SLANT CURVES IN 3-DIMENSIONAL ALMOST CONTACT METRIC GEOMETRY

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Dedicated to the memory of Franki Dillen

ABSTRACT. This work has two purposes. The first purpose is to give a survey on slant curves in 3-dimensional almost contact metric geometry. The second purpose is to study slant curves in 3-dimensional solvable Lie groups equipped with natural left invariant almost contact metric structure.

INTRODUCTION

In classical differential geometry of spatial curves, there are some nice classes of curves. The most simplest and fundamental one is the class of helices.

A spatial curve is said to be a *helix* if it has constant curvature and constant torsion.

Straight lines and circles are regarded as helices with curvature 0, and non-zero constant curvature and zero torsion, respectively.

As a generalization of the class of helices, the class of curves of constant slope have been paid much attention of classical differential geometers.

A spatial curve is said to be a *curve of constant slope* (also called a *cylindrical helix*) if its tangent vector field has a constant angle θ with a fixed direction called the *axis*.

The second name is derived from the fact that there exists a cylinder in Euclidean 3-space on which the curve moves in such a way that it cuts each ruling at a constant angle (see [60, pp. 72–73]).

These curves are classically characterized by Bertrand-Lancret-de Saint Venant Theorem (see [60], [68]):

> *A unit speed curve in Euclidean 3-space $\mathbb{E}^3$ with nonzero curvature is of constant slope if and only if the ratio of the torsion τ and the curvature κ is constant.*

For a curve of constant slope with nonzero curvature, the ratio τ/κ is called the *Lancret invariant* of the curve of constant slope.

Barros [3] generalized the above characterization due to Bertrand-Lancret-de Saint Venant to curves in 3-dimensional space forms. Corresponding results for 3-dimensional Lorentzian space forms are obtained by Ferrández [11]. Moreover Ferrández, Giménez and Lucas [12], [13] investigated Bertrand-Lancret-de Saint Venant problem for null curves in Minkowski 3-space. (See also [14], [18]).

According to Thurston, there are eight simply connected model spaces in 3-dimensional geometries:

- The Euclidean 3-space $\mathbb{E}^3$, the 3-sphere $\mathbb{S}^3$, the hyperbolic 3-space $\mathbb{H}^3$,
- the product spaces $\mathbb{S}^2 \times \mathbb{R}$, $\mathbb{H}^2 \times \mathbb{R}$,
- the Heisenberg group Nil_3, the universal covering $\widetilde{\mathrm{SL}_2\mathbb{R}}$ of $\mathrm{SL}_2\mathbb{R}$,
- the space Sol_3.

In recent years, differential geometry of surfaces in those model spaces has been extensively studied.

It should be remarked that every orientable Riemannian 3-manifold admits almost contact structure compatible to the metric (*cf.* [9]).

These eight model spaces admit *invariant* almost contact structure compatible to the metric . The resulting almost contact metric 3-manifolds are homogeneous almost contact metric 3-manifolds. Moreover they are *normal* except Sol_3. In particular, $\mathbb{S}^3$, Nil_3 and $\widetilde{\mathrm{SL}_2\mathbb{R}}$ are homogeneous normal contact metric 3-manifolds of constant holomorphic sectional curvature, *i.e.*, *Sasakian space forms*. The hyperbolic 3-space is the only Kenmotsu manifold in Thurston's list.

Thus it would be interesting to study curves and surfaces in these seven model spaces by using normal almost contact metric structure.

On the other hand, in 3-dimensional *contact geometry* and *topology*, Legendre curves (curves tangent to the contact distribution) play a central role (*cf.* [65]).

As a generalization of Legendre curve, the notion of slant curves was introduced in [17].

A unit speed curve γ in an almost contact metric manifold $(M; \varphi, \xi, \eta, g)$ is said to be *slant* if its tangent vector field makes constant *contact angle* θ with ξ, *i.e.*, $\cos\theta := \eta(\gamma')$ is constant along γ.

In our previous paper [17], we studied slant curves in Sasakian 3-manifols. In [18], we have shown that biharmonic curves in Sasakian space forms are slant.

Since then slant curves have been paid much attention of differential geometers of almost contact manifolds.

Slant curves in Sasakian 3-manifolds are of particular interest. In fact, Cabrerizo, Fernández and Gómez,[10] showed that contact magnetic curves are slant helices. In [45], Munteanu and the first named author of this paper studied periodicity of magnetic curves in 3-dimensional Sasakian space forms of constant holomorphic sectional curvature> -3.

Wełyczko [78] studied *almost Legendre curves*, *i.e.*, slant curves with contact angle $\pi/2$ in normal almost contact metric 3-manifolds. See also Srivastava [67]. Călin, Crasmareanu and Munteanu [12] studied slant curves with proper mean curvature vector field in normal almost contact metric 3-manifold of type $(0, \beta)$. In

particular they have given explicit parametrization of slant curves in the hyperbolic 3-space equipped with natural homogeneous normal almost contact metric structure (Kenmotsu structure of constant curvature).

Next, Călin and Crasmareanu [11] studied slant curves in normal almost contact metric 3-manifolds. The present authors studied almost Legendre curves in normal almost contact metric 3-manifolds with proper mean curvature vector field [39]. Suh, Lee and the second named author studied Legendre curves in Sasakian 3-manifolds whose mean curvature vector field satisfies C-parallel or C-proper condition [53]. Güvenç and Özgür studied slant curves satisfying C-parallel or C-proper conditions in trans-Sasakian manifolds of arbitrary dimension [32]. Hou and Sun studied slant curves in the unit tangent sphere bundles equipped with standard contact metric structures [33].

Here we would like to point out that in 1963, Tashiro and Tachibana have been studied special kind of slant curves called C-loxodromes in Sasakian manifold [73]. For the precise definition of C-loxodrome, see Definition 5.1. It should be remarked that the class of slant curves is larger than the class of C-loxodoromes in Sasakian manifolds. See Section 5.3.

Moreover, in 1995, Blair, Dillen, Verstraelen and Vrancken gave a variational characterization of slant curves in K-contact manifolds [8].

There is another viewpoint for slant curve geometry. Since on 3-dimensional *contact* metric manifolds, the associated almost CR-structures are automatically integrable, submanifold geometry of contact metric 3-manifolds with respect to CR-structure (pseudo-Hermitian structure) is also an interesting subject. Here the pseudo-Hermitain geometry of submanifolds in contact metric 3-manifolds is formulated as submanifold geometry with respect to *Tanaka-Webster connection* instead of Levi-Civita connection.

In [20], Cho and second named author of this paper started to investigate slant curves in contact metric 3-manifolds with respect to Tanaka-Webster connection. In [52] the second named author continued her study on slant curves of Sasakian 3-manifolds in pseudo-Hermitian geometry. Özgür and Güvenç studied slant curves with proper mean curvature vector field of contact metric 3-manifolds in pseudo-Hermitian geometry [61]. For other related works on pseudo-Hermitian geometry of curves and surfaces, we refer to [19, 37, 38].

In addition in our previous paper [41], we extended pseudo-Hermitian geometry of slant curves in contact metric 3-manifolds to those in normal *almost contact* metric 3-manifolds.

In contrast to contact case, the Levi-form of almost contact metric manifolds may be degenerate. In addition Tanaka-Webster connection is defined under contact condition. Thus to develop submanifold geometry in almost contact metric 3-manifolds analogues to pseudo-Hermitian geometry, we need to introduce appropriate linear connections. For this purpose, in [40], we have introduced a linear connection on almost contact metric manifolds. Our connection coincides with Tanaka-Webster connection when the structure is (integrable) contact metric. In [41], the present authors studied slant curves in normal almost contact metric 3-manifolds with proper mean curvature vector field with respect to the linear connection mentioned above.

This paper has two purposes. The first purpose is to give a survey on slant curves in 3-dimensional almost contact metric geometry. The second purpose is to study slant curves in the space Sol_3. More precisely, in the final section we

study slant curves in certain 2-parameter family of solvable Lie groups equipped with natural left invariant almost contact metric structure. This 2-parameter family includes, Euclidean 3-space (cosymplectic manifold), hyperbolic 3-space (Kenmotsu manifold) as well as the space Sol_3.

Throughout this paper we denote by $\Gamma(E)$ the space of all smooth sections of a vector bundle E.

1. Preliminaries

1.1. Let (M, g) be a 3-dimensional Riemannian manifold with Levi-Civita connection ∇. We denote by $\mathrm{O}(M)$ the orthonormal frame bundle of M.

A unit speed curve $\gamma : I \to M$ is said to be a *geodesic* if it satisfies $\nabla_{\gamma'}\gamma' = 0$.

More generally, γ is said to be a *Frenet curve* if there exists an orthonormal frame field $\mathcal{E} = (E_1, E_2, E_3)$ along γ such that

$$(1.1) \qquad \nabla_{\gamma'}\mathcal{E} = \mathcal{E} \begin{pmatrix} 0 & -\kappa_1 & 0 \\ \kappa_1 & 0 & -\kappa_2 \\ 0 & \kappa_2 & 0 \end{pmatrix}$$

for some *non-negative* functions κ_1 and κ_2. Here the vector field E_1 is the unit tangent vector field γ'. For a unit speed curve with non-vanishing acceleration $\nabla_{\gamma'}\gamma'$, the *first curvature* κ_1 is defined by the formula

$$\kappa_1 = |\nabla_{\gamma'}\gamma'|.$$

The second unit vector field E_2 is thus obtained by

$$\nabla_{\gamma'}\gamma' = \kappa_1 E_2.$$

The curve $\gamma : I \to M$ induces a principal bundle $\gamma^*\mathrm{O}(M)$ over I. One can see that $\mathcal{E}$ is a section of $\gamma^*\mathrm{O}(M)$.

Next the second curvature κ_2 and the third unit vector field E_3 are defined by the formula

$$\kappa_2 = |\nabla_{\gamma'}E_2 + \kappa_1 E_1|, \quad \nabla_{\gamma'}E_2 + \kappa_1 E_1 = \kappa_2 E_3.$$

From these one can check that E_3 satisfies

$$\nabla_{\gamma'}E_3 = -\kappa_2 E_2.$$

1.2. Next we assume that (M^3, g) is *oriented*. Denote by $\mathrm{d}v_g$ the volume element induced by the metric g compatible to the orientation. Thus for every positively oriented local coordinate system (x_1, x_2, x_3), $\mathrm{d}v_g$ is expressed as

$$\mathrm{d}v_g = \sqrt{\det(g_{ij})}\, \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3, \quad g_{ij} = g(\partial/\partial x_i, \partial/\partial x_j).$$

The volume element $\mathrm{d}v_g$ defines a vector product operation $\times$ on each tangent space T_pM by the rule

$$\mathrm{d}v_g(X, Y, Z) = g(X \times Y, Z), \quad X, Y, Z \in T_pM.$$

We can define an endomorphism field $(X \wedge Y)$ by

$$(X \wedge Y)Z = Z \times (X \times Y).$$

Then by elementary linear algebra, we have

$$(X \wedge Y)Z = g(Y, Z)X - g(Z, X)Y.$$

1.3. Now let $\gamma(s)$ be a unit curve in the oriented Riemannian 3-manifold $(M^3, g, \mathrm{d}v_g)$ with non-vanishing acceleration $\nabla_{\gamma'}\gamma'$. Then we put $\kappa := |\nabla_{\gamma'}\gamma'|$. We can take a unit normal vector field N by the formula $\nabla_{\gamma'}\gamma' = \kappa N$. Next define a unit vector field B by $B = T \times N$. Here $T = \gamma'$. In this way we obtain an orthonormal frame field $\mathcal{F} = (T, N, B)$ along γ which is *positively oriented*, that is, $\mathrm{d}v_g(T, N, B) = 1$. The orthonormal frame field $\mathcal{F}$ is called the *Frenet frame field* and satisfies

(1.2)
$$\nabla_{\gamma'}\mathcal{F} = \mathcal{F} \begin{pmatrix} 0 & -\kappa & 0 \\ \kappa & 0 & -\tau \\ 0 & \tau & 0 \end{pmatrix}$$

for some function τ. The functions κ and τ are called the *curvature* and *torsion* of γ, respectively. By definition $\mathcal{F}$ is a section of $\gamma^*\mathrm{SO}(M)$. Here $\mathrm{SO}(M)$ is a positive orthonormal frame bundle of M.

The ordinary differential equation (1.2) is called the *Frenet-Serret formula* of γ. The unit vector fields T, N and B are called the *tangent vector field, principal normal vector field* and *binormal vector field* of γ, respectively.

Put $E_1 = T$, $E_2 = N$ and $\kappa_1 = \kappa$. Then we obtain an orthonormal frame field $\mathcal{E} = (E_1, E_2, E_3)$ as in Section 1.1. The Frenet frame field $\mathcal{F}$ is related to $\mathcal{E}$ by $E_3 = \epsilon B$ and $\kappa_2 = \epsilon\tau$ with $\epsilon = \pm 1$. Here ϵ is determined by the formula $\epsilon = \mathrm{d}v_g(E_1, E_2, E_3)$.

Fundamental theorems of curve theory in (M^3, g, dv) is formulated as follows.

Theorem 1.1 (uniqueness theorem). *Let γ_1, $\gamma_2 : I \to M$ be unit speed curves in an oriented Riemannian 3-manifold $(M^3, g, \mathrm{d}v_g)$ with curvatures and torsions (κ_1, τ_1), (κ_2, τ_2), respectively. Then γ_1 is congruent to γ_2 under orientation preserving isometries if and only if $(\kappa_1, \tau_1) = (\kappa_2, \tau_2)$.*

Theorem 1.2 (existence theorem). *Let $\kappa(s) > 0$ and $\tau(s)$ be smooth functions defined on an interval I. Then there exits a unit speed curve $\gamma : I \to M$ in an oriented Riemannian 3-manifold $(M, g, \mathrm{d}v_g)$ with curvature κ and torsion τ.*

Based on these fundamental theorems, it is natural to take positive orthonormal frame fields along unit speed curves in *oriented* Riemannian 3-manifolds.

Throughout this paper we take positive orthonormal frame field for unit speed curves in oriented Riemannian 3-manifolds.

For more informations on Frenet curves, we refer to [66, Chapter 7, B.].

1.4. Here we recall some vector bundle calculus of curves for our later use.

Let $\gamma(s)$ be a unit curve in an oriented Riemannian 3-manifold (M^3, g). Then the vector bundle γ^*TM is defined by

$$\gamma^*TM = \bigcup_{s \in I} T_{\gamma(s)}M.$$

A section $X \in \Gamma(\gamma^*TM)$ is called a vector field along γ. The Levi-Civita connection ∇ induces a connection ∇^γ on γ^*TM by

$$\nabla^\gamma_{\frac{\mathrm{d}}{\mathrm{d}s}} X = \nabla_{\gamma'} X.$$

One can see that $(\gamma^*TM, \gamma^*g, \nabla^\gamma)$ is a Riemannian vector bundle over I, *i.e.*, $\nabla^\gamma(\gamma^*g) = 0$.

The *mean curvature vector field* H of γ is a section of γ^*TM defined by $H = \nabla_{\gamma'}\gamma' = \kappa N$. By definition, geodesics are unit speed curves with vanishing mean curvature vector field.

Lemma 1.1. *Let (M, g) be an oriented Riemannian 3-manifold and γ a unit speed curve. Then we have*

$$\nabla_{\gamma'} H = -\kappa^2 T + \kappa' N + \kappa\tau B,$$
$$\nabla_{\gamma'}\nabla_{\gamma'} H = -3\kappa\kappa' T + (\kappa'' - \kappa^3 - \kappa\tau^2)N + (2\kappa'\tau + \kappa\tau')B.$$

The *Laplace-Beltrami operator* Δ of $(\gamma^* TM, \nabla^\gamma)$ is defined by

$$\Delta = -\nabla^\gamma_{\frac{d}{ds}}\nabla^\gamma_{\frac{d}{ds}} = -\nabla_{\gamma'}\nabla_{\gamma'}.$$

Thus for any $X \in \Gamma(\gamma^* TM)$, we have

$$\Delta X = -\nabla_{\gamma'}\nabla_{\gamma'} X.$$

A vector field X along γ is said to be *proper* if it satisfies $\Delta X = \lambda X$ for some function λ. In particular, X is said to be *harmonic* if $\Delta X = 0$. Thus Lemma 1.1 implies the following fundamental result.

Proposition 1.1. *Let γ be a unit speed curve in an oriented Riemannian 3-manifold (M, g). Then γ has proper mean curvature vector field $(\Delta H = \lambda H)$ if and only if γ is a geodesic $(\lambda = 0)$ or a helix satisfying $\kappa^2 + \tau^2 = \lambda$.*

1.5. The *normal bundle* $T^\perp\gamma$ of the curve γ is given by

$$T^\perp\gamma = \bigcup_{s \in I} T_s^\perp\gamma, \quad T_s^\perp\gamma = \mathbb{R}N(s) \oplus \mathbb{R}B(s).$$

The *normal connection* $\nabla^\perp$ is the connection in $T^\perp\gamma$ defined by

$$\nabla^\perp_{\gamma'} X = \nabla_{\gamma'} X - g(\nabla_{\gamma'} X, T)T$$

for any section $X \in \Gamma(\gamma^* TM)$.

