(r_1,\dots,r_n)pf(r_1,\dots,r_n)+f(r_1,\dots,r_n)^2q))\\
&-(pf(r_1,\dots,r_n)^2+f(r_1,\dots,r_n)qf(r_1,\dots,r_n))=0. \qquad (2.13)
\end{aligned}$$

This implies

$$pf(r_1,\dots,r_n)^2+f(r_1,\dots,r_n)^2(q-1)+f(r_1,\dots,r_n)(p+q)f(r_1,\dots,r_n)=0,$$

for all $r_1,\dots,r_n\in U$. Then by Lemma 2.2, one of the following holds:

(1) $p,q-1\in C$ and $p+q-1=-(p+q)\in C$. In this case we have $F(x)=ax+xb=(a+b)x$ and $G(x)=px+xq=(p+q)x$ for all $x\in R$, with $2(p+q)=1$ which is our conclusion (1).

(2) $f(x_1,\dots,x_n)^2$ is central valued on R and $p+q-1=-(p+q)\in C$. In this case, we have $F(x)=ax+xb=(a+b)x$ and $G(x)=px+xq$ for all $x\in R$ with $p+q\in C$ and $2(p+q)=1$, which is our conclusion (2).

Case 2: Let $p\in C$ and $q\in C$.

Then by (2.5), U satisfies

$$\begin{aligned}
&(af(r_1,\dots,r_n)^2+f(r_1,\dots,r_n)^2b)\\
&-(af(r_1,\dots,r_n)+f(r_1,\dots,r_n)b)(p+q)f(r_1,\dots,r_n)\\
&-(p+q)f(r_1,\dots,r_n)(af(r_1,\dots,r_n)+f(r_1,\dots,r_n)b)=0. \qquad (2.14)
\end{aligned}$$

This can be written as

$$\begin{aligned}
&a(1-p-q)f(r_1,\dots,r_n)^2+f(r_1,\dots,r_n)^2b(1-p-q)\\
&-(f(r_1,\dots,r_n)(a+b)(p+q)f(r_1,\dots,r_n)=0 \qquad (2.15)
\end{aligned}$$

for all $r_1,\dots,r_n\in U$. Then by Lemma 2.2, one of the following holds:

(1) $a(1-p-q),b(1-p-q)\in C$ and $a(1-p-q)+b(1-p-q)=(a+b)(p+q)\in C$. Since $G\neq Id$, $p+q\neq 1$ and hence $a,b\in C$. Then conclusion follows by Case 1.

(2) $f(x_1,\dots,x_n)^2$ is central valued on R and $a(1-p-q)+b(1-p-q)=(a+b)(p+q)\in C$. This implies $2(p+q)(a+b)=a+b$. Since $G\neq 0$, $0\neq p+q\in C$. Hence $(a+b)(p+q)\in C$ yields $a+b\in C$. Thus $2(p+q)(a+b)=a+b$ gives $(2(p+q)-1)(a+b)=0$. This implies either $a+b=0$ or $2(p+q)=1$. When $a+b=0$, $F(x)=ax+xb=[a,x]$

for all $x \in R$, $G(x) = px + xq = (p+q)x$ for all $x \in R$, which is conclusion (3). On the other hand when $2(p+q) = 1$, then $F(x) = ax + xb$ for all $x \in R$ with $a + b \in C$ and $G(x) = px + xq = (p+q)x$ for all $x \in R$ with $2(p+q) = 1$, which is our conclusion (4).

Case 3: Let $a \in C$ and $q \in C$.

Then by (2.5), we have

$$f(r_1, \dots, r_n)^2(a+b) \\ = f(r_1, \dots, r_n)(a+b)(p+q)f(r_1, \dots, r_n) + (p+q)f(r_1, \dots, r_n)^2(a+b),$$

for all $r_1, \dots, r_n \in U$.

