More Identities for Fibonacci and Lucas Octonions

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Abstract

In this paper, we give a new approach to obtain identities for Fibonacci and Lucas octonions.

Keywords: Fibonacci Octonions; Lucas Octonions, octonions

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1. Introduction

O is an eight-dimensional non-associative and non-commutative division algebra over $\mathbb{R}$. A natural basis of this algebra as a space over $\mathbb{R}$ is formed by the elements

$$e_0 = 1, \ e_1 = i, \ e_2 = j, \ e_3 = k, \ e_4 = e, \ e_5 = ie, \ e_6 = je, \ e_7 = ke$$

where $i^2 = j^2 = k^2 = -1$, $ij = k = -ji$, $jk = i = -kj$, $ki = j = -ik$ and $ijk = -1$.

The multiplication table for the basis of O is

.	1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
1	1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	e_1	-1	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6
e_2	e_2	$-e_3$	-1	e_1	e_6	e_7	$-e_4$	$-e_5$
e_3	e_3	e_2	$-e_1$	-1	e_7	$-e_6$	e_5	$-e_4$
e_4	e_4	$-e_5$	e_6	$-e_7$	-1	e_1	e_2	e_3
e_5	e_5	e_4	$-e_7$	e_6	$-e_1$	-1	$-e_3$	e_2
e_6	e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	-1	$-e_1$
e_7	e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	-1

Together with this notation, a octonion p takes the form

$$p = \sum_{i=0}^{7} p_i e_i$$

where $p_i \in \mathbb{R}$. Every octonion p can be written as the sum of two parts, real and imaginary parts.

$$p = \operatorname{Re}(p) + \operatorname{Im}(p).$$

This definition gives that $\operatorname{Re}(p) = p_0$ and $\operatorname{Im}(p) = \sum_{i=1}^{7} p_i e_i$. The conjugate of $p, \overline{p}$, is defined

$$\overline{p} = \operatorname{Re}(p) - \operatorname{Im}(p).$$

For $p, q \in O$, the identities

$$\overline{\overline{p}} = p, \ \overline{(p+q)} = \overline{p} + \overline{q}, \ \overline{pq} = \overline{q}\,\overline{p}$$

---

follow.

For $n \geq 2$, the Fibonacci and Lucas sequences are defined as

$$F_n = F_{n-1} + F_{n-2}, \quad F_0 = 0, F_1 = 1$$

and

$$L_n = L_{n-1} + L_{n-2}, \quad L_0 = 2, L_1 = 1,$$

respectively. There are lots of amazing identities belongs to Fibonacci and Lucas numbers. For the details, we refer the book of T. Koshy ([3]).

Keçilioğlu and Akkuş [2] gave the definitions of the Fibonacci and Lucas octonions as

$$Q_n = \sum_{j=0}^{7} F_{n+j}e_j, \ H_n = \sum_{j=0}^{7} L_{n+j}e_j,$$

respectively. Moreover, they present several identities about the Fibonacci and Lucas octonions. The binet formulas of these octonions are used in their paper to get identities.

Up to now, the identities for both Fibonacci quaternion Lucas quaternions and Fibonacci octonion Lucas octonions are obtained by the using binet formulas or special identities for Fibonacci and Lucas numbers. Irmak [1] gives an alternative approach to obtain identities. He used the matrix methods for Fibonacci and Lucas quaternions.

In this paper, we give several identities for Fibonacci and Lucas octonions by using Irmak's approach.

Before going further, we need a square matrix W as

$$W = \begin{pmatrix} Q_1 & Q_0 \\ Q_0 & Q_{-1} \end{pmatrix}$$

Since the algebra O is non-commutative, we need two different determinant definition. Let A be a octonion square matrix. Denote A by,

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

where $a_{ij} \in O$ for $i = 1, 2$ and $j = 1, 2$. The determinant of A, $\det A$, is defined by

$$\begin{aligned} \det A &= \det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \\ &= a_{11}a_{22} - a_{12}a_{21}. \end{aligned} \tag{1.1}$$

The above definition is called rule "multiplication from above to down below". Since the set of all quaternion is not commutative, another product direction can be defined. Namely, The definition

$$\begin{aligned} Det A &= \det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \\ &= a_{22}a_{11} - a_{21}a_{12} \end{aligned} \tag{1.2}$$

is called the rule "multiplication from down below to above" (For details, see the book [4], section 9.11.)