The Laplace-Beltrami operator $\Delta^\perp = -\nabla^\perp_{\gamma'}\nabla^\perp_{\gamma'}$ of the vector bundle $(T^\perp\gamma, \nabla^\perp)$ is called the *normal Laplacian.*

1.6. During their studies on Euclidean submanifolds with pointwise k-planar normal sections, Arslan and West [2] introduced the notion of submanifold of $AW(k)$ type. Arslan and Özgür studied curves of $AW(k)$-type in Euclidean space.

Remarkably, the notion of "$AW(k)$-type submanifold" makes sense for submanifolds in arbitrary Riemannian manifolds.

Definition 1.1 ([1])**.** A Frenet curve $\gamma(s)$ in a Riemannian 3-manifold is said to be of type:

(1) $AW(1)$ if $N^{(3)}(s)^\perp = 0$,
(2) $AW(2)$ if $\|N^{(2)}(s)^\perp\|^2 N^{(3)}(s)^\perp = g(N^{(3)}(s)^\perp, N^{(2)}(s)^\perp)N^{(2)}(s)^\perp$,
(3) $AW(3)$ if $\|N^{(1)}(s)^\perp\|^2 N^{(3)}(s)^\perp = g(N^{(3)}(s)^\perp, N^{(1)}(s)^\perp)N^{(1)}(s)^\perp$,

where

$$N^{(1)}(s)^\perp = (\nabla_{\gamma'}\gamma')^\perp,$$
$$N^{(2)}(s)^\perp = (\nabla_{\gamma'}\nabla_{\gamma'}\gamma')^\perp$$
$$N^{(3)}(s)^\perp = (\nabla_{\gamma'}\nabla_{\gamma'}\nabla_{\gamma'}\gamma')^\perp.$$

1.7. Let (M, D) be a manifold with a linear connection. A curve $\gamma : I \to M$ is said to be a D-geodesic if $D_{\gamma'}\gamma' = 0$. In case, M is a Riemannian manifold and D is the Levi-Civita connection, then a unit speed curve γ is a D-geodesic if and only if it is a geodesic in usual sense. We define the tension field $\tau(\gamma; D)$ of γ with respect to D by $\tau(\gamma; D) := D_{\gamma'}\gamma'$.

In [19] D-biharmonicity of curves was introduced. Denote by T^D and R^D the torsion and curvature tensor field of D.

Definition 1.2. A curve $\gamma : I \to (M, D)$ is said to be D-*biharmonic* if it satisfies

$$(1.3) \qquad D_{\gamma'}D_{\gamma'}\tau(\gamma; D) - T^D(\gamma', D_{\gamma'}\tau(\gamma; D)) - (D_{\gamma'}T^D)(\gamma', \tau(\gamma; D))$$
$$+ R^D(\tau(\gamma; D), \gamma')\gamma' = 0.$$

Short calculation shows that (1.3) is rewritten as

$$D_{\gamma'}D_{\gamma'}\tau(\gamma; D) + D_{\gamma'}T^D(\tau(\gamma; D), \gamma') + R^D(\tau(\gamma; D), \gamma')\gamma' = 0.$$

For biharmonic maps in Riemannian geometry we refer to a survey [54] by Montaldo and Oniciuc.

2. Almost contact manifolds

2.1. Let M be a manifold of odd dimension $m = 2n + 1$. Then M is said to be an *almost contact manifold* if its structure group $\mathrm{GL}_m\mathbb{R}$ of the linear frame bundle is reducible to $\mathrm{U}(n) \times \{1\}$. This is equivalent to existence of a tensor field φ of type $(1,1)$, a vector field ξ and a 1-form η satisfying

$$\varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1.$$

From these conditions one can deduce that

$$\varphi\xi = 0, \quad \eta \circ \varphi = 0.$$

Moreover, since $\mathrm{U}(n) \times \{1\} \subset \mathrm{SO}(2n + 1)$, M admits a Riemannian metric g satisfying

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y)$$

for all $X, Y \in \mathfrak{X}(M)$. Here $\mathfrak{X}(M) = \Gamma(TM)$ denotes the Lie algebra of all smooth vector fields on M. Such a metric is called an *associated metric* of the almost contact manifold $M = (M, \varphi, \xi, \eta)$. With respect to the associated metric g, η is metrically dual to ξ, that is

$$g(X, \xi) = \eta(X)$$

for all $X \in \mathfrak{X}(M)$. A structure (φ, ξ, η, g) on M is called an *almost contact metric structure*, and a manifold M equipped with an almost contact metric structure is said to be an *almost contact metric manifold*. A plane section Π at a point p of an almost contact metric manifold M is said to be *holomorphic* if it is invariant under φ_p. The sectional curvature function $\mathcal{H}$ of holomorphic plane sections are called the *holomorphic sectional curvature* (also called φ-sectional curvature).

2.2. The *fundamental 2-form* Φ of $(M, \varphi, \xi, \eta, g)$ is defined by

$$\Phi(X, Y) = g(X, \varphi Y), \quad X, Y \in \mathfrak{X}(M).$$

An almost contact metric manifold M is said to be a *contact metric manifold* if $\Phi = \mathrm{d}\eta$. On a contact metric manifold, η is a *contact form*, *i.e.*, $(\mathrm{d}\eta)^n \wedge \eta \neq 0$. Thus every contact metric manifold is orientable. Here we recall the followong fundamental fact ([7, Theorem 4.6]).

Proposition 2.1. *On a contact metric manifold* $(M^{2n+1}, \varphi, \xi, \eta, g)$, *the volume element* $\mathrm{d}v_g$ *induced from the associated metric* g *is related to the contact form* η *by*

$$\mathrm{d}v_g = \frac{(-1)^n}{2^n n!} \eta \wedge (\mathrm{d}\eta)^n.$$

2.3. On the direct product manifold $M \times \mathbb{R}$ of an almost contact metric manifold and the real line $\mathbb{R}$, any tangent vector field can be represented as the form $(X, \lambda \mathrm{d}/\mathrm{d}t)$, where $X \in \mathfrak{X}(M)$ and λ is a function on $M \times \mathbb{R}$ and t is the Cartesian coordinate on the real line $\mathbb{R}$.

Define an almost complex structure J on $M \times \mathbb{R}$ by

$$J(X, \lambda \mathrm{d}/\mathrm{d}t) = (\varphi X - \lambda \xi, \eta(X) \mathrm{d}/\mathrm{d}t).$$

If J is integrable then M is said to be *normal*.

Equivalently, M is normal if and only if

$$[\varphi, \varphi](X, Y) + 2\mathrm{d}\eta(X, Y)\xi = 0,$$

where $[\varphi, \varphi]$ is the *Nijenhuis torsion* of φ defined by

$$[\varphi, \varphi](X, Y) = [\varphi X, \varphi Y] + \varphi^2[X, Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y]$$

for any $X, Y \in \mathfrak{X}(M)$.

For more details on almost contact metric manifolds, we refer to Blair's monograph [7].

2.4. For an arbitrary almost contact metric 3-manifold M, we have ([57]):

$$(2.1) \qquad (\nabla_X \varphi)Y = g(\varphi \nabla_X \xi, Y)\xi - \eta(Y)\varphi \nabla_X \xi,$$

where ∇ is the Levi-Civita connection on M. Moreover, we have

$$\mathrm{d}\eta = \eta \wedge \nabla_\xi \eta + \alpha \Phi, \quad \mathrm{d}\Phi = 2\beta \eta \wedge \Phi,$$

where α and β are the functions defined by

$$(2.2) \qquad \alpha = \frac{1}{2}\mathrm{Trace}\,(\varphi \nabla \xi), \quad \beta = \frac{1}{2}\mathrm{Trace}\,(\nabla \xi) = \frac{1}{2}\mathrm{div}\,\xi.$$

Olszak [57] showed that an almost contact metric 3-manifold M is normal if and only if $\nabla \xi \circ \varphi = \varphi \circ \nabla \xi$ or, equivalently,

$$(2.3) \qquad \nabla_X \xi = -\alpha \varphi X + \beta(X - \eta(X)\xi), \quad X \in \Gamma(TM).$$

We call the pair (α, β) the *type* of a normal almost contact metric 3-manifold M.

Using (2.1) and (2.3) we note that the covariant derivative $\nabla \varphi$ of a 3-dimensional normal almost contact metric manifold is given by

$$(2.4) \qquad (\nabla_X \varphi)Y = \alpha(g(X, Y)\xi - \eta(Y)X) + \beta(g(\varphi X, Y)\xi - \eta(Y)\varphi X).$$

Moreover M satisfies

$$2\alpha\beta + \xi(\alpha) = 0.$$

Thus if α is a nonzero constant, then $\beta = 0$. In particular, a normal almost contact metric 3-manifold is said to be

- *cosymplectic* (or *coKähler*) *manifold* if $\alpha = \beta = 0$,
- *quasi-Sasakian manifold* if $\beta = 0$ and $\xi(\alpha) = 0$.
- *α-Sasakian manifold* if α is a nonzero constant and $\beta = 0$,
- *β-Kenmotsu manifold* if $\alpha = 0$ and β is a nonzero constant.

1-Sasakian manifolds and 1-Kenmotsu manifolds are simply called *Sasakian manifolds* and *Kenmotsu manifolds*, respectively. Sasakian manifolds are characterized as normal contact metric 3-manifolds.

Sasakian manifolds of constant holomorphic sectional curvature are called *Sasakian space forms*. One can verify that every cosymplectic manifold satisfies

$$\mathrm{d}\eta = 0, \quad \mathrm{d}\Phi = 0.$$

More generally, almost contact metric manifolds with closed η and closed Φ are called *almost cosymplectic manifolds* [31]. An almost cosymplectic manifold is cosymplectic if and only if it is normal.

Remark 2.1. An almost contact metric manifold M is said to be a *locally conformal almost cosymplectic manifold* (resp. *locally conformal cosymplectic manifold*) if there exists an open covering $\{U_\lambda\}_{\lambda \in \Lambda}$ together with smooth functions $\sigma_\lambda \in C^\infty(U_\lambda)$ such that the structure $(\varphi, e^{\sigma_\lambda}\xi, e^{-\sigma_\lambda}\eta, e^{-2\sigma_\lambda}g)$ is almost cosymplectic (resp. cosymplectic) over U_λ. Olszak showed the following characterization for normal almost contact metric manifolds satisfying (2.4) with $\alpha = 0$.

Proposition 2.2 ([58])**.** *An almost contact metric manifold M is normal locally conformal almost cosymplectic manifold if and only if M satisfies* (2.4) *with function β, $\mathrm{d}\beta \wedge \eta = 0$ and $\alpha = 0$. In such a case the almost contact metric manifold is locally conformal cosymplectic.*

Remark 2.2 (*f-Kenmotsu manifolds*)*.* Olszak and Roşca [59] showed that on almost contact metric manifold M of dimension $2n + 1 > 3$ satisfying (2.4) with $\alpha = 0$ automatically satisfy the equation $\mathrm{d}\beta \wedge \eta = 0$. But this does not hold in general when $\dim M = 3$. Clearly when β is a constant, this condition holds. Based on these observations, in 3-dimensional case, Olszak and Roşca introduced the notion of f-Kenmotsu manifold as follows:

Definition 2.1 ([59])**.** An almost contact metric manifold M is said to be an *f-Kenmotsu manifold* if it satisfies

$$(\nabla_X \varphi)Y = f(g(\varphi X, Y)\xi - \eta(Y)\varphi X),$$

where f is a function satisfying $\mathrm{d}f \wedge \eta = 0$.

2.5. As a generalization of the class of Kenmotsu manifold, the notion of almost Kenmotsu manifold was introduced.

An almost contact metric manifold $(M, \varphi, \xi, \eta, g)$ is said to be an *almost β-Kenmotsu manifold* if it satiesfies

$$\mathrm{d}\eta = 0, \quad \mathrm{d}\Phi = 2\beta\eta \wedge \Phi$$

for some constant $\beta \neq 0$. Almost 1-Kenmotsu manifolds are refered as to *almost Kenmotsu manifolds*. One can see that an almost β-Kenmotsu manifold is normal if and only if it is a β-Kenmotsu manifold.

Remark 2.3. In [47], an almost contact metric manifold M is called an *almost β-cosymplectic manifold* if it satisfies $\mathrm{d}\eta = 0$ and there exist a real number β such that $\mathrm{d}\Phi = 2\beta\eta \wedge \Phi$.

Remark 2.4. An arbitrary almost contact metric 3-manifold M satisfies (*cf.* [57, Proposition 1]):

$$\mathrm{d}\Phi = 2\beta\,\eta \wedge \Phi.$$

Here the function β is defined by $\beta = \operatorname{div}\xi/2$ as before. Next η is closed if and only if $\nabla\xi$ is self-adjoint with respect to g. Thus M is almost β-Kenmotsu if and only if $\nabla\xi$ is self-adjoint and $\operatorname{div}\xi$ is non-zero constant.

Remark 2.5. An almost contact metric manifold M is said to be

- *nearly Sasakian* if

$$(\nabla_X\varphi)Y + (\nabla_Y\varphi)X = 2g(X,Y)\xi - \eta(X)Y - \eta(Y)X$$

   for all X, $Y \in \mathfrak{X}(M)$.
- *nearly cosymplectic* if $(\nabla_X\varphi)X = 0$ for all $X \in \mathfrak{X}(M)$.

In case $\dim M = 3$, nearly Sasakian manifolds are automatically Sasakian. Analogously, every nearly cosymplectic 3-manifold is cosymplectic (see [46]).

3. CR-MANIFOLDS

3.1. An *almost CR-structure* S of a smooth manifold M is a complex vector subbundle $S \subset T^{\mathbb{C}}M$ of the complexified tangent bundle of M satisfying $S \cap \overline{S} = \{0\}$. A manifold M equipped with an almost CR-structure is called an *almost CR-manifold*.

An almost CR-structure S is said to be *integrable* if it satisfies the *integrability condition*:

$$[\Gamma(S), \Gamma(S)] \subset \Gamma(S).$$

In such a case, (M, S) is called a *CR-manifold*.

Now let $M = (M, \varphi, \xi, \eta, g)$ be an almost contact metric manifold. Then we define an almost CR-structure S of M by

$$S = \{X - \sqrt{-1}\varphi X \mid X \in \Gamma(D)\}$$

with

$$D = \{X \in TM \mid \eta(X) = 0\}.$$

We call S the *almost CR-structure* associated to (φ, ξ, η, g). Note that when $\dim M = 3$, the associated almost CR-structure S is automatically integrable.

One can easily check that the associated almost CR-structures of normal almost contact metric manifolds are integrable.

Assume that M is a contact metric manifold. Define a section L of $\Gamma(D^* \otimes D^*)$ by

$$L(X, Y) = -\mathrm{d}\eta(X, \varphi Y).$$

Then L is positive definite on $D \otimes D$ and called the *Levi-form* of M. When the associated almost CR-structure S is integrable, the resulting CR-manifold (M, S) is called a *strongly pseudo-convex CR-manifold* or *strongly pseudo-convex pseudo-Hermitian manifold*.

Proposition 3.1 ([72])**.** *Let M be a contact metric manifold. Then its associated almost CR-structure is integrable if and only if $\mathcal{Q} = 0$. Here the tensor field $\mathcal{Q}$ is defined by*

$$(3.1) \qquad \mathcal{Q}(Y, X) := (\nabla_X\varphi)Y - g((I+h)X, Y)\xi - \eta(X)\eta(Y)\xi + \eta(Y)(I+h)X,$$

where $h = \pounds_\xi\varphi/2$. The tensor field $\mathcal{Q}$ is called the Tanno tensor field.

Thus on a strongly pseudo-convex CR-manifolds, the following formula holds:

$$(3.2) \qquad (\nabla_X \varphi)Y = g((I+h)X, Y)\xi - \eta(Y)(I+h)X$$

for all vector fields X and Y. The formula (3.2) implies

$$\nabla_X \xi = -\varphi(I+h)X, \quad X \in \mathfrak{X}(M).$$

Remark 3.1. A contact metric 3-manifold M is Sasakian if and only if $h = 0$.

Problem 1. Let M be a contact metric manifold. Then its associated CR-structure is integrable if and only if M satisfies (3.2). Thus every contact metric 3-manifold satisfies (3.2). In almost contact setting can we prove the following statement ?

> Let M be an almost contact metric manifold. Then its associated CR-structure is integrable if and only if M satisfies (2.1) ?