This can be written as

$$(p+q-1)f(r_1, \dots, r_n)^2(a+b) + f(r_1, \dots, r_n)(a+b)(p+q)f(r_1, \dots, r_n) = 0.$$

Then by Lemma 2.4, one of the following holds:

(1) $p+q-1, (a+b)(p+q) \in C$ and $(p+q-1)(a+b) + (a+b)(p+q) = 0$. This implies $p+q \in C$. Since $G \neq 0$, $p+q \neq 0$ and hence $0 \neq a+b \in C$. Hence $(p+q-1)(a+b) + (a+b)(p+q) = 0$ yields $2(p+q) = 1$. Thus in this case we have $F(x) = ax + xb = x(a+b) = (a+b)x$ and $G(x) = px + xq = (p+q)x$ for all $x \in R$ with $2(p+q) = 1$, which is our conclusion (1).

(2) $a+b, (a+b)(p+q) \in C$ and $(p+q-1)(a+b) + (a+b)(p+q) = 0$. Since $a \in C$, $a+b \in C$ yields $b \in C$. Since $F \neq 0$, $a+b \neq 0$ and thus $(a+b)(p+q) \in C$ implies $p+q \in C$. Hence, $(p+q-1)(a+b)+(a+b)(p+q) = 0$ yields $2(p+q) = 1$. Thus in this case we have $F(x) = ax + xb = x(a+b) = (a+b)x$ and $G(x) = px + xq = (p+q)x$ for all $x \in R$ with $2(p+q) = 1$, which is our conclusion (1).

(3) $f(x_1, \dots, x_n)^2$ is central valued on R and $(p+q-1)(a+b) = -(a+b)(p+q) \in C$. Thus in this case we have $F(x) = ax + xb = x(a+b)$ for all $x \in R$ and $G(x) = px + xq = (p+q)x$ for all $x \in R$, which is our conclusion (5).

Case 4: Let $b \in C$ and $p \in C$.

Then by (2.5), we have

$$(a+b)f(r_1, \dots, r_n)^2 \\ = (a+b)f(r_1, \dots, r_n)^2(p+q) + f(r_1, \dots, r_n)(p+q)(a+b)f(r_1, \dots, r_n),$$

for all $r_1, \dots, r_n \in U$. This can be written as

$$(a+b)f(r_1, \dots, r_n)^2(p+q-1) + f(r_1, \dots, r_n)(p+q)(a+b)f(r_1, \dots, r_n) = 0$$

for all $r_1, \dots, r_n \in U$. Then by Lemma 2.4, one of the following holds:

(1) $a+b, (p+q)(a+b) \in C$ and $(a+b)(p+q-1) + (p+q)(a+b) = 0$. Since $b \in C$, $a+b \in C$ yields $a \in C$. Since $F \neq 0$, $a+b \neq 0$ and thus $(p+q)(a+b) \in C$ implies $p+q \in C$. Hence, $(a+b)(p+q-1) + (p+q)(a+b) = 0$ yields $2(p+q) = 1$. Thus in this case we have $F(x) = (a+b)x$ and $G(x) = (p+q)x$ for all $x \in R$ with $2(p+q) = 1$, which is our conclusion (1).

(2) $p+q-1, (p+q)(a+b) \in C$ and $(a+b)(p+q-1) + (p+q)(a+b) = 0$. Since $p \in C$, $p+q-1 \in C$ yields $q \in C$. Since $G \neq 0$, $p+q \neq 0$ and thus $(p+q)(a+b) \in C$ implies $a+b \in C$. Hence, $(a+b)(p+q-1) + (p+q)(a+b) = 0$ yields $2(p+q) = 1$. Thus in this case we have $F(x) = (a+b)x$ and $G(x) = (p+q)x$ for all $x \in R$ with $2(p+q) = 1$, which is our conclusion (1).

(3) $f(x_1, \dots, x_n)^2$ is central valued on R and $(a+b)(p+q-1) = -(p+q)(a+b) \in C$. Thus in this case we have $F(x) = ax + xb = (a+b)x$ for all $x \in R$ and $G(x) = px + xq = x(p+q)$ for all $x \in R$, which is our conclusion (6).