The Fibonacci matrix U is known that

$$U = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

which yields that

$$U^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}.$$

If we multiply the matrices W and U^n, we get Fibonacci octonion matrix (entries are Fibonacci octonions). Namely, we obtain the following fact

$$WU^n = U^nW = \begin{pmatrix} Q_{n+1} & Q_n \\ Q_n & Q_{n-1} \end{pmatrix}. \tag{1.3}$$

2. Main Result

First theorem is about the Cassini identity belongs to Fibonacci and Lucas octonions. Since there are two different determinant definitions of a octonion matrix, we give the different Cassini identities.

Theorem 2.1. *(First type Cassini identity) For $n \geq 1$, we have the identities*

$$Q_{n-1}Q_{n+1} - Q_n^2 = (-1)^n \left(2Q_1 - (14e_5 + 12e_6 + 7e_7)\right)$$

and

$$H_{n-1}H_{n+1} - H_n^2 = 5(-1)^{n-1} \left(2Q_1 - (14e_5 + 12e_6 + 7e_7)\right).$$

Proof. By (1.3)

$$
\begin{aligned}
U^n W &= \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix} \begin{pmatrix} Q_1 & Q_0 \\ Q_0 & Q_{-1} \end{pmatrix} \\
&= \begin{pmatrix} Q_{n+1} & Q_n \\ Q_n & Q_{n-1} \end{pmatrix}
\end{aligned}
\tag{2.1}
$$

follows. After taking determinant (1.2) together with the identity $F_{n+1}F_{n-1} - F_n^2 = (-1)^n$, we obtain

$$
\begin{aligned}
Q_{n-1}Q_{n+1} - Q_n^2 &= \left(F_{n+1}F_{n-1} - F_n^2\right)\left(Q_{-1}Q_1 - Q_0^2\right) \\
&= (-1)^n \left(2Q_1 - (14e_5 + 12e_6 + 7e_7)\right).
\end{aligned}
$$

as claimed. Since $F_{n-1} + F_{n+1} = L_n$, then we write

$$
\begin{aligned}
\left(U^{n-1} + U^{n+1}\right) W &= \begin{pmatrix} L_{n+1} & L_n \\ L_n & L_{n-1} \end{pmatrix} \begin{pmatrix} Q_1 & Q_0 \\ Q_0 & Q_{-1} \end{pmatrix} \\
&= \begin{pmatrix} H_{n+1} & H_n \\ H_n & H_{n-1} \end{pmatrix}.
\end{aligned}
$$

Applying the equation (1.2) gives that

$$
\begin{aligned}
H_{n-1}H_{n+1} - H_n^2 &= \left(L_{n+1}L_{n-1} - L_n^2\right)\left(Q_{-1}Q_1 - Q_0^2\right) \\
&= 5(-1)^{n-1} \left(2Q_1 - (14e_5 + 12e_6 + 7e_7)\right).
\end{aligned}
$$

$\square$

Now, we use the first definition of determinant (1.1) to obtain the second type Cassini identity.

Theorem 2.2. *(Second type Cassini identity) For $n \geq 1$, we have the identities*

$$Q_{n+1}Q_{n-1} - Q_n^2 = (-1)^n \left(2Q_{-1} + 14e_5 + 18e_6 + 7e_7\right)$$

and

$$H_{n+1}H_{n-1} - H_n^2 = 5(-1)^{n-1} \left(2Q_{-1} + 14e_5 + 18e_6 + 7e_7\right).$$

Proof. Since the equation (2.1) holds, then the definition 1.1 yields that

$$
\begin{aligned}
Q_{n+1}Q_{n-1} - Q_n^2 &= \left(F_{n-1}F_{n+1} - F_n^2\right)\left(Q_1Q_{-1} - Q_0^2\right) \\
&= (-1)^n \left(2Q_{-1} + 14e_5 + 18e_6 + 7e_7\right).
\end{aligned}
$$

Similarly, we get

$$H_{n+1}H_{n-1} - H_n^2 = 5(-1)^{n-1} \left(2Q_{-1} + 14e_5 + 18e_6 + 7e_7\right).$$

$\square$

Theorem 2.3. *For $n, m \geq 1$ integers, then*

$$Q_{m+n} = F_{m+1}Q_n + F_m Q_{n-1}$$

follows.