Note that Călin and Ispas [13] considered almost contact metric manifolds satisfying the following condition (which is slightly different from (2.1)):

$$(\nabla_X \varphi)Y = \varphi(\nabla_{\varphi X}\varphi)Y - g(\nabla_{\varphi X}\xi, Y)\xi.$$

3.2. In the study of strongly pseudo-convex CR-manifolds, the linear connection $\hat{\nabla}$ introduced by Tanaka and Webster is highly useful:

$$(3.3) \qquad \hat{\nabla}_X Y = \nabla_X Y + \eta(X)\varphi Y + \{(\nabla_X \eta)Y\}\xi - \eta(Y)\nabla_X \xi.$$

Here ∇ is the Levi-Civita connection of the associated metric. The linear connection $\hat{\nabla}$ is referred as the *Tanaka-Webster connection* ([71, 77]). It should be remarked that the Tanaka-Webster connection has *non-vanishing* torsion $\hat{T}$:

$$\hat{T}(X, Y) = 2g(X, \varphi Y)\xi + \eta(Y)\varphi hX - \eta(X)\varphi hY.$$

With respect to the Tanaka-Webster connection all the structure tensor fields (φ, ξ, η, g) are parallel, *i.e.*,

$$\hat{\nabla}\varphi = 0, \ \hat{\nabla}\xi = 0, \ \hat{\nabla}\eta = 0, \ \hat{\nabla}g = 0.$$

4. Curve theory in almost contact metric 3-manifolds

4.1. Let $(M, \varphi, \xi, \eta, g)$ be a contact metric 3-manifold. Then as we have seen before, the volume element dv_g derived from the associated metric g is related to the contact form η by

$$(4.1) \qquad dv_g = -\frac{1}{2}\eta \wedge \Phi.$$

Here Φ is the fundamental 2-form. Even if M is non-contact, M is orientable by the 3-form $-\eta \wedge \Phi/2$ and the volume element dv_g coincides with this 3-form. Thus hereafter we orient the almost contact metric 3-manifold M by $dv_g = -\eta \wedge \Phi/2$. With respect to this orientation, the vector product $\times$ is computed as

$$(4.2) \qquad X \times Y = -\Phi(X, Y)\xi + \eta(X)\varphi Y - \eta(Y)\varphi X.$$

With respect to this orientation, we have

$$dv_g(X, \varphi X, \xi) = 1$$

for any unit vector field X orthogonal to ξ (equivalently, $X \times \varphi X = \xi$). Camcı called the vector product operation $\times$ given in (4.2) the new *extended cross product*. However the operation $\times$ coincides with the vector product induced by dv_g and hence *not* a new operation.

4.2. Let $\gamma(s)$ be a unit speed curve in an almost contact metric 3-manifold $M = (M, \varphi, \xi, \eta, g)$. Denote by $\theta(s)$ the the angle function between the tangent vector field γ' and ξ, *i.e.*, $\eta(\gamma'(s)) = \cos \theta(s)$ with $0 \leq \theta \leq \pi$.

Definition 4.1. A unit speed curve γ in an almost contact metric 3-manifold is said to be *slant* if its contact angle θ is constant.

By definition, slant curves with constant angle 0 are trajectories of ξ. Slant curves with constant angle $\pi/2$ are called *almost Legendre curves* or *almost contact curves*. J. Wełyczko [78] studied almost contact curves in normal almost contact metric 3-manifolds. See also [39].

When M is a contact metric 3-manifold, almost Legendre curves are traditionally called *Legendre curves*(*cf.* [4]). Legendre curves play crucial roles in 3-dimensional contact geometry and contact topology.

Slant curves appear naturally in differential geometry of Sasakian 3-manifolds. For instance, biharmonic curves in 3-dimensional Sasakian space forms are slant helices [19]. Moreover the normal flowlines of the magnetic field associated to the Reeb vector field of Sasakian 3-manifolds are slant helices [9, 10] (see also section 5.3).

Example 4.1 (Euclidean helices)**.** Let $\mathbb{E}^3(x, y, z)$ be the Euclidean 3-space with metric $\langle \cdot, \cdot \rangle = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$. Then the standard cosymplectic structure associated to g is defined by

$$\eta = \mathrm{d}z, \ \xi = \frac{\partial}{\partial z}, \ \varphi \frac{\partial}{\partial x} = \frac{\partial}{\partial y}, \ \varphi \frac{\partial}{\partial y} = -\frac{\partial}{\partial x}.$$

Now let $\gamma(s)$ be a slant helix with constant contact angle θ. Then γ is congruent to the following *model helix*:

$$\gamma(s) = (a \cos(s/c), a \sin(s/c), bs/c),$$

where $a > 0$, $b \neq 0$ and $c = \sqrt{a^2 + b^2} > 0$ are constants. The Frenet frame of γ is given by

$$\mathcal{F} = (T, N, B) = \begin{pmatrix} -(a/c)\sin(s/c) & -\cos(s/c) & (b/c)\sin(s/c) \\ (a/c)\cos(s/c) & -\sin(s/c) & -(b/c)\cos(s/c) \\ b/c & 0 & a/c \end{pmatrix}.$$

One can see that $\det \mathcal{F} = 1$. The curvature and torsion of γ are $\kappa = a/c^2 > 0$ and $\tau = b/c^2 \neq 0$. One can see that γ has constant contact angle θ with $\cos \theta = b/c$. In particular every almost Legendre helix is congruent to the circle

$$\gamma(s) = (c \cos(s/c), c \sin(s/c), 0)$$

of curvature $1/c > 0$ and torsion 0.

5. Slant curves in Sasakian 3-manifolds

5.1. Let γ be a unit speed curve in a contact metric 3-manifold M. We have

$$\xi \times T = \varphi T, \ \ \xi \times N = \varphi N, \ \ \xi \times B = \varphi B.$$

Since ξ is expressed as

$$\xi = \eta(T)T + \eta(N)N + \eta(B)B$$

along γ, we have

$$\varphi T = -\eta(N)B + \eta(B)N, \quad \varphi N = -\eta(B)T + \eta(T)B, \quad \varphi B = -\eta(T)N + \eta(N)T.$$

Differentiating $\eta(T)$, $\eta(N)$ and $\eta(B)$ along γ, we have

$$\eta(T)' = \kappa\eta(N) - g(T, \varphi hT),$$
$$\eta(N)' = -\kappa\eta(T) + (\tau - 1)\eta(B) - g(N, \varphi hT),$$
$$\eta(B)' = -(\tau - 1)\eta(N) - g(B, \varphi hT).$$

For example, we compute

$$\begin{aligned}
\eta(N)' = g(N, \xi)' &= g(\nabla_{\gamma'} N, \xi) + g(N, \nabla_{\gamma'}\xi) \\
&= g(-\kappa T + \tau B, \xi) - g(N, \varphi(I + h)T) \\
&= -\kappa\eta(T) + \tau\eta(B) - g(N, \varphi(I + h)T) \\
&= -\kappa\eta(T) + \tau\eta(B) - g(N, \varphi T) - g(N, \varphi hT) \\
&= -\kappa\eta(T) + (\tau - 1)\eta(B) - g(N, \varphi hT).
\end{aligned}$$

Now we assume that M is Sasakian. Then we have

$$\eta(T)' = \kappa\eta(N),$$
$$\eta(N)' = -\kappa\eta(T) + (\tau - 1)\eta(B),$$
$$\eta(B)' = -(\tau - 1)\eta(N).$$

Proposition 5.1 ([19]). *A non-geodesic unit speed curve γ in a Sasakian 3-manifold is a slant curve if and only if $\eta(N) = 0$.*

Moreover the second equation shows that on a non-geodesic slant curve, $\kappa\eta(T) = (\tau - 1)\eta(B)$. The third equation implies that $\eta(B)$ is constant along γ. Hence we obtain

Proposition 5.2 ([19]). *The ratio of κ and $\tau - 1$ is constant along a non-geodesic slant curve in a Sasakian 3-manifold.*

Conversely we have the following result.

Proposition 5.3 ([19], see also [53]). *Let $\gamma(s)$ be a unit speed curve in a Sasakian 3-manifold satisfying $\eta(N) = 0$ and the ratio of κ and $\tau - 1$ is constant. Then γ is a slant curve.*

In [14] Camcı exhibited the following example.

Example 5.1. Define a function $\sigma(s)$ by

$$\sigma(s) = \frac{1}{2}(1 - \cos(2\sqrt{2}s))$$

and define a curve $\gamma(s)$ by the ODE

$$x'(s) = -2\sqrt{2 - \sigma(s)^2}\sin\theta(s)$$

with

$$\theta'(s) = -2\sigma(s) + \frac{2}{1 + \sigma(s)}.$$

Then the resulting curve satisfies $(\tau - 1)/\kappa = $ constant. However this curve is not slant. In fact,

$$\eta(N)' = \frac{1}{2}\sigma''(s) = 2\cos(2\sqrt{2}s).$$

Thus γ does not satisfy $\eta(N) = 0$. Hence his example is not a counterexample to [19].

Remark 5.1 (Legendre curves). Let γ be a non-geodesic Legendre curve in a Sasakian 3-manifold. Then differentiating $\eta(\gamma') = 0$ along γ, we have

$$0 = g(\gamma', \xi)' = g(\kappa N, \xi) + g(T, \nabla_{\gamma'}\xi) = \kappa g(N, \xi) - g(T, \varphi T) = \kappa \eta(N).$$

Thus we have $\eta(N) = 0$. Since N is ortogonal to both T and ξ, N is expressed as $N = \epsilon \varphi T$ with $\epsilon = \pm 1$. Then

$$\begin{aligned}
\nabla_{\gamma'} N &= \epsilon\{(\nabla_{\gamma'}\varphi)T + \varphi(\nabla_{\gamma'}T)\} \\
&= \epsilon\{g(\gamma', \gamma')\xi + \varphi(\kappa N)\} = \epsilon(\xi + \kappa \varphi N) \\
&= -\kappa T + \epsilon \xi.
\end{aligned}$$

Since (T, N, B) is positively oriented, we have $1 = \mathrm{dv}_g(T, N, B) = \mathrm{dv}_g(T, \epsilon\varphi T, B)$. Namely, $B = \epsilon T \times \varphi T$. On the other hand, B is computed as

$$\begin{aligned}
B = T \times N &= -\Phi(T, N)\xi + \eta(T)\varphi N - \eta(N)\varphi T \\
&= -g(T, \varphi N)\xi = -g(T, \varphi(\epsilon\varphi T))\xi = \epsilon \xi.
\end{aligned}$$

Hence we obtain

$$\nabla_{\gamma'} N = -\kappa T + B.$$

This formula should coincides with

$$\nabla_{\gamma'} N = -\kappa T + \tau B.$$

Hence we have $\tau = 1$.

Corollary 5.1 ([4]). *Every Legendre curve in a Sasakian 3-manifold has constant torsion 1.*

Example 5.2 (Legendre helices in $\mathbb{S}^3$). Let $\mathbb{C}^2$ be the complex Euclidean 2-space with complex structure J:

$$J(z_1, z_2) = \sqrt{-1}(z_1, z_2).$$

Identify $\mathbb{C}^2$ with Euclidean 4-space $\mathbb{E}^4$ via the isomorphism

$$(z_1, z_2) \longmapsto (x_1, y_1, x_2, y_2), \quad z_j = x_j + \sqrt{-1}y_j, \ j = 1, 2.$$

Then J corresponds to the linear transformation

$$(x_1, y_1, x_2, y_2) \longmapsto (-y_1, x_1, -y_2, x_2).$$

Now let us introduce a Sasakian structure on the unit 3-sphere $\mathbb{S}^3 \subset \mathbb{C}^2 = \mathbb{E}^4$. We take the unit normal vector field $\boldsymbol{n}$ of $\mathbb{S}^3$ in $\mathbb{C}^2$ by $\boldsymbol{n} = \boldsymbol{x}$, where $\boldsymbol{x}$ is the position vector field of $\mathbb{C}^2$. Then the Levi-Civita connections D of $\mathbb{C}^2$ and ∇ of $\mathbb{S}^3$ are related by the *Gauss formula*:

$$D_X Y = \nabla_X Y - \langle X, Y \rangle \boldsymbol{n}.$$

Define the vector field ξ on $\mathbb{S}^3$ by $\xi = -J\boldsymbol{n}$ and set $\eta = g(\xi, \cdot)$, where g is the metric of $\mathbb{S}^3$. Then η is a contact form on $\mathbb{S}^3$ and ξ is the Reeb vector field. The associated endomorphism field φ is the restriction of J to $\mathbb{S}^3$. It is well known that $\mathbb{S}^3$ equipped with this structure is a Sasakian manifold of constant curvature 1.

Now let a, b and ϕ be constants such that

(5.1) $$a^2 \cos^2 \phi + b^2 \sin^2 \phi = 1.$$

Then
$$\gamma(s) = (\cos\phi\cos(as), \cos\phi\sin(as), \sin\phi\cos(bs), \sin\phi\sin(bs))$$
is an arclength parametrized curve in $\mathbb{S}^3$ (see [30]). One can see that γ lies in the flat torus $\mathbb{T}^2$ of constant mean curvature $\cot(2\phi)$ given by the equations $x_1^2 + y_1^2 = \cos^2\phi$ and $x_2^2 + y_2^2 = \sin^2\phi$. The tangent vector field T is
$$T = (-a\cos\phi\sin(as), a\cos\phi\cos(as), -b\sin\phi\sin(bs), b\sin\phi\cos(bs)).$$

From this equation and the Gauss formula we get
$$\nabla_{\gamma'}\gamma' = \gamma'' + \gamma = \begin{pmatrix} (1-a^2)\cos\phi\cos(as) \\ (1-a^2)\cos\phi\sin(as) \\ (1-b^2)\sin\phi\cos(bs) \\ (1-b^2)\sin\phi\sin(bs) \end{pmatrix}.$$

Thus the curvature κ is computed as
$$\begin{aligned} \kappa^2 &= |\nabla_{\gamma'}\gamma'|^2 \\ &= a^4\cos^2\phi + b^4\sin^2\phi - 1 \\ &= a^2(1 - b^2\sin^2\phi) + b^2(1 - a^2\cos^2\phi) - 1 \\ &= a^2 + b^2 - a^2b^2 - 1 = (a^2 - 1)(1 - b^2). \end{aligned}$$

Hereafter we assume that $\kappa \neq 0$. Then the principal normal N is given by
$$N = \frac{1}{\sqrt{(a^2-1)(1-b^2)}} \begin{pmatrix} (1-a^2)\cos\phi\cos(as) \\ (1-a^2)\cos\phi\sin(as) \\ (1-b^2)\sin\phi\cos(bs) \\ (1-b^2)\sin\phi\sin(bs) \end{pmatrix}.$$

On the other hand, the Reeb vector field along γ is given by
$$\xi_\gamma = \begin{pmatrix} \cos\phi\sin(as) \\ -\cos\phi\cos(as) \\ \sin\phi\sin(bs) \\ -\sin\phi\cos(bs) \end{pmatrix}.$$

Hence the contact angle θ is computed as
$$(5.2) \qquad\qquad \cos\theta = \eta(T) = -(a\cos^2\phi + b\sin^2\phi).$$

Hence γ is a slant curve. In particular, γ is a Legendre curve if and only if $a\cos^2\phi + b\sin^2\phi = 0$.

Next we compute the torsion τ of γ. The square τ^2 of the torsion is given by $\tau^2 = |\nabla_{\gamma'}N + \kappa T|^2$. Since κ is constant, we have
$$\nabla_{\gamma'}N = \frac{1}{\kappa}(\gamma'' + \gamma)' = \frac{1}{\sqrt{(a^2-1)(1-b^2)}} \begin{pmatrix} -a(1-a^2)\cos\phi\sin(as) \\ a(1-a^2)\cos\phi\cos(as) \\ -b(1-b^2)\sin\phi\sin(bs) \\ b(1-b^2)\sin\phi\cos(bs) \end{pmatrix}.$$

Thus we get
$$\tau^2 = \left(\frac{(1-a^2)a}{\kappa} + a\kappa\right)^2 \cos^2\phi + \left(\frac{(1-b^2)b}{\kappa} + b\kappa\right)^2 \sin^2\phi = (ab)^2.$$

Thus B has the form

$$B = \frac{\varepsilon}{ab\sqrt{(a^2-1)(1-b^2)}} \begin{pmatrix} -a(1-a^2)b^2\cos\phi\sin(as) \\ a(1-a^2)b^2\cos\phi\cos(as) \\ -b(1-b^2)a^2\sin\phi\sin(bs) \\ b(1-b^2)a^2\sin\phi\cos(bs) \end{pmatrix}, \quad \varepsilon = \pm 1.$$

Next computing the determinant of (T, N, B, γ), we get $\det(T, N, B, \gamma) = -\varepsilon$. Hence we have $\varepsilon = -1$. Thus we get

$$\nabla_{\gamma'}B = \frac{-1}{ab\sqrt{(a^2-1)(1-b^2)}} \begin{pmatrix} -a^2(1-a^2)b^2\cos\phi\cos(as) \\ -a^2(1-a^2)b^2\cos\phi\sin(as) \\ -b^2(1-b^2)a^2\sin\phi\cos(bs) \\ -b^2(1-b^2)a^2\sin\phi\sin(bs) \end{pmatrix} = abN.$$

From the Frenet-Serret formula, we have $\tau = -ab$.