□

Proof of the Main Theorem. In [12, Theorem 3], Lee proved that every generalized derivation g on a dense right ideal of R can be uniquely extended to a generalized derivation of U and thus can be assumed to be defined on the whole U with the form $g(x) = ax + d(x)$ for some $a \in U$ and d is a derivation of U. In the light of this, we may assume that there exist $a, b \in U$ and derivations d, δ of U such that $F(x) = ax + d(x)$ and $G(x) = bx + \delta(x)$. Since I, R, and U satisfy the same generalized polynomial identities (see [3]) as well as the same differential identities (see [14]), without loss of generality, to prove our results, we may assume $F(f(x_1,\dots,x_n))^2 = F(f(x_1,\dots,x_n))G(f(x_1,\dots,x_n)) + G(f(x_1,\dots,x_n))F(f(x_1,\dots,x_n))$ for all $x_1,\dots,x_n \in U$.

If F and G both are inner generalized derivations of R, then by Lemma 2.9 we obtain our conclusions. Thus we assume that not both of F and G are inner. Hence U satisfies

$$\begin{aligned} & af(x_1,\dots,x_n)^2 + d(f(x_1,\dots,x_n)^2) \\ &= (af(x_1,\dots,x_n) + d(f(x_1,\dots,x_n)))(bf(x_1,\dots,x_n) + \delta(f(x_1,\dots,x_n))) \\ &+(bf(x_1,\dots,x_n) + \delta(f(x_1,\dots,x_n)))(af(x_1,\dots,x_n) + d(f(x_1,\dots,x_n))) \end{aligned} \tag{2.16}$$

for all $(x_1,\dots,x_n \in U$, where d, δ are two derivations on U not both are inner.

Case 1: Assume that d and δ are C-dependent modulo inner derivations of U i.e., $\alpha d + \beta\delta = ad_q$, where $\alpha, \beta \in C$.

Subcase 1.i: Suppose $\alpha = 0$. Then $\delta(x) = [p, x]$, where $p = \beta^{-1}q$. Obviously d is not an inner derivation of U. From (2.16) we obtain that U satisfies

$$\begin{aligned} & af(x_1,\dots,x_n)^2 + d(f(x_1,\dots,x_n))f(x_1,\dots,x_n) + f(x_1,\dots,x_n)d(f(x_1,\dots,x_n)) \\ &= (af(x_1,\dots,x_n) + d(f(x_1,\dots,x_n)))(bf(x_1,\dots,x_n) + [p, f(x_1,\dots,x_n)]) \\ &+(bf(x_1,\dots,x_n) + [p, f(x_1,\dots,x_n)])(af(x_1,\dots,x_n) + d(f(x_1,\dots,x_n)). \end{aligned} \tag{2.17}$$

Let $f^d(x_1,\dots,x_n)$ be the polynomials obtained from $f(x_1,\dots,x_n)$ replacing each coefficients α_σ with $d(\alpha_\sigma)$. Then we have

$$d(f(x_1,\dots,x_n)) = f^d(x_1,\dots,x_n) + \sum_i f(x_1,\dots,d(x_i),\dots,x_n).$$

Thus (2.17) gives

$$\begin{aligned} & af(x_1,\dots,x_n)^2 + (f^d(x_1,\dots,x_n) + \sum_i f(x_1,\dots,d(x_i),\dots,x_n))f(x_1,\dots,x_n) \\ & +f(x_1,\dots,x_n)(f^d(x_1,\dots,x_n) + \sum_i f(x_1,\dots,d(x_i),\dots,x_n)) \\ &= (af(x_1,\dots,x_n) + f^d(x_1,\dots,x_n) + \sum_i f(x_1,\dots,d(x_i),\dots,x_n)) \\ & \cdot(bf(x_1,\dots,x_n) + [p, f(x_1,\dots,x_n)]) + (bf(x_1,\dots,x_n) + [p, f(x_1,\dots,x_n)]) \\ & \cdot(af(x_1,\dots,x_n) + f^d(x_1,\dots,x_n) + \sum_i f(x_1,\dots,d(x_i),\dots,x_n)). \end{aligned} \tag{2.18}$$