Proof. The fact $U^{m+n}F = U^m(U^nF)$ yields that

$$
\begin{aligned}
U^{m+n}W &= \begin{pmatrix} Q_{m+n+1} & Q_{m+n} \\ Q_{m+n} & Q_{m+n-1} \end{pmatrix} \\
&= U^m(U^nW) \\
&= \begin{pmatrix} F_{m+1} & F_m \\ F_m & F_{m-1} \end{pmatrix}\begin{pmatrix} Q_{n+1} & Q_n \\ Q_n & Q_{n-1} \end{pmatrix}.
\end{aligned}
\tag{2.2}
$$

Equating the first row second column entries of two matrix in (2.2) yields that

$$
Q_{m+n} = F_{m+1}Q_n + F_mQ_{n-1}.
$$

$\square$

Since $F_{m-2} + F_{m-1} = F_m$ and $Q_{m+n+1} + Q_{m+n-1} = H_{m+n}$ holds, we get the following identity as a corollary.

Corollary 2.1. *For positive integers m and n, we get*

$$
H_{m+n} = L_{m+1}Q_n + L_mQ_{n-1}
\tag{2.3}
$$

Theorem 2.4. *For $n, m \geq 1$ integers, we have*

$$
Q_{m+1}Q_n + Q_mQ_{n-1} = \sum_{i=0}^{7} Q_{m+n+i}e_i.
$$

Proof. The identity $F(T^{m+n}F) = (FT^m)(T^nF)$ yields that

$$
\begin{aligned}
&\begin{pmatrix} Q_1 & Q_0 \\ Q_0 & Q_{-1} \end{pmatrix}\begin{pmatrix} Q_{m+n+1} & Q_{m+n} \\ Q_{m+n} & Q_{m+n-1} \end{pmatrix} \\
&= \begin{pmatrix} Q_{m+1} & Q_m \\ Q_m & Q_{m-1} \end{pmatrix}\begin{pmatrix} Q_{n+1} & Q_n \\ Q_n & Q_{n-1} \end{pmatrix}
\end{aligned}
\tag{2.4}
$$

If we equalize the first row and second column elements in the equation (2.4), we obtain the claimed result. $\square$

We deduce the following matrix equation system by the equation (2.3).

$$
\begin{pmatrix} H_{m+n+1} & H_{m+n} \\ H_{m+n} & H_{m+n-1} \end{pmatrix} = \begin{pmatrix} L_{m+1} & L_m \\ L_m & L_{m-1} \end{pmatrix}\begin{pmatrix} Q_{n+1} & Q_n \\ Q_n & Q_{n-1} \end{pmatrix}.
\tag{2.5}
$$

If we multiply the equation (2.5) with the matrix W from left side, we deduce

$$
\begin{aligned}
&\begin{pmatrix} Q_1 & Q_0 \\ Q_0 & Q_{-1} \end{pmatrix}\begin{pmatrix} H_{m+n+1} & H_{m+n} \\ H_{m+n} & H_{m+n-1} \end{pmatrix} \\
&= \left(\begin{pmatrix} Q_1 & Q_0 \\ Q_0 & Q_{-1} \end{pmatrix}\begin{pmatrix} L_{m+1} & L_m \\ L_m & L_{m-1} \end{pmatrix}\right)\begin{pmatrix} Q_{n+1} & Q_n \\ Q_n & Q_{n-1} \end{pmatrix} \\
&= \begin{pmatrix} H_{m+1} & H_m \\ H_m & H_{m-1} \end{pmatrix}\begin{pmatrix} Q_{n+1} & Q_n \\ Q_n & Q_{n-1} \end{pmatrix}.
\end{aligned}
$$

Equating the first row and second column element, we obtain the following theorem.