Now we concentrate on Legendre helices. Assume that γ is Legendre, then from the equation (5.2), we get

$$N = \begin{pmatrix} a\cos\phi\cos(as) \\ a\cos\phi\sin(as) \\ b\sin\phi\cos(bs) \\ b\sin\phi\sin(bs) \end{pmatrix} = -\varphi T.$$

In this case, the binormal vector field is given by $B = -\xi_\gamma$. One can see that

$$\det(T, N, B, \gamma)$$
$$= \begin{vmatrix} -a\cos\phi\sin(as) & a\cos\phi\cos(as) & -\cos\phi\sin(as) & \cos\phi\cos(as) \\ a\cos\phi\cos(as) & a\cos\phi\sin(as) & \cos\phi\cos(as) & \cos\phi\sin(as) \\ -b\sin\phi\sin(as) & b\sin\phi\cos(bs) & -\sin\phi\sin(bs) & \sin\phi\cos(bs) \\ b\sin\phi\cos(bs) & b\sin\phi\sin(bs) & \sin\phi\cos(bs) & \sin\phi\sin(bs) \end{vmatrix} = 1.$$

From the equation (5.1), (5.2) and $\sin^2\theta + \cos^2\theta = 1$, we have $ab = -1$. Hence $\mathcal{F} = (T, -\varphi T, -\xi_\gamma)$ is a positive orthonormal frame field along γ. The torsion τ is computed by $\tau = -ab = 1$. This fact is confirmed also by the formula

$$-\tau N = \nabla_{\gamma'}B = -\nabla_{\gamma'}\xi = \varphi T = -N.$$

Note that in case $\tau = -ab = -1$, $\eta(T) = 0$ can not hold [1]. In fact the Legendre condition $a\cos^2\phi + b\sin^2\phi = 0$ is not compatible with (5.1) under the condition $ab = 1$.

5.2. Variational characterization. Let M be a Sasakian 3-manifold. Denote by $\mathcal{M}_L$ the space of closed curves in M with length L. Blair, Dillen, Verstraelen and Vrancken studied the variational problem for the length functional

$$\mathcal{L} : \mathcal{M}_L \to \mathbb{R}^+; \quad \mathcal{L}(\gamma) = \int_0^L \sqrt{g(\gamma'(s), \gamma'(s))}\, ds$$

under the variations $\gamma^{(t)}$ of the form:

(5.3) $$\gamma^{(t)}(s) := \exp_{\gamma(s)}(tf(s)\xi_{\gamma(s)}).$$

---

[1]Under the conditions $a^2\cos^2\phi + b^2\sin^2\phi = 1$ and $\tau = -ab = -1$, Legendre condition implies $\cos^2\phi = -1/(1-a^2)$ and $\sin^2\phi = a^2/(1-a^2)$. On the other hand, when $\tau = 1$, we obtain $\cos^2\phi = 1/(1+a^2)$ and $\sin^2\phi = a^2/(1+a^2)$.

Then the first variation formula for $\mathcal{L}$ is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \mathcal{L}(\gamma^{(t)}) = -\int_0^L f(s)\frac{\mathrm{d}}{\mathrm{d}s}\cos\theta(s)\,\mathrm{d}s.$$

This first variational formula implies the following variational characterization for slant curves:

Theorem 5.1 ([8])**.** *A unit speed closed curve γ in a Sasakian 3-manifold M is a critical point of the length functional under the variation of the form* (5.3) *if and only if γ is a slant curve.*

5.3. Magnetic curves. Magnetic curves represent, in physics, the trajectories of the charged particles moving on a Riemannian manifold under the action of magnetic fields. A *magnetic field F* on a Riemannian manifold (M, g) is a closed 2-form and the *Lorentz force* associated to F is an endomorphism field L defined by

$$g(LX, Y) = F(X, Y), \quad X, Y \in \Gamma(TM).$$

The magnetic trajectories of F are curves γ satisfying the *Lorentz equation*:

$$\nabla_{\gamma'}\gamma' = L\gamma'.$$

One can see that every magnetic trajectory has constant speed. Unit speed magnetic curves are called *normal magnetic curves*.

Now let us consider magnetic curves in a Sasakian 3-manifold M with magnetic field $F_{\xi,q} = -q\Phi$. Here q is a constant (and called the *strength* of $F_{\xi,q}$). Then its Lorentz force $L_{\xi,q}$ is $q\varphi$. This magnetic field is called the *contact magnetic field* of strength q. The magnetic curve equation is

$$\nabla_{\gamma'}\gamma' = q\varphi\gamma'.$$

Proposition 5.4. *Let γ be a normal magnetic curve in a Sasakian 3-manifold M with respect to the contact magnetic field F_ξ. Then γ is a slant curve.*

(*Proof.*) Direct computation shows that

$$
\begin{aligned}
g(\gamma', \xi)' &= g(\nabla_{\gamma'}\gamma', \xi) + g(\gamma', \nabla_{\gamma'}\xi) \\
&= qg(\varphi\gamma', \xi) + g(\gamma', -\varphi\gamma') = 0.
\end{aligned}
$$

Hence γ is a slant curve. $\square$

Now let us study curvature and torsion of a *non-geodesic* normal magnetic curve γ with respect to $F_{\xi,q}$. Then as we have seen above, γ is a slant curve with constant contact angle θ.

By the vector product operation, the magnetic curve equation can be rewritten as

$$\nabla_{\gamma'}\gamma' = (q\xi) \times \gamma'.$$

Thus we have

$$q\xi \times \gamma' = \kappa N.$$

Hence

$$\kappa^2 = q^2 g(\xi \times \gamma', \xi \times \gamma') = q^2(1 - \cos^2\theta) = q^2\sin^2\theta.$$

Thus γ has constant curvature $\kappa = |q\sin\theta| > 0$. Since $\eta(N) = 0$, we have

$$N = \frac{\varepsilon}{|\sin\theta|}\varphi\gamma',$$

where $\varepsilon = q/|q|$ is the signature of q. The binormal B is computed as

$$B = \gamma' \times N = \frac{\varepsilon}{|\sin\theta|}\gamma' \times \varphi\gamma' = \frac{\varepsilon}{|\sin\theta|}(\xi - \cos\theta\gamma').$$

Inserting this into the equation $\nabla_{\gamma'}B = -\tau N$, we have

$$\tau = 1 + q\cos\theta.$$

Hence γ is a slant helix. Note that the ratio of κ and $\tau - 1$ is computed as

$$\frac{\tau - 1}{\kappa} = \varepsilon\frac{\cos\theta}{|\sin\theta|}.$$

Theorem 5.2 ([9, 10]). *The normal flowlines of contact magnetic field $F_{\xi,1}$ in a Sasakian 3-manifold are slant helices with curvature $\kappa = |\sin\theta|$ and torsion $\tau = 1 + \cos\theta$.*

Conversely let γ be a *non-geodesic* slant helix with constant curvature κ and torsion τ. Since γ is a slant curve, we have $\eta(N) = 0$. So N is orthogonal to both γ' and ξ. Thus N has the form $N = \lambda\xi \times \gamma'$. This implies

$$1 = |N| = |\lambda||\sin\theta|.$$

Hence λ is a constant. Note that since we assumed that γ is non-geodesic, $\sin\theta \neq 0$. Thus N has the form

$$N = \frac{\varepsilon}{|\sin\theta|}\varphi\gamma'.$$

This formula implies that

$$\nabla_{\gamma'}\gamma' = q\varphi\gamma'$$

with

$$q = \frac{\varepsilon}{|\sin\theta|}\kappa.$$

Hence γ is a normal magnetic curve with respect to the contact magnetic field of strength q.

The binormal B is given by

$$B = \gamma' \times N = \frac{\varepsilon}{|\sin\theta|}(\xi - \cos\theta\,\gamma').$$

The torison of γ is computed as

$$\tau = 1 + \frac{\varepsilon\kappa}{|\sin\theta|}\cos\theta.$$

Theorem 5.3. *Let γ be a non-geodesic slant helix in a Sasakian 3-manifold. Then γ is a normal magnetic curve with respect to the contact magnetic field $F_{\xi,q}$ with strength $q = \varepsilon\kappa/|\sin\theta|$.*

Periodicity of contact magnetic trajectories in 3-dimensional Sasakian space form of constant holomorphic sectional curvature ≥ 1 was investigated in [45]. For contact magnetic trajectories in Sasakian space forms of general dimension, we refer to [26].

In [73], Tashiro and Tachibana introduced the notion of C-loxodrome.

Definition 5.1. A unit speed curve γ in a Sasakian manifold is said to be a C-*loxodrome* if it satisfies

$$\nabla_{\gamma'}\gamma' = r\eta(\gamma')\varphi\gamma'.$$

Here r is a constant.

Clearly, every C-loxodrome has constant contact angle $\cos\theta = \eta(\gamma')$. Thus a C-loxodrome is a normal magnetic curve with respect to $F_{\xi,q}$ with strength $q = r\cos\theta$. It should be remarked that the notion of C-loxodrome is not equivalent to that of contact magnetic curve or slant curve. In fact, if a C-loxodrome γ has constant contact angle $\pi/2$, namely, γ is a Legendre curve, then γ should be a Legendre geodesic.

Yanamoto [80] investigated C-loxodromes in the unit 3-sphere $\mathbb{S}^3$ equipped with canonical Sasakian structure.

A differomorphism f on a Sasakian manifold is said to be a CL-transformation if it carries C-loxodromes to C-loxodromes [73]. Takamatsu and Mizusawa studied infinitesimal CL-transformations on compact Sasakian manifolds [69]. As an analogue of Wely's conformal curvature tensor field, Koto and Nagao introduced CL-curvature tensor field for Sasakian manifolds. The CL-curvature tensor field is invariant under CL-transformations. Koto and Nagao showed that Sasakian space forms are characterized as Sasakian manifolds with vanishing CL-curvature tensor fields [50].

Remark 5.2. Every proper biharmonic curve in a 3-dimensional Sasakian space form of constant holomorphic sectional curvature H are slant helices satisfying

$$\kappa^2 + \tau^2 = 1 + (H-1)\sin^2\theta.$$

Inserting $\kappa = \sin\theta$ and $\tau = 1 + \cos\theta$ in the left hand side of this ODE, we have

$$\kappa^2 + \tau^2 = \sin^2\theta + (1+\cos\theta)^2 = 2(1+\cos\theta).$$

Hence we get

$$(H-1)\cos^2\theta + 2\cos\theta + (2-H) = 0.$$

In case $H = 1$, we have $\cos\theta = -1/2$.

Assume that $H \neq 1$, $t = \cos\theta$ is a solution to $(H-1)t^2 + 2t + (2-H) = 0$. Since $1 - (2-H)(H-1) = (H - \frac{3}{2})^2 + \frac{3}{4} > 0$, this quadratic equation has real roots if and only if

$$-1 \leq \frac{-1 \pm \sqrt{H^2 - 3H + 3}}{H - 1} \leq 1.$$

One can check that for any $H \in \mathbb{R} \setminus \{1\}$, we have

$$-1 < \frac{-1 + \sqrt{H^2 - 3H + 3}}{H - 1} < 1.$$

Note that

$$\lim_{H \to 1} \frac{-1 + \sqrt{H^2 - 3H + 3}}{H - 1} = -\frac{1}{2}.$$

On the other hand, for any $H \in \mathbb{R} \setminus \{1\}$, we have

$$\left| \frac{-1 - \sqrt{H^2 - 3H + 3}}{H - 1} \right| > 1.$$

Thus we notice the following fact.

Proposition 5.5. *For any real number H different from 1, there exist normal magnetic helix which is proper biharmonic in the 3-dimensional Sasakian space form of constant holomorphic sectional curvature H with constant contact angle*

$$\theta = \cos^{-1} \frac{-1 + \sqrt{H^2 - 3H + 3}}{H - 1}.$$

6. C-PARALLEL CONDITIONS

Let γ be a unit speed curve in an almost contact metric 3-manifold. Then γ is said to have *η-parallel* (or *C-parallel*) mean curvature vector field if

$$g(\nabla_{\gamma'} H, X) = 0$$

for all $X \in \Gamma(\gamma^* TM)$ orthogonal to ξ. One can see that γ has η-parallel mean curvature vector field if and only if there exists a function λ along γ such that $\nabla_{\gamma'} H = \lambda \xi$.

Theorem 6.1 ([53]). *Let γ be a slant curve in a Sasakian 3-manifold. Then γ has a η-parallel mean curvature vector field if and only if γ is a geodesic ($\lambda = 0$) or a helix with $\kappa = \sqrt{-\lambda \cos \theta}$ and $\tau = \lambda \sin \theta / \sqrt{-\lambda \cos \theta}$.*

Corollary 6.1 ([53]). *Let γ be a Legendre curve in a Sasakian 3-manifold. Then γ satisfies $\nabla_{\gamma'} H = \lambda \xi$ if and only if γ is a Legendre geodesic.*

7. CANONICAL CONNECTIONS

7.1. Let $M = (M, \varphi, \xi, \eta, g)$ be an almost contact metric manifold. Define a tensor field $A = A^t$ of type $(1, 2)$ by

$$(7.1) \qquad A^t_X Y = -\frac{1}{2} \varphi(\nabla_X \varphi)Y - \frac{1}{2}\eta(Y)\nabla_X \xi - t\eta(X)\varphi Y + (\nabla_X \eta)(Y)\xi,$$

for all vector fields X and Y. Here t is a real constant. We define a linear connection $\widetilde{\nabla}^t$ on M by

$$\widetilde{\nabla}^t_X Y = \nabla_X Y + A^t_X Y.$$

We call the connection $\widetilde{\nabla}^t$ the *canonical connection* of M. Note that the connection ∇^0 is the (φ, ξ, η)-connection introduced by Sasaki and Hatakeyama in [64]. Moreover $\widetilde{\nabla}^1$ was introduced by Cho [16]. When M is a strongly pseudo-convex CR-manifold, (3.2) implies that

$$\widetilde{\nabla}^t_X Y = \nabla_X Y - t\eta(X)\varphi Y + \eta(Y)\varphi(I + h)X - g(\varphi(I + h)X, Y)\xi.$$

This formula shows that when M is a strongly pseudo-convex CR-manifold, $\widetilde{\nabla}^t|_{t=-1}$ is the *Tanaka-Webster connection*. The canonical connection $\widetilde{\nabla}^t$ on an almost contact metric manifold satisfies the following conditions:

$$\widetilde{\nabla}\varphi = 0, \quad \widetilde{\nabla}\xi = 0, \quad \widetilde{\nabla}\eta = 0, \quad \widetilde{\nabla}g = 0.$$

Remark 7.1 (Generalized Tanaka-Webster connection). Let M be a contact metric manifold. Tanno introduced the following linear connection on M ([72]):

$$^T\nabla_X Y := \nabla_X Y + \eta(X)\varphi Y + \eta(Y)\varphi(I + h)X - g(\varphi(I + h)X, Y)\xi.$$

This linear connection is called the *generalized Tanaka-Webster connection*. In case, the associated almost CR-structure S is integrable, generalized Tanaka-Webster connection coincides with our canonical connection $\widetilde{\nabla}^t|_{t=-1}$. The generalized Tanaka-Webster connection does not coincide with $\widetilde{\nabla}^t|_{t=-1}$ if S is non-integrable. In fact, ξ, η and g are parallel with respect to $^T\nabla$ but for φ, $^T\nabla$ satisfies

$$(^T\nabla_X \varphi)Y = \mathcal{Q}(Y, X)$$

holds. Here $\mathcal{Q}$ is the Tanno tensor field. Hence we notice that on a contact metric manifold M, $^T\nabla = \widetilde{\nabla}^t|_{t=-1}$ if and only if its associated CR-structure is integrable.

7.2. Normal almost contact metric 3-manifolds. In this subsection we assume that M is a normal almost contact metric 3-manifold (or more generally, trans-Sasakian manifold of general dimension) of type (α, β). Then (7.1) is reduced to

$$(7.2) \qquad \begin{aligned} A_X^t Y &= \alpha\{g(X, \varphi Y)\xi + \eta(Y)\varphi X\} \\ &\quad + \beta\{g(X, Y)\xi - \eta(Y)X\} - t\eta(X)\varphi Y. \end{aligned}$$

The torsion tensor field $\mathfrak{T}^t$ of $\widetilde{\nabla}^t$ is given by

$$(7.3) \qquad \begin{aligned} \widetilde{\mathfrak{T}}^t(X, Y) &= \alpha\{2g(X, \varphi Y)\xi - \eta(X)\varphi Y + \eta(Y)\varphi X\} \\ &\quad + \eta(X)(\beta Y - t\varphi Y) - \eta(Y)(\beta X - t\varphi X). \end{aligned}$$

In particular, for a Kenmotsu manifold ($\alpha = 0$ and $\beta = 1$) the above equation reduce as follows:

$$(7.4) \qquad \begin{aligned} A_X^t Y &= -\eta(Y)X - t\eta(X)\varphi Y + g(X, Y)\xi, \\ \widetilde{\mathfrak{T}}^t(X, Y) &= \eta(X)(Y - t\varphi Y) - \eta(Y)(X - t\varphi X). \end{aligned}$$

For a Sasakian manifold ($\alpha = 1$ and $\beta = 0$) we get

$$(7.5) \qquad \begin{aligned} A_X^t Y &= g(X, \varphi Y)\xi + \eta(Y)\varphi X - t\eta(X)\varphi Y, \\ \widetilde{\mathfrak{T}}^t(X, Y) &= 2g(X, \varphi Y)\xi - (1 + t)\eta(X)\varphi Y + (1 + t)\eta(Y)\varphi X. \end{aligned}$$

On a Sasakian 3-manifolds, canonical connection $\widetilde{\nabla}^t$ coincides with the linear connection introduced by Okumura. In particular, $\widetilde{\nabla}^1$ is called the *Okumura connection* [55].