Since d is outer derivation, by Kharchenko's theorem [11], we have that U satisfies

$$\begin{aligned}
af(x_1,\ldots,x_n)^2 + (f^d(x_1,\ldots,x_n) + \sum_i f(x_1,\ldots,y_i,\ldots,x_n))f(x_1,\ldots,x_n)\\
+f(x_1,\ldots,x_n)((f^d(x_1,\ldots,x_n) + \sum_i f(x_1,\ldots,y_i,\ldots,x_n))\\
= (af(x_1,\ldots,x_n) + f^d(x_1,\ldots,x_n) + \sum_i f(x_1,\ldots,y_i,\ldots,x_n))\\
\cdot(bf(x_1,\ldots,x_n) + [p, f(x_1,\ldots,x_n)]) + (bf(x_1,\ldots,x_n) + [p, f(x_1,\ldots,x_n)])\\
\cdot(af(x_1,\ldots,x_n) + f^d(x_1,\ldots,x_n) + \sum_i f(x_1,\ldots,y_i,\ldots,x_n)).
\end{aligned} \tag{2.19}$$

Particularly, U satisfies the blended component,

$$\begin{aligned}
\sum_i f(x_1,\ldots,y_i,\ldots,x_n)f(x_1,\ldots,x_n) + f(x_1,\ldots,x_n)\sum_i f(x_1,\ldots,y_i,\ldots,x_n)\\
= \sum_i f(x_1,\ldots,y_i,\ldots,x_n)(bf(x_1,\ldots,x_n) + [p, f(x_1,\ldots,x_n)])\\
+(bf(x_1,\ldots,x_n) + [p, f(x_1,\ldots,x_n)])\sum_i f(x_1,\ldots,y_i,\ldots,x_n).
\end{aligned} \tag{2.20}$$

In particular, for $y_1 = x_1$, $y_2 = y_3 = \ldots = y_n = 0$, we get from above

$$\begin{aligned}
2f(x_1,\ldots,x_n)^2 = f(x_1,\ldots,x_n)(bf(x_1,\ldots,x_n) + [p, f(x_1,\ldots,x_n)])\\
+(bf(x_1,\ldots,x_n) + [p, f(x_1,\ldots,x_n)])f(x_1,\ldots,x_n),
\end{aligned} \tag{2.21}$$

which gives

$$(b+p)f(x_1,\ldots,x_n)^2 - f(x_1,\ldots,x_n)^2(p+2) + f(x_1,\ldots,x_n)bf(x_1,\ldots,x_n) = 0.$$

Then by Lemma 2.2, one of the following holds:

(1) $b+p, p+2, b \in C$ and $(b+p)-(p+2) = -b$. This implies $p \in C$ and $b = 1$. Thus in this case we have $G(x) = bx + [p, x] = x$ for all $x \in R$, a contradiction.

(2) $f(x_1,\ldots,x_n)^2$ is central valued on R and $(b+p)-(p+2) = -b \in C$. This gives $b = 1$. In this case, we have from (2.17) that U satisfies

$$\begin{aligned}
af(x_1,\ldots,x_n)^2 + d(f(x_1,\ldots,x_n)^2)\\
= (af(x_1,\ldots,x_n) + d(f(x_1,\ldots,x_n)))(f(x_1,\ldots,x_n) + [p, f(x_1,\ldots,x_n)])\\
+(f(x_1,\ldots,x_n) + [p, f(x_1,\ldots,x_n)])(af(x_1,\ldots,x_n)\\
+d(f(x_1,\ldots,x_n))).
\end{aligned} \tag{2.22}$$

This implies

$$\begin{aligned}
0 = af(x_1,\ldots,x_n)[p, f(x_1,\ldots,x_n)] + d(f(x_1,\ldots,x_n))[p, f(x_1,\ldots,x_n)]\\
+f(x_1,\ldots,x_n)af(x_1,\ldots,x_n) + [p, f(x_1,\ldots,x_n)]af(x_1,\ldots,x_n)\\
+[p, f(x_1,\ldots,x_n)]d((f(x_1,\ldots,x_n)).
\end{aligned}$$