Theorem 2.5. *For $m, n \geq 1$, we get*

$$
H_{m+1}Q_n + H_mQ_{n-1} = \sum_{i=0}^{7} H_{m+n+1+i}e_i.
$$

Let define the conjugate matrix of W as

$$\widetilde{W} = \left(\begin{array}{cc} \widetilde{Q}_1 & \widetilde{Q}_0 \\ \widetilde{Q}_0 & \widetilde{Q}_{-1} \end{array} \right).$$

From now on, we give both first and second type Cassini identities for the conjugate Fibonacci and Lucas octonions.

Theorem 2.6. *The identities*

$$\begin{array}{rcl}
\widetilde{Q}_{n-1}\widetilde{Q}_{n+1} - \widetilde{Q}_n^2 & = & (-1)^n \left(2 - 2e_2 - 2e_3 - 4e_4 - 20e_5 - 22e_6 - 23e_7\right) \\
\widetilde{Q}_{n+1}\widetilde{Q}_{n-1} - \widetilde{Q}_n^2 & = & (-1)^n \left(2 - 2e_1 - 4e_2 - 6e_3 - 10e_4 - 2e_5 - 8e_6 - 35e_7\right) \\
\widetilde{H}_{n-1}\widetilde{H}_{n+1} - \widetilde{H}_n^2 & = & 5\,(-1)^{n-1} \left(2 - 2e_2 - 2e_3 - 4e_4 - 20e_5 - 22e_6 - 23e_7\right)
\end{array}$$

and

$$\widetilde{H}_{n+1}\widetilde{H}_{n-1} - \widetilde{H}_n^2 = 5\,(-1)^{n-1} \left(2 - 2e_1 - 4e_2 - 6e_3 - 10e_4 - 2e_5 - 8e_6 - 35e_7\right)$$

holds for $n \geq 1$.

Proof. The identity

$$\begin{array}{rcl}
\widetilde{W}U^n & = & \left(\begin{array}{cc} \widetilde{Q}_1 & \widetilde{Q}_0 \\ \widetilde{Q}_0 & \widetilde{Q}_{-1} \end{array} \right) \left(\begin{array}{cc} F_{n+1} & F_n \\ F_n & F_{n-1} \end{array} \right) \\
& = & \left(\begin{array}{cc} \widetilde{Q}_{n+1} & \widetilde{Q}_n \\ \widetilde{Q}_n & \widetilde{Q}_{n-1} \end{array} \right).
\end{array}$$

After taking determinant according to the first and second rules, we obtain first and second identities. To obtain the third and fourth ones, we follow the proofs of Theorem 2.1 and Theorem 2.2. □

In order to cut unnecessary repetition, we get the following identities together with the matrices $W, \widetilde{W}$ and U.

Theorem 2.7. *For the integers m and n,*

1) The identity $\widetilde{W}\left(U^{m+n}\right)W = \left(\widetilde{W}U^m\right)\left(U^n W\right)$ yields that

$$Q_{m+n+1} - \sum_{s=1}^{7} Q_{m+n+1+s}e_s = \widetilde{Q}_{m+1}Q_{n+1} + \widetilde{Q}_m Q_n.$$

2) The identity $\left(\widetilde{W}U^{m+n}\right)W = \left(\widetilde{W}U^m\right)\left(U^n W\right)$ gives

$$\sum_{s=0}^{7} \widetilde{Q}_{m+n+1+s}e_s = \widetilde{Q}_{m+1}Q_{n+1} + \widetilde{Q}_m Q_n.$$

3) By the identities $\left(WU^{m+n}\right)\widetilde{W} = \left(WU^m\right)\left(U^n\widetilde{W}\right)$ and $W\left(U^{m+n}\widetilde{W}\right) = \left(WU^m\right)\left(U^n\widetilde{W}\right)$

$$\begin{array}{rcl}
\displaystyle\sum_{s=0}^{7} \widetilde{Q}_{m+n+1+s}e_s & = & Q_{m+1}\widetilde{Q}_{n+1} + Q_m\widetilde{Q}_n \\
Q_{m+n+1} - \displaystyle\sum_{s=1}^{7} Q_{m+n+1+s}e_s & = & Q_{m+1}\widetilde{Q}_{n+1} + Q_m\widetilde{Q}_n
\end{array}$$

hold, respectively.

3. Acknowledgement

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