8. SLANT CURVES IN PSEUDO-HERMITIAN GEOMETRY

8.1. Let $\gamma = \gamma(s) : I \to M^3$ be a curve parameterized by arc-length in normal almost contact metric 3-manifold M^3. We may define the Frenet frame fields $\tilde{F} = (\widetilde{T}, \widetilde{N}, \widetilde{B})$ along γ with respect to the canonical connection $\widetilde{\nabla}^t$, since $\widetilde{\nabla}^t$ is a metrical connection. Then $\tilde{F}$ satisfies the following Frenet-Serret equations with respect to $\widetilde{\nabla}^t$:

$$(8.1) \qquad \begin{cases} \widetilde{\nabla}_{\gamma'}^t \widetilde{T} = & \widetilde{\kappa}\widetilde{N} \\ \widetilde{\nabla}_{\gamma'}^t \widetilde{N} = -\widetilde{\kappa}\widetilde{T} & + \widetilde{\tau}\widetilde{B} \\ \widetilde{\nabla}_{\gamma'}^t \widetilde{B} = & -\widetilde{\tau}\widetilde{N} \end{cases}$$

where $\widetilde{\kappa} = |\widetilde{\nabla}_{\gamma'}^t \widetilde{T}|$ is the *pseudo-Hermitian curvature* of γ and $\widetilde{\tau}$ its *pseudo-Hermitian torsion* for the canonical connection $\widetilde{\nabla}^t$. A non-geodesic curve γ is said to be a *pseudo-Hermitian circle* if $\widetilde{\kappa}$ is nonzero constant and $\widetilde{\tau} = 0$. A *pseudo-Hermitian helix* is a non-geodesic curve with nonzero constant pseudo-Hermitian curvature $\widetilde{\kappa}$ and pseudo-Hermitian torsion $\widetilde{\tau}$. The vector field $\widetilde{H} = \widetilde{\nabla}_{\gamma'}^t \gamma'$ is called the *pseudo-Hermitian mean curvature vector field* of γ.

8.2. Contact metric 3-manifolds. In [61], Özgür and Güvenç studied slant curves in contact metric 3-manifold in terms of Tanaka-Webster connection $\hat{\nabla} = \widetilde{\nabla}^t|_{t=-1}$. In this subsection we generalize results in [61] to slant curves in a contact metric 3-manifold M equipped with $\widetilde{\nabla}^t$.

We start with the following result which is a slight modification of [61].

Proposition 8.1 (*cf.* [20],[61])**.** *Let γ be a non-geodesic unit speed curve in a contact metric 3-manifold $(M, \widetilde{\nabla}^t)$. Then γ is a Legendre curve if and only if $\tilde{\tau} = 0$.*

Lemma 8.1. *Let γ be a curve in an almost contact metric 3-manifold M. Then*

$$(8.2) \qquad \widetilde{\nabla}^{t\perp}_{\gamma'} \widetilde{\nabla}^{t\perp}_{\gamma'} \gamma' = \widetilde{\kappa}' \widetilde{N} + \widetilde{\kappa}\widetilde{\tau}\widetilde{B},$$

$$(8.3) \qquad \widetilde{\nabla}^{t\perp}_{\gamma'} \widetilde{\nabla}^{t\perp}_{\gamma'} \widetilde{\nabla}^{t\perp}_{\gamma'} \gamma' = (\widetilde{\kappa}'' - \widetilde{\kappa}\widetilde{\tau}^2)\widetilde{N} + (2\widetilde{\kappa}'\widetilde{\tau} + \widetilde{\kappa}\widetilde{\tau}')\widetilde{B}.$$

In pseudo-Hermitian geometry, we introduce the following notion.

Definition 8.1. In a contact metric 3-manifold $(M^3, \widetilde{\nabla}^t)$, a vector field X normal to a unit speed curve γ is said to be *pseudo-Hermitian parallel* if $\widetilde{\nabla}^{t\perp}_{\dot\gamma} X = 0$.

By using the equation (8.2) we get

Proposition 8.2. *Let γ be a non-geodesic unit speed curve in a contact metric 3-manifold $(M, \widetilde{\nabla}^t)$. Then γ has pseudo-Hermitian parallel mean curvature vector field if and only if γ is a pseudo-Hermitian circle.*

8.3. Normal almost contact metric 3-manifolds. In this subsection, we assume that M is a *normal* almost contact metric 3-manifold.

Proposition 8.3 ([11])**.** *A Frenet curve γ is a slant curve in a normal almost contact metric 3-manifold M of type (α, β) if and only if γ satisfies*

$$(8.4) \qquad \eta(N) = -\frac{\beta}{\kappa} \sin^2 \theta.$$

If $\beta > 0$ then a necessary condition for γ to be slant is

$$(8.5) \qquad |\sin\theta| \le \min\left\{\sqrt{\frac{\kappa}{\beta}}, \frac{\kappa}{\beta}, 1\right\}.$$

We suppose that γ is a non-geodesic slant curve in a normal almost contact metric 3-manifold M of type (α, β). Then γ can not be an integral curve of ξ. We find the following orthonormal frame field in normal almost contact metric 3-manifold M along γ:

$$(8.6) \qquad \epsilon_1 = T = \gamma', \quad \epsilon_2 = \frac{\varphi\gamma'}{|\sin\theta|}, \quad \epsilon_3 = \frac{\xi - \cos\theta\gamma'}{|\sin\theta|}.$$

Note that $\xi = \cos\theta\epsilon_1 + |\sin\theta| \epsilon_3$.

Then for a slant curve γ in normal almost contact metric 3-manifold M we have

$$(8.7) \qquad \begin{cases} \nabla_{\gamma'}\epsilon_1 = \delta |\sin\theta| \epsilon_2 - \beta |\sin\theta| \epsilon_3, \\ \nabla_{\gamma'}\epsilon_2 = -\delta |\sin\theta| \epsilon_1 + (\alpha + \delta\cos\theta)\epsilon_3, \\ \nabla_{\gamma'}\epsilon_3 = \beta |\sin\theta| \epsilon_1 - (\alpha + \delta\cos\theta)\epsilon_2, \end{cases}$$

where $\delta = g(\nabla_{\gamma'}\gamma', \varphi\gamma')/\sin^2\theta$.

We define $\widetilde{T} = T$ then from the definition of $\widetilde{\nabla}^t$ we get

$$(8.8) \qquad \begin{aligned} \widetilde{\nabla}^t_{\gamma'}\widetilde{T} = \widetilde{\nabla}^t_{\gamma'}\gamma' &= \nabla_{\gamma'}\gamma' + |\sin\theta| ((\alpha - t)\cos\theta\epsilon_2 + \beta\epsilon_3) \\ &= |\sin\theta| (\delta + (\alpha - t)\cos\theta)\epsilon_2. \end{aligned}$$

Thus we can write

$$(8.9) \qquad \widetilde{\nabla}^t_{\gamma'}\widetilde{T} = \widetilde{\kappa}\widetilde{N},$$

with $\widetilde{\kappa} = |\sin\theta(\delta+(\alpha-t)\cos\theta)|$, $\widetilde{N} = \epsilon_2$, where $\widetilde{N}$ is a $\widetilde{\nabla}^t$-principal normal vector field.

Differentiation $\widetilde{N}$ along γ

$$(8.10) \qquad \widetilde{\nabla}^t_{\gamma'}\widetilde{N} = -\widetilde{\kappa}\widetilde{T} + \cos\theta(\delta + (\alpha - t)\cos\theta)\epsilon_3,$$

which implies $\widetilde{\tau} = \cos\theta(\delta+(\alpha-t)\cos\theta)$ and $\widetilde{B} = \epsilon_3$. The Frenet frame field satisfies

$$(8.11) \qquad \varphi\widetilde{T} = |\sin\theta|\widetilde{N}, \ \ \varphi\widetilde{N} = -|\sin\theta|\widetilde{T} + \cos\theta\widetilde{B}, \ \ \varphi\widetilde{B} = -\cos\theta\widetilde{N}.$$

From (8.8) we get

Proposition 8.4. *Let γ be a slant curve in a normal almost contact metric 3-manifold M of type (α, β). Then γ is $\widetilde{\nabla}^t$-geodesic if and only if γ satisfies*
$$\nabla_{\gamma'}\gamma' = \beta\cos\theta\gamma' - (\alpha - t)\cos\theta\varphi\gamma' - \beta\xi.$$

In particular, for an almost Legendre curve we have ([40])

Proposition 8.5. *Let γ be an almost Legendre curve in a normal almost contact metric 3-manifold M of type (α, β). Then γ is $\widetilde{\nabla}^t$-geodesic if and only if γ satisfies $\nabla_{\gamma'}\gamma' = -\beta\xi$.*

Corollary 8.1. *Let γ be an almost Legendre curve in a quasi-Sasakian 3-manifold M. Then γ is $\widetilde{\nabla}^t$-geodesic if and only if γ it is an almost Legendre geodesic with respect to the Levi-Civita connection.*

9. Bianchi-Cartan-Vranceanu spaces

Let μ be a real number and set
$$\mathcal{D} = \left\{(x, y, z) \in \mathbb{R}^3(x, y, z) \mid 1 + \mu(x^2 + y^2) > 0 \right\}.$$

Note that $\mathcal{D}$ is the whole $\mathbb{R}^3(x, y, z)$ for $\mu \geq 0$. On the region $\mathcal{D}$, we equip the following Riemannian metric:

$$(9.1) \qquad g_{\lambda,\mu} = \frac{dx^2 + dy^2}{\{1 + \mu(x^2 + y^2)\}^2} + \left(dz + \frac{\lambda}{2}\frac{ydx - xdy}{1 + \mu(x^2 + y^2)}\right)^2.$$

Take the following orthonormal frame field on $(\mathcal{D}, g_c)$:

$$\begin{aligned}
u_1 &= \{1 + \mu(x^2 + y^2)\}\frac{\partial}{\partial x} - \frac{\lambda y}{2}\frac{\partial}{\partial z}, \\
u_2 &= \{1 + \mu(x^2 + y^2)\}\frac{\partial}{\partial y} + \frac{\lambda x}{2}\frac{\partial}{\partial z}, \\
u_3 &= \frac{\partial}{\partial z}.
\end{aligned}$$

Then the Levi-Civita connection ∇ of this Riemannian 3-manifold is described as

$$\nabla_{u_1}u_1 = 2\mu\, yu_2, \ \ \nabla_{u_1}u_2 = -2\mu\, yu_1 + \frac{\lambda}{2}u_3, \ \ \nabla_{u_1}u_3 = -\frac{\lambda}{2}u_2,$$

$$(9.2) \qquad \nabla_{u_2}u_1 = -2\mu\, xu_2 - \frac{\lambda}{2}u_3, \ \ \nabla_{u_2}u_2 = 2\mu\, xu_1, \ \ \nabla_{u_2}u_3 = \frac{\lambda}{2}u_1,$$

$$\nabla_{u_3}u_1 = -\frac{\lambda}{2}u_2, \ \ \nabla_{u_3}u_2 = \frac{\lambda}{2}u_1, \ \ \nabla_{u_3}u_3 = 0.$$

$$(9.3) \qquad [u_1, u_2] = -2\mu\, yu_1 + 2\mu\, xu_2 + \lambda u_3, \ \ [u_2, u_3] = [u_3, u_1] = 0.$$

Define the endomorphism field φ by

$$\varphi u_1 = u_2, \ \varphi u_2 = -u_1, \ \varphi u_3 = 0.$$

The dual one-form η of the vector field $\xi = u_3$ is a contact form on $\mathcal{D}$ and satisfies

$$d\eta(X, Y) = \frac{\lambda}{2} g(X, \varphi Y), \ \ X, Y \in \mathfrak{X}(\mathcal{D}).$$

Moreover the structure (φ, ξ, η, g) is a normal almost contact metric 3-manifold of type $(\lambda/2, 0)$. The normal almost contact metric 3-manifold $(\mathcal{D}, g_c)$ is of constant holomorphic sectional curvature $\mathcal{H} = 4\mu - 3\lambda^2/4$. (*cf.* [5], [70]). In particular, if we choose $\lambda = 2$, then $\mathcal{M}^3(2, \mu)$ is a Sasakian manifold of constant holomorphic sectional curvature $\mathcal{H} = 4\mu - 3$.

Hereafter we denote this Riemannian 3-manifold $(\mathcal{D}, g_{\lambda,\mu})$ by $\mathcal{M}^3(\lambda, \mu)$. The 2-parameter family of Riemannian 3-manifolds $\{\mathcal{M}^3(\lambda, \mu) \,|\, \lambda, \mu \in \mathbb{R}\}$ is classically known by L. Bianchi [6], E. Cartan [15] and G. Vranceanu [76] (See also Kobayashi [48]). The Riemannian manifolds $\mathcal{M}^3(\lambda, \mu)$ are called the *Bianchi-Cartan-Vranceanu spaces*. This 2-parameter family includes all the Riemannian metric with 4 or 6-dimensional isometry group other than constant negative curvature metrics. More precisely $\mathcal{M}^3(\lambda, \mu)$ is (locally) isometric to one of the following spaces.

- $\mu = \lambda = 0$: Euclidean 3-space $\mathbb{R}^3$,
- $\mu = 0, \ \lambda \neq 0$: The Heisenberg group Nil,
- $\mu > 0, \ \lambda \neq 0$: The special unitary group SU(2),
- $\mu < 0, \ \lambda \neq 0$: The universal covering of $\mathrm{SL}_2\mathbb{R}$,
- $\mu > 0, \ \lambda = 0$: Product space $\mathbb{S}^2(4\mu) \times \mathbb{R}$,
- $\mu < 0, \ \lambda = 0$: Product space $\mathbb{H}^2(4\mu) \times \mathbb{R}$,
- $4\mu = \lambda^2$: The 3-sphere $\mathbb{S}^3(\mu)$ of curvature μ.

9.1. From (7.5), the canonical connection $\widetilde{\nabla}^t$ of the Bianchi-Cartan-Vranceanu space is described as

$$\widetilde{\nabla}_{u_1} u_1 = 2\mu \, y u_2, \ \ \widetilde{\nabla}_{u_1} u_2 = -2\mu \, y u_1, \ \ \widetilde{\nabla}_{u_1} u_3 = 0,$$

(9.4) $$\widetilde{\nabla}_{u_2} u_1 = -2\mu \, x u_2, \ \ \widetilde{\nabla}_{u_2} u_2 = 2\mu \, x u_1, \ \ \widetilde{\nabla}_{u_2} u_3 = 0,$$

$$\widetilde{\nabla}_{u_3} u_1 = -(t + \lambda/2) u_2, \ \ \widetilde{\nabla}_{u_3} u_2 = (t + \lambda/2) u_1, \ \ \widetilde{\nabla}_{u_3} u_3 = 0.$$

By using the above data, we calculate the curvature tensor $\widetilde{R}(X, Y) = [\widetilde{\nabla}_X, \widetilde{\nabla}_Y] - \widetilde{\nabla}_{[X,Y]}$. Then we find that

(9.5) $$\widetilde{R}^t(u_1, u_2) u_1 \ = \ \left\{ -4\mu + \lambda(t + \frac{\lambda}{2}) \right\} u_2,$$

(9.6) $$\widetilde{R}(u_1, u_2) u_2 \ = \ -\left\{ -4\mu + \lambda(t + \frac{\lambda}{2}) \right\} u_1,$$

all others are zero. Hence the holomorphic sectional curvature $\widetilde{\mathcal{H}}^t$ with respect to $\widetilde{\nabla}^t$ is given by

$$\widetilde{\mathcal{H}}^t = g(\widetilde{R}^t(u_1, u_2) u_2, u_1) = 4\mu - \lambda(t + \frac{\lambda}{2}).$$

9.2. Now we compute the biharmonicity equation:

$$(9.7) \qquad \begin{cases} \widetilde{H} = \widetilde{\nabla}^t_{\gamma'} \gamma', \\ \widetilde{\nabla}^t_{\gamma'} \widetilde{\nabla}^t_{\gamma'} \widetilde{H} + \widetilde{\nabla}^t_{\gamma'} \widetilde{\mathfrak{T}}(\widetilde{H}, \gamma') + \widetilde{R}^t(\widetilde{H}, \gamma')\gamma' = 0. \end{cases}$$

of a *slant curve* γ in a normal almost contact metric 3-manifold with respect to the canonical connection $\widetilde{\nabla}^t$.