It gives

$$\begin{aligned}
0 = af(x_1,\ldots,x_n)[p, f(x_1,\ldots,x_n)]\\
+(f^d(x_1,\ldots,x_n) + \sum_i f(x_1,\ldots,y_i,\ldots,x_n))[p, f(x_1,\ldots,x_n)]\\
+f(x_1,\ldots,x_n)af(x_1,\ldots,x_n) + [p, f(x_1,\ldots,x_n)]af(x_1,\ldots,x_n)\\
+[p, f(x_1,\ldots,x_n)](f^d(x_1,\ldots,x_n) + \sum_i f(x_1,\ldots,y_i,\ldots,x_n)).
\end{aligned}$$

In particular, U satisfies the blended component

$$\sum_i f(x_1,\ldots,y_i,\ldots,x_n)[p,f(x_1,\ldots,x_n)] \\ +[p,f(x_1,\ldots,x_n)]\sum_i f(x_1,\ldots,y_i,\ldots,x_n)=0. \tag{2.23}$$

Putting $y_i=[q,x_i]$ in (2.23), where $q\notin C$, we have that U satisfies

$$[q,f(x_1,\ldots,x_n)][p,f(x_1,\ldots,x_n)]+[p,f(x_1,\ldots,x_n)][q,f(x_1,\ldots,x_n)]=0. \tag{2.24}$$

Then by Lemma 2.3, $p\in C$. Thus $G(x)=bx+[p,x]=x$ for all $x\in R$, a contradiction.

Subcase 1.ii: Suppose $\alpha\neq 0$, then $\alpha d+\beta\delta=ad_q$ gives $d=\mu\delta+ad_c$ for some $\mu\in C$ and $c\in U$. Then we can assume that δ is not an inner derivation, otherwise d and δ both will be inner derivations, a contradiction. From (2.16), U satisfies

$$\begin{aligned}
&af(x_1,\ldots,x_n)^2+\mu\delta(f(x_1,\ldots,x_n)^2)+[c,f(x_1,\ldots,x_n)^2]\\
&=\Big(af(x_1,\ldots,x_n)+\mu\delta(f(x_1,\ldots,x_n))+[c,f(x_1,\ldots,x_n)]\Big)\\
&\cdot\Big(bf(x_1,\ldots,x_n)+\delta(f(x_1,\ldots,x_n))\Big)\\
&+\Big(bf(x_1,\ldots,x_n)+\delta(f(x_1,\ldots,x_n))\Big)\\
&\cdot\Big(af(x_1,\ldots,x_n)+\mu\delta(f(x_1,\ldots,x_n))\\
&+[c,f(x_1,\ldots,x_n)]\Big),
\end{aligned}$$

that is,

$$\begin{aligned}
&af(x_1,\ldots,x_n)^2+\mu(f^\delta(x_1,\ldots,x_n)+\sum_i f(x_1,\ldots,\delta(x_i),\ldots,x_n))f(x_1,\ldots,x_n)\\
&+\mu f(x_1,\ldots,x_n)(f^\delta(x_1,\ldots,x_n)+\sum_i f(x_1,\ldots,\delta(x_i),\ldots,x_n))+[c,f(x_1,\ldots,x_n)^2]\\
&=\Big(af(x_1,\ldots,x_n)+\mu(f^\delta(x_1,\ldots,x_n)+\sum_i f(x_1,\ldots,\delta(x_i),\ldots,x_n))+[c,f(x_1,\ldots,x_n)]\Big)\\
&\cdot\Big(bf(x_1,\ldots,x_n)+f^\delta(x_1,\ldots,x_n)+\sum_i f(x_1,\ldots,\delta(x_i),\ldots,x_n)\Big)\\
&+\Big(bf(x_1,\ldots,x_n)+(f^\delta(x_1,\ldots,x_n)+\sum_i f(x_1,\ldots,\delta(x_i),\ldots,x_n))\Big)\\
&\cdot\Big(af(x_1,\ldots,x_n)+\mu(f^\delta(x_1,\ldots,x_n)+\sum_i f(x_1,\ldots,\delta(x_i),\ldots,x_n))\\
&+[c,f(x_1,\ldots,x_n)]\Big).
\end{aligned}$$

Then by Kharchenko's theorem [11], we have that U satisfies

$$
\begin{aligned}
& af(x_1,\ldots,x_n)^2+\mu(f^\delta(x_1,\ldots,x_n)+\sum_i f(x_1,\ldots,y_i,\ldots,x_n))f(x_1,\ldots,x_n) \\
& +\mu f(x_1,\ldots,x_n)(f^\delta(x_1,\ldots,x_n)+\sum_i f(x_1,\ldots,y_i,\ldots,x_n))+[c,f(x_1,\ldots,x_n)^2] \\
= & \Big(af(x_1,\ldots,x_n)+\mu(f^\delta(x_1,\ldots,x_n)+\sum_i f(x_1,\ldots,y_i,\ldots,x_n))+[c,f(x_1,\ldots,x_n)]\Big) \\
& \cdot\Big(bf(x_1,\ldots,x_n)+f^\delta(x_1,\ldots,x_n)+\sum_i f(x_1,\ldots,y_i,\ldots,x_n)\Big) \\
& +\Big(bf(x_1,\ldots,x_n)+(f^\delta(x_1,\ldots,x_n)+\sum_i f(x_1,\ldots,y_i,\ldots,x_n))\Big) \\
& \cdot\Big(af(x_1,\ldots,x_n)+\mu(f^\delta(x_1,\ldots,x_n)+\sum_i f(x_1,\ldots,y_i,\ldots,x_n)) \\
& +[c,f(x_1,\ldots,x_n)]\Big).
\end{aligned} \tag{2.25}
$$

In particular, for $x_1=0$, U satisfies

$$0=\mu f(x_1,\ldots,x_n)^2+\mu f(x_1,\ldots,x_n)^2, \tag{2.26}$$

that is, $2\mu f(x_1,\ldots,x_n)^2=0$. Since $char(R)\neq 2$, U satisfies $\mu f(x_1,\ldots,x_n)^2=0$. This implies that either $\mu=0$ or $f(x_1,\ldots,x_n)^2=0$. Now $f(x_1,\ldots,x_n)^2=0$, implies $f(x_1,\ldots,x_n)=0$ for all $x_1,\ldots,x_n\in U$, a contradiction. Hence we have $\mu=0$. Thus (2.25) reduces to

$$
\begin{aligned}
af(x_1,\ldots,x_n)^2= & \Big(af(x_1,\ldots,x_n)+[c,f(x_1,\ldots,x_n)]\Big) \\
& \cdot\Big(bf(x_1,\ldots,x_n)+f^\delta(x_1,\ldots,x_n)+\sum_i f(x_1,\ldots,y_i,\ldots,x_n)\Big) \\
& +\Big(bf(x_1,\ldots,x_n)+(f^\delta(x_1,\ldots,x_n)+\sum_i f(x_1,\ldots,y_i,\ldots,x_n))\Big) \\
& \cdot\Big(af(x_1,\ldots,x_n)+[c,f(x_1,\ldots,x_n)]\Big).
\end{aligned}
$$

In particular, U satisfies blended components

$$
\begin{aligned}
& (af(x_1,\ldots,x_n)+[c,f(x_1,\ldots,x_n)])\sum_i f(x_1,\ldots,y_i,\ldots,x_n) \\
& +\sum_i f(x_1,\ldots,y_i,\ldots,x_n)(af(x_1,\ldots,x_n)+[c,f(x_1,\ldots,x_n)])=0.
\end{aligned} \tag{2.27}
$$

For $y_1=x_1$ and $y_2=y_3=,\ldots,=y_n=0$, U satisfies

$$
\begin{aligned}
& (af(x_1,\ldots,x_n)+[c,f(x_1,\ldots,x_n)])f(x_1,\ldots,x_n) \\
& +f(x_1,\ldots,x_n)(af(x_1,\ldots,x_n)+[c,f(x_1,\ldots,x_n)])=0,
\end{aligned} \tag{2.28}
$$

that is

$$(a+c)f(x_1,\ldots,x_n)^2-f(x_1,\ldots,x_n)^2c+f(x_1,\ldots,x_n)af(x_1,\ldots,x_n)=0,$$

for all $x_1,\ldots,x_n\in U$. Then by Lemma 2.2, we have one of the followings:

(1) $a+c,c,a\in C$ and $2a=0$. Thus $a=0$. In this case $F(x)=ax+[c,x]=0$ for all $x\in U$, a contradiction.