First, the pseudo-Hermitain mean curvature vector field $\widetilde{H}$ is given by $\widetilde{H} = \tilde{\kappa}\widetilde{N}$. By using (8.1), (8.11) and (9.5) we calculate

$$(9.8) \qquad \widetilde{\nabla}^t_{\gamma'} \widetilde{\nabla}^t_{\gamma'} \widetilde{H} + \widetilde{R}^t(\widetilde{H}, \gamma')\gamma'$$
$$= -3\tilde{\kappa}\tilde{\kappa}'\widetilde{T} + \left\{ \tilde{\kappa}'' - \tilde{\kappa}\left(\tilde{\kappa}^2 + \tilde{\tau}^2 - \widetilde{\mathcal{H}}^t \sin^2\theta \right) \right\}\widetilde{N} + (2\tilde{\kappa}'\tilde{\tau} + \tilde{\kappa}\tilde{\tau}')\widetilde{B}.$$

We compute the torsion terms of the biharmonic equation. By using (7.3), we get the following formulas:

$$\widetilde{\mathfrak{T}}^t(\widetilde{H}, \gamma') = \tilde{\kappa}\,\widetilde{\mathfrak{T}}^t(\widetilde{N}, \widetilde{T}),$$
$$\widetilde{\mathfrak{T}}^t(\widetilde{N}, \widetilde{T}) = 2\alpha g(\widetilde{N}, \varphi\widetilde{T})\xi - \cos\theta\beta\widetilde{N} + (\alpha + t)\cos\theta\,\varphi\widetilde{T}.$$

Next, by using (8.11), we have

$$\widetilde{\mathfrak{T}}^t(\widetilde{N}, \widetilde{T}) = -(\alpha + t)\cos\theta\,|\sin\theta|\,\widetilde{T} + (2\alpha|\sin\theta| - \beta\cos\theta)\widetilde{N} + (\alpha + t)\cos^2\theta\,\widetilde{B}.$$

Thus we obtain

$$\mathfrak{T}^t(\widetilde{N}, \widetilde{T}) + \tilde{\kappa}\,\widetilde{\nabla}^t_{\gamma'}\widetilde{\mathfrak{T}}^t(\widetilde{N}, \widetilde{T})$$
$$= -(\alpha + t)\cos\theta\,|\sin\theta|\,\tilde{\kappa}'\,\widetilde{T} + (2\alpha|\sin\theta| - \beta\cos\theta)\,\tilde{\kappa}'\,\widetilde{N} + (\alpha + t)\cos^2\theta\,\tilde{\kappa}'\,\widetilde{B}$$
$$\quad + \kappa\left\{-(\alpha + t)\cos\theta\,|\sin\theta|\right\}\,(\tilde{\kappa}\,\widetilde{N}) + (2\alpha|\sin\theta| - \beta\cos\theta)(-\tilde{\kappa}\widetilde{T} + \tilde{\tau}\widetilde{B})$$
$$\quad + (\alpha + t)\cos^2\theta\,(-\tilde{\tau}\,\widetilde{N})$$
$$= \cos\theta\{(\alpha - t)|\sin\theta|\tilde{\kappa}' + \beta\tilde{\kappa}^2\}\widetilde{T}$$
$$\quad + \{-\beta\tilde{\kappa}' + |\sin\theta|\cos\theta(\alpha - t)\tilde{\kappa}^2 - (2\alpha\sin^2\theta + (\alpha + t)\cos^2\theta)\tilde{\kappa}\tilde{\tau}\}\widetilde{N}$$
$$\quad + \{(2\alpha\sin^2\theta + (\alpha + t)\cos^2\theta)\tilde{\kappa}' - \cos\theta\beta\tilde{\kappa}\tilde{\tau}\}\widetilde{B}.$$

Using the relation

$$2\alpha\sin^2\theta + (\alpha + t)\cos^2\theta = 2\alpha - (\alpha - t)\cos^2\theta,$$

we get

$$\widetilde{\nabla}^t_{\gamma'}\widetilde{\mathfrak{T}}^t(\widetilde{H}, \gamma')$$
$$= \cos\theta\{(\alpha - t)|\sin\theta|\tilde{\kappa}' + \beta\tilde{\kappa}^2\}\widetilde{T}$$
$$\quad + [-\cos\theta\{\beta\tilde{\kappa}' - (\alpha - t)\tilde{\kappa}(|\sin\theta|\,\tilde{\kappa} + \cos\theta\,\tilde{\tau})\} - 2\alpha\tilde{\kappa}\tilde{\tau}]\widetilde{N}$$
$$\quad + [2\alpha\,\tilde{\kappa}' - \cos\theta\{(\alpha - t)\cos\theta\,\tilde{\kappa}' + \beta\,\tilde{\kappa}\tilde{\tau}\}]\widetilde{B}.$$

Since γ is slant, we have $\cos\theta\tilde{\kappa} = |\sin\theta|\tilde{\tau}$, by using this relation

$$2\alpha\tilde{\kappa}\tilde{\tau} = 2\alpha(\cos^2\theta + \sin^2\theta)\tilde{\kappa}\tilde{\tau} = 2\alpha\cos\theta\tilde{\kappa}(|\sin\theta|\tilde{\kappa} + \cos\theta\tilde{\tau}).$$

Hence we get

$$
\begin{aligned}
\widetilde{\nabla}^t_{\gamma'}\widetilde{\mathfrak{T}}(\widetilde{\tau}^t(\gamma),\gamma') = {} & \cos\theta\{|\sin\theta\,|\,(\alpha-t)\widetilde{\kappa}' + \beta\widetilde{\kappa}^2\}\widetilde{T} \\
& - \cos\theta\{\beta\widetilde{\kappa}' + (\alpha-t)\widetilde{\kappa}(\widetilde{\kappa}\,|\sin\theta\,| + \widetilde{\tau}\cos\theta)\}\widetilde{N} \\
& + \{2\alpha\widetilde{\kappa}' - \cos\theta((\alpha-t)\widetilde{\kappa}'\cos\theta + \beta\widetilde{\kappa}\widetilde{\tau})\}\widetilde{B}.
\end{aligned}
$$

Proposition 9.1. *Let γ be a slant curve in a normal almost contact metric 3-manifold. Then γ is $\widetilde{\nabla}^t$-biharmonic if and only if its curvature and torsion satisfies the following system of ODE's:*

$$(9.9) \qquad -3\widetilde{\kappa}\widetilde{\kappa}' + \cos\theta\{(\alpha-t)|\sin\theta|\widetilde{\kappa}' + \beta\widetilde{\kappa}^2\} = 0,$$

$$(9.10) \quad \widetilde{\kappa}'' - \widetilde{\kappa}(\widetilde{\kappa}^2 + \widetilde{\tau}^2 - \widetilde{\mathcal{H}}^t\sin^2\theta) - \cos\theta\{\beta\widetilde{\kappa}' + (\alpha+t)\widetilde{\kappa}(|\sin\theta|\widetilde{\kappa} + \cos\theta\widetilde{\tau})\} = 0,$$

$$(9.11) \qquad 2\widetilde{\kappa}'\widetilde{\tau} + \widetilde{\kappa}\widetilde{\tau}' + 2\alpha\widetilde{\kappa}' - \cos\theta\{(\alpha-t)\cos\theta(\widetilde{\kappa}' + \beta\widetilde{\kappa}\widetilde{\tau})\} = 0.$$

Now let us consider non-geodesic slant curves in the Bianchi-Cartan-Vranceanu space. Since $\alpha = \frac{\lambda}{2}$ and $\beta = 0$, (9.9) reduces to

$$\widetilde{\kappa}' = 0 \ \ \text{or} \ \ \widetilde{\kappa} = \frac{1}{3}(\lambda/2 - t)\cos\theta|\sin\theta|.$$

In both cases, $\widetilde{\kappa}$ is a nonzero constant. Then (9.11) implies that $\widetilde{\tau}$ is constant. The equation (9.10) reduces to

$$\widetilde{\kappa}^2 + \widetilde{\tau}^2 - \widetilde{\mathcal{H}}^t\sin^2\theta + (\lambda/2 + t)\cos\theta(|\sin\theta|\widetilde{\kappa} + \cos\theta\widetilde{\tau}) = 0.$$

Using the slant condition again, this equation is rewritten as

$$(9.12) \qquad \widetilde{\kappa}^2 + \widetilde{\tau}^2 = \widetilde{\mathcal{H}}^t\sin^2\theta - (\lambda/2 + t)\widetilde{\tau}.$$

Remark 9.1. From (8.10), we have

$$\widetilde{\tau} = \cos\theta(\delta + (\lambda/2 - t)\cos\theta)$$

with $\delta = g(\nabla_{\gamma'}\gamma', \varphi\gamma')/\sin^2\theta$. Thus (9.12) can be rewritten as

$$\widetilde{\kappa}^2 + \widetilde{\tau}^2 = \widetilde{\mathcal{H}}^t - (\lambda/2 + t)\cos\theta\,(\delta + (\lambda/2 - t)\cos\theta).$$

By using the slant condition $\cos\theta\widetilde{\kappa} = |\sin\theta|\widetilde{\tau}$ again, (9.12) can be rewritten as

$$(9.13) \qquad \widetilde{\kappa}^2 - (\lambda/2 + t)\cos\theta|\sin\theta|\widetilde{\kappa} - \widetilde{\mathcal{H}}^t\sin^4\theta = 0.$$

Thus $\widetilde{\kappa}$ is a positive solution to this equation. The discriminant of this equation is given by

$$
\begin{aligned}
\mathcal{D} &= \sin^2\theta\{(\lambda/2 - t)^2\cos^2\theta + 4\widetilde{\mathcal{H}}^t\sin^2\theta\} \\
&= \sin^2\theta\{(\lambda/2 + t)^2\cos^2\theta - 4\lambda(\lambda/2 + t) + 16\mu\}.
\end{aligned}
$$

Thus if $(\lambda/2 + t)^2\cos^2\theta - 4\lambda(\lambda/2 + t) + 16\mu \geq 0$, $\widetilde{\kappa}$ is a positive function which has the form:

$$\widetilde{\kappa} = \frac{1}{2}|\sin\theta|\left\{\cos\theta(\lambda/2 + t) \pm \sqrt{(\lambda/2 + t)^2\cos^2\theta + 4\widetilde{\mathcal{H}}^t}\right\}.$$

Now we consider the case

$$\widetilde{\kappa} = \frac{1}{3}(\lambda/2 - t)\cos\theta|\sin\theta|.$$

Inserting this into (9.13), we have

$$(9.14) \qquad \sin^2\theta\{2\cos^2\theta(\lambda/2 - t)^2 + 9\widetilde{\mathcal{H}}^4\sin^2\theta\} = 0.$$

Theorem 9.1. *If γ is a slant curve in Bianchi-Cartan-Vranceanu space, then γ is $\widetilde{\nabla}^t$-biharmonic if and only if γ is $\widetilde{\nabla}^t$-geodesic or a $\widetilde{\nabla}^t$-helix satisfying (9.12).*

If we choose $t = \lambda/2$ in (9.13), we have

$$\widetilde{\kappa} = \widetilde{\mathcal{H}}^t\sin^4\theta = 4\mu\sin^4\theta.$$

On the other hand, if $t = \lambda/2$ in (9.14), we get $\widetilde{\kappa} = 0$. In addition, $\sin\theta = 0$ or $\widetilde{\mathcal{H}} = 0$. Thus we obtain

Corollary 9.1. *Let γ be a slant curve in a Sasakian space form $\mathcal{M}^3(2, \mu)$ of constant holomorphic sectional curvature $4\mu - 3$. Then γ is biharmonic with respect to Tanaka-Webster connection $\hat{\nabla} = \widetilde{\nabla}^t|_{t=-1}$ if and only if it is a geodesic with respect to $\hat{\nabla}$ or a $\hat{\nabla}$-helix whose pseudo-Hermitian curvature and torsion are given by*

$$\widetilde{\kappa} = 2\sqrt{\mu}\sin^2\theta, \ \widetilde{\tau} = 2\sqrt{\mu}\cos\theta.$$

In case $\mu \leq 0$, $\hat{\nabla}$-biharmonic slant curves are slant $\hat{\nabla}$-geodesics.

Corollary 9.2. *Let γ be a slant curve in a Bianchi-Cartan-Vranceanu space $\mathcal{M}^3(0, \mu)$. Then γ is biharmonic with respect to $\widetilde{\nabla}^0$ if and only if it is a geodesic with respect to $\widetilde{\nabla}^0$-geodesic or a $\widetilde{\nabla}^0$-helix with*

$$\widetilde{\kappa} = 2\sqrt{\mu}\sin^2\theta, \ \widetilde{\tau} = 2\sqrt{\mu}\cos\theta.$$

In case $\mu \leq 0$, $\hat{\nabla}$-biharmonic slant curves are slant $\hat{\nabla}$-geodesics.

In particular, if γ is an almost Legendre curve, then we have

Theorem 9.2 ([40])**.** *If γ is an almost contact curve in Bianchi-Cartan-Vranceanu space, then γ is $\widetilde{\nabla}^t$-biharmonic if and only if γ is a geodesic or a $\widetilde{\nabla}^t$-circle satisfying $\widetilde{\kappa}^2 = \widetilde{\mathcal{H}}^t$.*

For explicit parametrizations of $\widetilde{\nabla}^t$-biharmonic almost Legendre curves in the Bianchi-Cartan-Vranceanu space, we refer to [40].

9.3. In [52], the second named author introduced the notion of "$AW(k)$-type" for Frenet curves in 3-dimensional strongly pseudo-convex CR-manifolds equipped with Tanaka-Webster connection. Moreover Özgür and Güvenç [61] studied slant curves of $AW(k)$-type in 3-dimensional strongly pseudo-convex CR-manifolds.

Definition 9.1. A Frenet curve $\gamma(s)$ in an almost contact metric 3-manifold $(M, \widetilde{\nabla}^t)$ equipped with $\widetilde{\nabla}^t$ is said to be of type:

(1) $AW(1)$ if $(\widetilde{\nabla}^t_{\gamma'}\widetilde{\nabla}^t_{\gamma'}\widetilde{\nabla}^t_{\gamma'}\gamma')^\perp = 0$,
(2) $AW(2)$ if $(\widetilde{\nabla}^t_{\gamma'}\widetilde{\nabla}^t_{\gamma'}\widetilde{\nabla}^t_{\gamma'}\gamma')^\perp$ is parallel to $(\widetilde{\nabla}^t_{\gamma'}\widetilde{\nabla}^t_{\gamma'}\gamma')^\perp$,
(3) $AW(3)$ if $(\widetilde{\nabla}^t_{\gamma'}\widetilde{\nabla}^t_{\gamma'}\widetilde{\nabla}^t_{\gamma'}\gamma')^\perp$ is parallel to $(\widetilde{\nabla}^t_{\gamma'}\gamma')^\perp$.

The following results were obtained.

Proposition 9.2 ([41])**.** *Let $\gamma(s)$ be a non-geodesic slant curve in an almost contact metric 3-manifold M equipped with canonical connection $\widetilde{\nabla} = \widetilde{\nabla}^t$. Then γ is of type $AW(1)$ with respect to $\widetilde{\nabla}$ if and only if it is an almost Legendre curve whose $\widetilde{\nabla}$-curvature is one of the following natural equations:*

(1)
$$\widetilde{\kappa}(s) = \mp\frac{\sqrt{2}}{s+c},$$

(2)
$$\widetilde{\kappa}(s) = \pm\sqrt{a}\left(\frac{1+\operatorname{cn}(\nu(s);1/\sqrt{2})}{1-\operatorname{cn}(\nu(s);1/\sqrt{2})}\right)^{\frac{1}{2}},$$

with $\nu(s) = \mp\sqrt{2a}(s+c) + 2K(1/\sqrt{2})$, or

(3)
$$\widetilde{\kappa}(s) = \frac{\sqrt{a}}{\operatorname{cn}\left(\sqrt{a}(s+c);1/\sqrt{2}\right)}.$$

Here a is a positive constant and $K(k)$ is the complete elliptic integral of the first kind defined by

$$K(1/\sqrt{2}) = \int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\theta}{\sqrt{1-\frac{1}{2}\sin^2\theta}}.$$

Proposition 9.3 ([41]). *Let γ be a non-geodesic slant curve in an almost contact metric 3-manifold. Then γ is of type $AW(2)$ with respect to the canonical connection $\widetilde{\nabla}$ if and only if it has $\widetilde{\nabla}$-torsion*

$$\widetilde{\tau} = \frac{\cos\theta}{\sqrt{-s^2 + as + b}}.$$

In case, γ satisfies $\cos\theta \neq 0$, then γ has $\widetilde{\nabla}$-curvature

$$\widetilde{\kappa} = \frac{|\sin\theta|}{\sqrt{-s^2 + as + b}}.$$

Proposition 9.4 ([41]). *Let γ be a non-geodesic slant curve in an almost contact metric 3-manifold. Then γ is of type $AW(3)$ with respect to the canonical connection $\widetilde{\nabla}$ if and only if it has constant $\widetilde{\nabla}$-torsion.*

10. SOLVABLE LIE GROUPS

10.1. In this section we study the following two-parameter family $\{G(c_1, c_2) \mid (c_1, c_2) \in \mathbb{R}^2\}$ of Lie groups:

$$G(c_1, c_2) := \left\{\begin{pmatrix} 1 & 0 & 0 & z \\ 0 & e^{c_1 z} & 0 & x \\ 0 & 0 & e^{c_2 z} & y \\ 0 & 0 & 0 & 1 \end{pmatrix} \middle| x, y, z \in \mathbb{R}\right\} \subset \mathrm{GL}_4\mathbb{R}$$

equipped with the left invariant metric

$$g = e^{-2c_1 z}\mathrm{d}x^2 + e^{-2c_2 z}\mathrm{d}y^2 + \mathrm{d}z^2.$$

The group operation of $G(c_1, c_2)$ is given explicitly by

(10.1) $$(x, y, z) * (\tilde{x}, \tilde{y}, \tilde{z}) = (x + e^{c_1 z}\tilde{x}, y + e^{c_2 z}\tilde{y}, z + \tilde{z}).$$

Remark 10.1. One can see that $G(c_1, c_2)$ is a Lie subgroup of the affine transformation group $A(3) := GL_3\mathbb{R} \ltimes \mathbb{R}^3 \subset GL_4\mathbb{R}$. Moreover, if $(c_1, c_2) \neq (0, 0)$, then $G(c_1, c_2)$ is isomorphic to the following Lie subgroup

$$G(c_1, c_2) := \left\{ \begin{pmatrix} e^{c_1 z} & 0 & x \\ 0 & e^{c_2 z} & y \\ 0 & 0 & 1 \end{pmatrix} \middle| x, y, z \in \mathbb{R} \right\} \subset GL_3\mathbb{R}$$

of the affine transformation group $A(2) = GL_2\mathbb{R} \ltimes \mathbb{R}^2$. This 2-parameter family of homogeneous spaces can be seen in [75]. Minimal surfaces in $G(c_1, c_2)$ have been studied in [34, 36, 43].