(2) $f(x_1,\dots,x_n)^2$ is central valued on R and $a \in C$ with $2a = 0$. This implies $a = 0$. Then by (2.27), U satisfies

$$[c, f(x_1,\dots,x_n)]\sum_i f(x_1,\dots,y_i,\dots,x_n)$$
$$+\sum_i f(x_1,\dots,y_i,\dots,x_n)[c, f(x_1,\dots,x_n)] = 0.$$

Replacing y_i with $[q, x_i]$ for some $q \notin C$, we get from above that U satisfies

$$[c, f(x_1,\dots,x_n)][q, f(x_1,\dots,x_n)] + [q, f(x_1,\dots,x_n)][c, f(x_1,\dots,x_n)] = 0. \quad (2.29)$$

By Lemma 2.3, $c \in C$. Then $F(x) = ax + [c, x] = 0$ for all $x \in R$, a contradiction.

Case 2: Let d and δ be linearly C-independent modulo inner derivations of U. Then from (2.16), U satisfies

$$\begin{aligned}
&af(x_1,\dots,x_n)^2 + (f^d(x_1,\dots,x_n) + \textstyle\sum_i f(x_1,\dots,d(x_i),\dots,x_n))f(x_1,\dots,x_n)\\
&+f(x_1,\dots,x_n)(f^d(x_1,\dots,x_n) + \textstyle\sum_i f(x_1,\dots,d(x_i),\dots,x_n))\\
&= (af(x_1,\dots,x_n) + f^d(x_1,\dots,x_n) + \textstyle\sum_i f(x_1,\dots,d(x_i),\dots,x_n))\\
&\cdot(bf(x_1,\dots,x_n) + f^\delta(x_1,\dots,x_n) + \textstyle\sum_i f(x_1,\dots,\delta(x_i),\dots,x_n))\\
&+(bf(x_1,\dots,x_n) + f^\delta(x_1,\dots,x_n) + \textstyle\sum_i f(x_1,\dots,\delta(x_i),\dots,x_n))\\
&\cdot(af(x_1,\dots,x_n) + (f^d(x_1,\dots,x_n) + \textstyle\sum_i f(x_1,\dots,d(x_i),\dots,x_n)) \qquad (2.30)
\end{aligned}$$

for all $x_1,\dots,x_n \in U$. Since d and δ are not inner, by Kharchenko's theorem [11], U satisfies

$$\begin{aligned}
&af(x_1,\dots,x_n)^2 + (f^d(x_1,\dots,x_n) + \sum_i f(x_1,\dots,y_i,\dots,x_n))f(x_1,\dots,x_n)\\
&+f(x_1,\dots,x_n)(f^d(x_1,\dots,x_n) + \sum_i f(x_1,\dots,y_i,\dots,x_n))\\
&= (af(x_1,\dots,x_n) + f^d(x_1,\dots,x_n) + \sum_i f(x_1,\dots,y_i,\dots,x_n))\\
&\cdot(bf(x_1,\dots,x_n) + f^\delta(x_1,\dots,x_n) + \sum_i f(x_1,\dots,z_i,\dots,x_n))\\
&+(bf(x_1,\dots,x_n) + f^\delta(x_1,\dots,x_n) + \sum_i f(x_1,\dots,z_i,\dots,x_n))\\
&\cdot(af(x_1,\dots,x_n) + f^d(x_1,\dots,x_n) + \sum_i f(x_1,\dots,y_i,\dots,x_n)).
\end{aligned}$$

In particular, for $x_1 = 0$, $z_1 = y_1$, we get $2f(y_1, x_2,\dots,x_n)^2 = 0$ implying $f(x_1,\dots,x_n)^2 = 0$ for all $x_1,\dots,x_n \in U$. It yields $f(x_1,\dots,x_n) = 0$ for all $x_1,\dots,x_n \in U$, a contradiction. Thus the proof of the theorem is completed.

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