The Lie algebra $\mathfrak{g}(c_1, c_2)$ of $G(c_1, c_2)$ is

$$\left\{ \begin{pmatrix} 0 & 0 & 0 & w \\ 0 & c_1 w & 0 & u \\ 0 & 0 & c_2 w & v \\ 0 & 0 & 0 & 0 \end{pmatrix} \middle| u, v, w \in \mathbb{R} \right\}.$$

Take an orthonormal basis

$$E_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & c_1 & 0 & 0 \\ 0 & 0 & c_2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

We denote by e_i the left invariant vector field on $G(c_1, c_2)$ which is obtained by left translation of E_i. Then we have

$$(10.2) \qquad e_1 = e^{c_1 z} \frac{\partial}{\partial x}, \quad e_2 = e^{c_2 z} \frac{\partial}{\partial y}, \quad e_3 = \frac{\partial}{\partial z},$$

$$(10.3) \qquad [e_1, e_2] = 0, \quad [e_2, e_3] = -c_2 \, e_2, \quad [e_3, e_1] = c_1 \, e_1.$$

Hence $G(c_1, c_2)$ is *solvable*. Indeed, the derived series $\{\mathfrak{D}^i\}$ of $\mathfrak{g}(c_1, c_2)$ is

$$\mathfrak{D}^1 = \begin{cases} \mathbb{R}E_1 \oplus \mathbb{R}E_2, & c_1 \neq 0, \ c_2 \neq 0, \\ \mathbb{R}E_1, & c_1 \neq 0, \ c_2 = 0, \\ \mathbb{R}E_2, & c_1 = 0, \ c_2 \neq 0, \\ \{\mathbf{0}\}, & c_1 = c_2 = 0, \end{cases} \qquad \mathfrak{D}^2 = \{\mathbf{0}\}.$$

Remark 10.2. The *derived series* $\{\mathfrak{D}^i\}$ of a real Lie algebra $\mathfrak{g}$ is a decreasing sequence of ideals defined by

$$\mathfrak{D}^0 = \mathfrak{g}, \quad \mathfrak{D}^{i+1} = [\mathfrak{D}^i, \mathfrak{D}^i].$$

A Lie algebra $\mathfrak{g}$ is said to be *solvable* if $\mathfrak{D}^i = \{\mathbf{0}\}$ for some $i > 0$.

The Levi-Civita connection ∇ of $G(c_1, c_2)$ is described as

$$\nabla_{e_1} e_1 = c_1 \, e_3, \quad \nabla_{e_1} e_2 = 0, \quad \nabla_{e_1} e_3 = -c_1 \, e_1,$$

$$(10.4) \qquad \nabla_{e_2} e_1 = 0, \quad \nabla_{e_2} e_2 = c_2 \, e_3, \quad \nabla_{e_2} e_3 = -c_2 \, e_2,$$

$$\nabla_{e_3} e_1 = 0, \quad \nabla_{e_3} e_2 = 0, \quad \nabla_{e_3} e_3 = 0.$$

These formulas show that the Lie algebra $\mathfrak{g}(c_1, c_2)$ is unimodular when and only when $c_1 + c_2 = 0$.

The Riemannian curvature tensor R of $G(c_1, c_2)$. is given by

$$(10.5) \quad \begin{aligned} R(e_1, e_2)e_1 &= c_1 c_2 e_2, & R(e_1, e_2)e_2 &= -c_1 c_2 e_1, \\ R(e_2, e_3)e_2 &= c_2^2 e_3, & R(e_2, e_3)e_3 &= -c_2^2 e_2, \\ R(e_3, e_1)e_3 &= c_1^2 e_1, & R(e_3, e_1)e_1 &= -c_1^2 e_3, \end{aligned}$$

the others are zero.

Define the endomorphism field φ by

$$\varphi e_1 = e_2, \ \ \varphi e_2 = -e_1, \ \ \varphi e_3 = 0.$$

Next we put $\xi = e_3$ and $\eta = \mathrm{d}z$. Then (φ, ξ, η, g) is an almost contact metric structure on $G(c_1, c_2)$ which is left invariant.

The holomorphic sectional curvature $\mathcal{H}$ with respect to the Levi-Civita connection ∇ is given by

$$\mathcal{H} = g(R(e_1, e_2)e_2, e_1) = -c_1 c_2.$$

One can check that $(G(c_1, c_2), \varphi, \xi, \eta, g)$ satisfies

$$\mathrm{d}\eta = 0, \ \ \mathrm{d}\Phi = 2\beta\eta \wedge \Phi, \ \ \beta = -\frac{1}{2}(c_1 + c_2).$$

Thus for $c_1 + c_2 \neq 0$, $G(c_1, c_2)$ is almost β-Kenmotsu. In case $c_1 + c_2 = 0$, $G(c_1, c_2)$ is almost cosymplectic.

The operator $h = \pounds_\xi \varphi / 2$ is computed as

$$he_1 = \frac{1}{2}(c_2 - c_1)e_2, \ \ he_2 = \frac{1}{2}(c_2 - c_1)e_1, \ \ he_3 = 0.$$

Here we introduce the operator $\hat{h}$ by $\hat{h} = h \circ \varphi$. Then we get

$$\hat{h}e_1 = \frac{c_2 - c_1}{2}e_1, \ \ \hat{h}e_2 = -\frac{c_2 - c_1}{2}e_2, \ \ \hat{h}e_3 = 0.$$

One can check the following fact.

Proposition 10.1. *The Lie group $G(c_1, c_2)$ is normal if and only if $\hat{h} = 0$, that is, $c_1 = c_2$.*

The covariant derivative $\nabla\xi$ is computed as

$$(10.6) \qquad \nabla_X \xi = \beta(X - \eta(X)\xi) + \hat{h}X = -\frac{1}{2}(c_1 + c_2)(X - \eta(X)\xi) + h\varphi X.$$

From this equation we have

$$\varphi\nabla_X \xi = \beta\varphi X + \varphi\hat{h}X.$$

Inserting this into (2.1), we get

$$(10.7) \qquad (\nabla_X \varphi)Y = g(\varphi(\beta I + \hat{h})X, Y)\xi - \eta(Y)\varphi(\beta I + \hat{h})X.$$

In case $c_1 + c_2 \neq 0$, we can introduce a new operator h' by

$$h' := -\frac{2}{c_1 + c_2}h \circ \varphi.$$

The covariant derivatives $\nabla\xi$ and $\nabla\varphi$ are rewritten as

$$\nabla_X \xi = \beta\{(I + h')X - \eta(X)\xi\},$$
$$(\nabla_X \varphi)Y = \beta\{g(\varphi(I + h')X, Y)\xi - \eta(Y)(I + h')X\}.$$

In case $\beta = -(c_1 + c_2)/2 = 0$, we have

$$\nabla_X \xi = \hat{h}X, \quad (\nabla_X \varphi)Y = g(\varphi\hat{h}X, Y)\xi - \eta(Y)\varphi\hat{h}X.$$

Example 10.1 ($G(0,0)$). The Lie group $G(0,0)$ is the Euclidean 3-space $(\mathbb{R}^3, +)$ equipped with usual abelian group structure. In this case the almost contact metric structure is cosymplectic.

Example 10.2 ($G(c,c)$). For $c_1 = c_2 = c \neq 0$, then $G(c,c)$ is the warped product model $\mathbb{R}(z) \times_{e^{-cz}} \mathbb{E}^2(x, y)$;

$$(\mathbb{R}^3(x, y, z), e^{-2cz}(\mathrm{d}x^2 + \mathrm{d}y^2) + \mathrm{d}z^2)$$

of the hyperbolic 3-space $\mathbb{H}^3(-c^2)$ of constant curvature $-c^2$. The almost contact metric structure satisfies

$$(\nabla_X \varphi)Y = -c(g(\varphi X, Y)\xi - \eta(Y)\varphi X)$$

for all vector fields X and Y on $G(c,c)$. Hence $G(c,c)$ is a $(-c)$-Kenmotsu manifold. In particular, hyperbolic 3-space $\mathbb{H}^3(-1) = G(-1,-1)$ is a *Kenmotsu manifold*. Note that hyperbolic 3-space does not admit contact metric structure. In addition, hyperbolic 3-space does not admit any other Lie group structure.

Example 10.3 ($c_1 c_2 = 0$). Choose $c_1 = 0$ and $c_2 = c \neq 0$. Then $G(0,c)$ is the Riemannian product of $\mathbb{R}(x)$ and the warped product model $\mathbb{R}(z) \times_{e^{-cz}} \mathbb{R}(y)$ of the hyperbolic plane $\mathbb{H}^2(-c^2)$. The structure is non-normal almost $(-c/2)$-Kenmotsu.

Example 10.4 ($G(c,-c)$). Choose $c_1 = -c_2 = c \neq 0$. Then $G(c,-c)$ is isomorphic to the identity component $SE(1,1)$ of the isometry group of the Minkowski plane $(\mathbb{R}^2(u,v), \mathrm{d}u\mathrm{d}v)$. In this case the almost contact metric structure is *almost cosymplectic* but not cosymplectic. This group provides an example of almost cosymplectic manifold that is not a (locally) Riemannian product. The Lie group $G(1,-1)$ is referred as the model space Sol_3 of solvgeometry in the sense of Thurston [74]. The almost cosymplectic structure on $G(c,-c)$ was investigated by Olszak [58].

Remark 10.3. Let $(M, \varphi, \xi, \eta, g)$ be an almost contact metric manifold. Then M satisfies the (κ, μ)-*nullity condition* provided there exist constants κ and μ such that

$$R(X, Y)\xi = (\kappa I + \mu h)(X \wedge Y)\xi$$

for all $X, Y \in \mathfrak{X}(M)$. In case M is almost β-Kenmotsu manifold, then $\kappa = -\beta^2$ and $h = 0$. In particular, if $\dim M = 3$, M is a β-Kenmotsu manifold. Note that $(\kappa, 0)$-nullity condition for almost cosymplectic manifolds was studied by Dacko [21]. Instead of (κ, μ)-nullity condition, the following condition have been studied by Dileo and Pastore [22], [24, 25].

Definition 10.1. An almost β-Kenmotsu manifold $(M, \varphi, \xi, \eta, g)$ satisfies the $(\kappa, \mu)'$-*nullity condition* provided there exist constants κ and μ such that

$$R(X, Y)\xi = (\kappa I + \mu h')(X \wedge Y)\xi$$

for all $X, Y \in \mathfrak{X}(M)$. Here the operator h' is defined by $h' = \hat{h}/\beta$. If M satisfies the $(\kappa, \mu)'$-nullity condition we have

- $\kappa \leq -\beta^2$.
- $\kappa = -\beta^2$, then $h' = 0$ (and hence $\hat{h} = 0$).
- $\kappa < -\beta^2$, then $\mu = -2\beta^2$.

More generally, almost Kenmotsu 3-manifolds satisfying

$$R(X,Y)\xi = (\kappa I + \mu h')(X \wedge Y)\xi, \quad X,Y \in \mathfrak{X}(M)$$

for some functions κ and μ are investigated by Saltarelli [62].

The Riemannian curvature of $G(c_1,c_2)$ satisfies the following formulas:

- $c_1 = c_2 = 0$: In this case $R = 0$.
- $c_1 + c_2 \neq 0$ and $c_1^2 + c_2^2 \neq 0$: In this case M satisfies the $(\kappa,\mu)'$-nullity condition with $\kappa = -(c_1^2 + c_2^2)/4$ and $\mu = -(c_1 + c_2)^2/2$.
  (1) In particular $G(c,c) = \mathbb{H}^3(-c^2)$ is $(-c)$-Kenmotsu manifold of constant curvature $-c^2$. Thus the curvature R satisfies $R(X,Y)\xi = -c^2(X \wedge Y)\xi$ for all vector fields X and Y.
  (2) In case $(c_1,c_2) = (0,c)$ or $(c_1,c_2) = (c,0)$ with $c \neq 0$, then $\kappa = -c^2/4$ and $\mu = -c^2/2$.
- $c_1 = -c_2 = c \neq 0$: In this case M satisfies $R(X,Y)\xi = -c^2(X \wedge Y)\xi$ for all vector fields X and Y.

Remark 10.4. Every almost β-Kenmotsu Lie group $G(c_1,c_2)$ with $\beta = (c_1+c_2)/2 \neq 0$ is pseudo-conformal to almost cosymplectic Lie group $G(c_1,c_2)$ for some $(\tilde{c}_1, \tilde{c}_2)$.

$$(\varphi, \xi, \eta, g) \longmapsto (\varphi, \xi, \eta, \tilde{g}),$$

with

$$\tilde{g} := e^{(c_1+c_2)z}g + (1 - e^{(c_1+c_2)z})\eta \otimes \eta,$$

the resulting almost contact Lie group with structure $(\varphi, \xi, \eta, \tilde{g})$ is $G(\tilde{c}_1, \tilde{c}_2)$ with $\tilde{c}_1 = -\tilde{c}_2 = (c_1 - c_2)/2$.

10.2. Slant curves in $G(c_1,c_2)$. Let γ be a slant curve in the solvable Lie group $G(c_1,c_2)$ with $\sin\theta \neq 0$ and take the orthonormal frame field

$$\epsilon_1 = \gamma', \quad \epsilon_2 = \frac{\varphi\gamma'}{|\sin\theta|}, \quad \epsilon_3 = \frac{\xi - \cos\theta}{|\sin\theta|}$$

along γ (see (8.6)).

Using (10.6) and (10.7) we have

(10.8)
$$\begin{cases} \nabla_{\gamma'}\epsilon_1 = \dfrac{b}{|\sin\theta|}\epsilon_2 + \dfrac{1}{|\sin\theta|}(a - \beta\sin^2\theta)\epsilon_3, \\[2mm] \nabla_{\gamma'}\epsilon_2 = -\dfrac{b}{|\sin\theta|}\epsilon_1 + \dfrac{1}{\sin^2\theta}(\sigma + b\cos\theta)\epsilon_3, \\[2mm] \nabla_{\gamma'}\epsilon_3 = -\dfrac{1}{|\sin\theta|}(a - \beta\sin^2\theta)\epsilon_1 - \dfrac{1}{\sin^2\theta}(\sigma + b\cos\theta)\epsilon_2, \end{cases}$$

where $a = g(\gamma', \varphi h\gamma')$, $b = g(\nabla_{\gamma'}\gamma', \varphi\gamma')$, $\sigma = g(\gamma', h\gamma')$, $\beta = -(c_1 + c_2)/2$.

From the first equation, we get

(10.9)
$$\kappa = \frac{1}{|\sin\theta|}\sqrt{(a - \beta\sin^2\theta)^2 + b^2}.$$

Proposition 10.2. *Let γ be a slant curve in the solvable Lie group $G(c_1,c_2)$. Then γ is a geodesic if and only if $(a - \beta\sin^2\theta)^2 + b^2 = 0$.*

Assume that γ is non-geodesic, then the principal normal is given by $N = \{\frac{b}{|\sin\theta|}\epsilon_2 + \frac{1}{|\sin\theta|}(a - \beta\sin^2\theta)\epsilon_3\}/\kappa$. Differentiating N by using (10.7) we get

$$\nabla_{\gamma'}N = -\frac{1}{\kappa}\{\frac{(a - \beta\sin^2\theta)^2 + b^2}{\sin^2\theta}\}\epsilon_1$$

(10.10)
$$+\frac{1}{\kappa|\sin\theta|}\{-\frac{\kappa'b}{\kappa} + b' - \frac{(a - \beta\sin^2\theta)(\sigma + b\cos\theta)}{\sin^2\theta}\}\epsilon_2$$

$$+\frac{1}{\kappa|\sin\theta|}\{-\frac{\kappa'(a - \beta\sin^2\theta)}{\kappa} + a' + \frac{b(\sigma + b\cos\theta)}{\sin^2\theta}\epsilon_3.$$

Since $h\gamma'$ is orthogonal to ξ, we have $h\gamma' = \sigma\gamma' + \nu\varphi\gamma'$. Applying φ we note that $a = -\nu$. From (10.10) we get

$$\tau B = \frac{1}{\kappa|\sin\theta|}\left[\{-\frac{\kappa'b}{\kappa} + b' - \frac{(a - \beta\sin^2\theta)(\sigma + b\cos\theta)}{\sin^2\theta}\}\epsilon_2\right.$$

(10.11)
$$\left. + \{-\frac{\kappa'(a - \beta\sin^2\theta)}{\kappa} + a' + \frac{b(\sigma + b\cos\theta)}{\sin^2\theta}\}\epsilon_3\right].$$

From $\kappa^2 = \frac{(a - \beta\sin^2\theta)^2 + b^2}{\sin^2\theta}$, we obtain

$$\tau = \frac{1}{\sin^2\theta}\left\{\frac{a'b - (a - \beta\sin^2\theta)b'}{\kappa^2} + (\sigma + b\cos\theta)\right\},$$

thus we have binormal vector field is $B = \frac{1}{\kappa|\sin\theta|}\{b\epsilon_3 - (a - \beta\sin^2\theta)\epsilon_2\}$. The ratio $\cos\theta/|\sin\theta|$ is computed as

$$\frac{\cos\theta}{|\sin\theta|} = \frac{\kappa^2(\tau\sin^2\theta - \sigma) - \{a'b - (a - \beta\sin^2\theta)b'\}}{\kappa b\sqrt{(a - \beta\sin^2\theta)^2 + b^2}}.$$

Now we consider almost Legendre curves in $G(c_1, c_2)$. In this case, the orthonormal frame field $\{\epsilon_1, \epsilon_2, \epsilon_3\}$ is simplified as $\epsilon_1 = \gamma'$, $\epsilon_2 = \varphi\gamma'$ and $\epsilon_3 = \xi$. The Frenet-Serret formula is reduced to

(10.12)
$$\begin{cases} \nabla_{\gamma'}\gamma' = (a - \beta)\xi + b\varphi\gamma', \\ \nabla_{\gamma'}\varphi\gamma' = \sigma\xi - b\gamma', \\ \nabla_{\gamma'}\xi = \beta\gamma' + h\varphi\gamma'. \end{cases}$$

The curvature is given by

(10.13)
$$\kappa = \sqrt{(a - \beta)^2 + b^2}.$$

From this formula, we obtain

Proposition 10.3. *Let γ be a Legendre curve in the solvable Lie group $G(c_1, c_2)$. Then γ is a geodesic if and only if $(a - \beta)^2 + b^2 = 0$.*

Assume that γ is non-geodesic, then we have

$$N = \frac{1}{\kappa}\{(a - \beta)\xi + b\varphi\gamma'\},$$

$$\tau = \frac{a'b - (a - \beta)b'}{\kappa^2} + \sigma,$$

$$B = \frac{1}{\kappa}\{b\xi - (a - \beta)\varphi\gamma'\}.$$

10.3. Legendre biharmonic curves in $G(c_1, c_2)$**.** Let $\gamma : I \to G(c_1, c_2)$ be a curve parametrized by arc-length with Frenet frame (T, N, B). Expand T, N, B as $T = T_1 e_1 + T_2 e_2 + T_3 e_3$, $N = N_1 e_1 + N_2 e_2 + N_3 e_3$, $B = B_1 e_1 + B_2 e_2 + B_3 e_3$ with respect to the basis $\{e_1, e_2, e_3\}$. Since (T, N, B) is positively oriented,

$$(10.14) \qquad B_1 = T_2 N_3 - T_3 N_2, B_2 = T_3 N_1 - T_1 N_3, B_3 = T_1 N_2 - T_2 N_1.$$

From these we get

$$R(N, T)T = -\{B_1 R(e_2, e_3) + B_2 R(e_3, e_1) + B_3 R(e_1, e_2)\}T.$$

Using the table of Riemannian curvature, we have

$$R(e_2, e_3)T = T_2 R(e_2, e_3)e_2 + T_3 R(e_2, e_3)e_3 = c_2^2(T_2 e_3 - T_3 e_2),$$

$$R(e_3, e_1)T = T_1 R(e_3, e_1)e_1 + T_3 R(e_3, e_1)e_3 = -c_1^2(T_1 e_3 - T_3 e_1),$$

$$R(e_1, e_2)T = T_1 R(e_1, e_2)e_1 + T_2 R(e_1, e_2)e_2 = c_1 c_2(T_1 e_2 - T_2 e_1).$$

Direct computation shows

$$
\begin{aligned}
&R(N, T)T \\
=\ & -B_1 c_2^2(T_2 e_3 - T_3 e_2) + B_2 c_1^2(T_1 e_3 - T_3 e_1) - B_3 c_1 c_2(T_1 e_2 - T_2 e_1) \\
=\ & -B_1 c_2^2(B_1 N - N_1 B) + B_2 c_1^2(-B_2 N + N_2 B) - B_3 c_1 c_2(B_3 N - N_3 B) \\
=\ & -(B_1^2 c_2^2 + B_2^2 c_1^2 + B_3^2 c_1 c_2)N + (N_1 B_1 c_2^2 + N_2 B_2 c_1^2 + N_3 B_3 c_1 c_2)B.
\end{aligned}
$$

Hence, the biharmonic equation for γ becomes

$$
\begin{aligned}
\mathcal{T}_2(\gamma) =\ & \nabla^3_T T + R(\kappa N, T)T \\
=\ & (-3\kappa\kappa')T + \{(\kappa'' - \kappa^3 - \kappa\tau^2) - \kappa(B_1^2 c_2^2 + B_2^2 c_1^2 + B_3^2 c_1 c_2)\}N \\
& + \{(2\tau\kappa' + \kappa\tau') + \kappa(N_1 B_1 c_2^2 + N_2 B_2 c_1^2 + N_3 B_3 c_1 c_2)\}B
\end{aligned}
$$

Theorem 10.1. *Let* $\gamma : I \to G(c_1, c_2)$ *be a curve parametrized by arc-length in the solvable Lie group* $G(c_1, c_2)$*. Then* γ *is a proper biharmonic curve if and only if*

$$(10.15) \qquad \begin{cases} \kappa = \text{constant} \neq 0, \\ \kappa^2 + \tau^2 = -B_1^2 c_2^2 - B_2^2 c_1^2 - B_3^2 c_1 c_2, \\ \tau' = -N_1 B_1 c_2^2 - N_2 B_2 c_1^2 - N_3 B_3 c_1 c_2. \end{cases}$$

In particular, If $c_1 = -c_2 = c$, then γ is a proper biharmonic curve if and only if

$$(10.16) \qquad \begin{cases} \kappa = \text{constant} \neq 0, \\ \kappa^2 + \tau^2 = c^2(2\eta(B)^2 - 1), \\ \tau' = 2c^2 \eta(B)\eta(N). \end{cases}$$

where $\eta(B) \neq 0$.

Remark 10.5. If $c_1 c_2 = 0$, then $\kappa^2 + \tau^2 = -B_1^2 c_2^2$ or $\kappa^2 + \tau^2 = -B_2^2 c_1^2$. Hence there does not exist biharmonic curve in $G(0, c)$.

If $c_1 = c_2 = c$, then $\kappa^2 + \tau^2 = -(B_1^2 + B_2^2 + B_3^2)c^2 = -c^2$. Hence there does not exist biharmonic curve in $G(c, c)$.

Next, we compute the principal normal N.

$$
\begin{aligned}
\kappa N = \nabla_{\gamma'} T =\ & \nabla_{\gamma'}(T_1 e_1 + T_2 e_2 + T_3 e_3) \\
=\ & (T_1' - c_1 T_1 T_3)e_1 + (T_2' - c_2 T_2 T_3)e_2 + (T_3' + c_1 T_1^2 + c_2 T_2^2)e_3.
\end{aligned}
$$

Hence

$$\kappa N_1 = T_1' - c_1 T_1 T_3,$$
(10.17)
$$\kappa N_2 = T_2' - c_2 T_2 T_3,$$
$$\kappa N_3 = T_3' + c_1 T_1^2 + c_2 T_2^2.$$

Differentiating N along γ

$$
\begin{aligned}
\nabla_T N &= \nabla_T(N_1 e_1 + N_2 e_2 + N_3 e_3) \\
&= (N_1' - c_1 T_1 N_3)e_1 + (N_2' - c_2 T_2 N_3)e_2 + (N_3' + c_1 T_1 N_1 + c_2 T_2 N_2)e_3.
\end{aligned}
$$

From this, we have

$$g(\nabla_T N, e_3) = N_3' + c_1 T_1 N_1 + c_2 T_2 N_2.$$

We assume that γ is an almost Legendre curve, *i.e.*, $T = T_1 e_1 + T_2 e_2 = \cos\alpha\, e_1 + \sin\alpha\, e_2$ for some function α. Then we have $T_1 N_1 + T_2 N_2 = 0$ and hence

(10.18) $$g(\nabla_T N, e_3) = N_3' + (c_1 - c_2)T_1 N_1.$$

On the other hand, using the Frenet-Serret equation,

(10.19) $$g(\nabla_T N, e_3) = g(-\kappa T + \tau B, e_3) = -\kappa T_3 + \tau B_3.$$

From the equation (10.18) and (10.19), for an almost Legendre curve γ we have

(10.20) $$N_3' + (c_1 - c_2)T_1 N_1 = \tau B_3.$$

Now we look for biharmonic almost Legendre curves. From (10.15), γ has constant curvature. By using (10.14), (10.17) and (10.20), we get

(10.21) $$\tau = -3(c_1 - c_2)\sin\alpha\cos\alpha.$$

Differentiating (10.21), we get

(10.22) $$\tau' = -3(c_1 - c_2)(\cos^2\alpha - \sin^2\alpha)\alpha'.$$

From the third equation of (10.15), (10.14) and (10.17), we have

(10.23) $$\tau' = \frac{1}{\kappa^2}(c_1 - c_2)(c_1^2\cos^4\alpha - c_2^2\sin^4\alpha)\alpha'.$$

From the equation (10.22) and (10.23), one can deduce that $c_1 = c_2$ or α is a constant.

For the case $c_1 = c_2$, as we have shown before, there does not exist biharmonic curve in $G(c, c)$. When α is a constant then the second equation of (10.15) is rewritten as

$$\kappa^2 + \tau^2 = -\frac{1}{\kappa^2}(c_1\cos^2\alpha + c_2\sin^2\alpha)^2(c_1^2\cos^2\alpha + c_2^2\sin^2\alpha) \le 0.$$

Thus γ cannot be proper biharmonic. Hence, there does not exist non-geodesic almost Legendre curve satisfying biharmonic condition (10.15) in the solvable Lie group $G(c_1, c_2)$.

Theorem 10.2. *There does not exist proper biharmonic almost Legendre curve in the solvable Lie group $G(c_1, c_2)$.*

10.4. Biharmonicity with respect to canonical connection. On the solvable Lie group $G(c_1, c_2)$, the canonical connection $\widetilde{\nabla}^t$ is given by

$$\widetilde{\nabla}^t_{e_3} e_1 = -te_2, \ \widetilde{\nabla}^t_{e_3} e_2 = te_1,$$

for other pair of i and j, we have $\widetilde{\nabla}^t_{e_i} e_j = 0$.

By using the above data, the curvature tensor $\widetilde{R}^t$ is computed as

$$(10.24) \qquad \widetilde{R}^t(e_i, e_j)e_k = 0, \ \ i, j, k = 1, 2, 3.$$

Thus the affine homogeneous space $(G(c_1, c_2), \widetilde{\nabla}^t)$ is flat for any $(c_1, c_2) \in \mathbb{R}^2$. Moreover we have

$$\widetilde{\mathfrak{T}}^t(X, Y) = -[X, Y] - t(\eta(X)\varphi Y - \eta(Y)\varphi X), \ \ \widetilde{\nabla}^t \widetilde{\mathfrak{T}}^t = 0$$

for all $X, Y \in \mathfrak{g}(\mu_1, \mu_2)$. In particular we have

$$(10.25) \qquad \widetilde{\mathfrak{T}}^t(e_1, e_3) = c_1 e_1 + te_2, \ \ \widetilde{\mathfrak{T}}^t(e_2, e_3) = c_2 e_2 - te_1.$$

Remark 10.6. Dileo introduced the following linear connection on almost β-Kenmotsu manifolds of arbitrary odd dimension with integrable associated CR-structure [22].

$$\nabla'_X Y = \nabla_X Y + \beta\{g((I + h')X, Y)\xi - \eta(Y)(I + h')X\}.$$

One can easily check that ∇' coincides with our connection $\widetilde{\nabla}^0$ on almost β-Kenmotsu manifolds with integrable associated CR-structure.

In particular the canonical connection $\nabla' = \widetilde{\nabla}^0$ on $G(c_1, c_2)$ with $c_1 + c_2 \neq 0$ satisfies

$$\widetilde{R}^0 = 0, \ \ \widetilde{\nabla}^0 \widetilde{\mathfrak{T}}^0 = 0, \ \ \widetilde{\mathfrak{T}}^0(X, Y) = -[X, Y]$$

for all $X, Y \in \mathfrak{g}(\mu_1, \mu_2)$. These formulas imply that the canonical connection $\widetilde{\nabla}^0$ coincides with the so-called *Cartan-Schouten's (−)-connection* [49].

Dileo showed the following classification.

Theorem 10.3 ([22]). *Let M be an almost β-Kenmotsu 3-manifold. Then M satisfies $\widetilde{\nabla}^0 \widetilde{\mathfrak{T}}^0 = 0$ and $\widetilde{\nabla}^0 \widetilde{R}^0 = 0$ if and only if M is locally isomomorphic to $G(\mu_1, \mu_2)$ as an almost contact metric manifold with $\beta = -(\mu_1 + \mu_2)/2 \neq 0$.*

The biharmonicity equation with respect to $\widetilde{\nabla}^t$ is given as follows.

Proposition 10.4. *A unit speed curve γ in the solvable Lie group $G(c_1, c_2)$ is biharmonic with respect to the canonical connection $\widetilde{\nabla}^t$ if and only if it satifies:*

$$(10.26) \qquad \begin{cases} \widetilde{H} = \widetilde{\nabla}^t_{\gamma'} \gamma', \\ \widetilde{\nabla}^t_{\gamma'} \widetilde{\nabla}^t_{\gamma'} \widetilde{H} + \widetilde{\nabla}^t_{\gamma'} \widetilde{\mathfrak{T}}(\widetilde{H}, \gamma') = 0. \end{cases}$$

Let us express the tangent vector field as $T(s) = \gamma'(s) = T_1(s)e_1 + T_2(s)e_2 + T_3(s)e_3$ as before. Then the pseudo-Hermitian mean curvature vector field $\widetilde{H}$ with respect to $\widetilde{\nabla}^t$ is computed as

$$\tilde{H} = (T_1' + tT_2 T_3)e_1 + (T_2' - tT_1 T_3)e_2 + T_3' e_3.$$

In case γ is an almost Legendre curve, then its tangent vector field is expressed as $\gamma'(s) = T_1(s)e_1 + T_2(s)e_2 = \cos\alpha(s)e_1 + \sin\alpha(s)e_2$ as before. Hence we obtain

Proposition 10.5. *If γ is an almost Legendre curve in $G(c_1, c_2)$, then γ is $\widetilde{\nabla}^t$-geodesic if and only if the tangent vector field has the form $\gamma'(s) = T_1 e_1 + T_2 e_2$, where T_1 and T_2 are constants satisfying $T_1^2 + T_2^2 = 1$.*

Note that the $\widetilde{\nabla}^t$-harmonicity of almost Legendre curves in $G(c_1, c_2)$ does not depend on t.

We obtain explicit parametric equations of almost Legendre $\widetilde{\nabla}^t$-geodesics. Let $\gamma(s) = (x(s), y(s), z(s))$ be a Frenet curve in $G(c_1, c_2)$. Then the tangent vector field $T(s) = \gamma'(s)$ of γ is represented by

$$(10.27) \qquad T(s) = \left(\frac{\mathrm{d}x}{\mathrm{d}s}, \frac{\mathrm{d}y}{\mathrm{d}s}, \frac{\mathrm{d}z}{\mathrm{d}s} \right) = \frac{\mathrm{d}x}{\mathrm{d}s} \frac{\partial}{\partial x} + \frac{\mathrm{d}y}{\mathrm{d}s} \frac{\partial}{\partial y} + \frac{\mathrm{d}z}{\mathrm{d}s} \frac{\partial}{\partial z}.$$

From (10.2), we have

$$\frac{\mathrm{d}x}{\mathrm{d}s} = e^{c_1 z} T_1, \;\; \frac{\mathrm{d}y}{\mathrm{d}s} = e^{c_1 z} T_2, \;\; \frac{\mathrm{d}z}{\mathrm{d}s} = 0.$$

Therefore we obtain

Theorem 10.4. *If an almost Legendre curve $\gamma(s)$ in $G(c_1, c_2)$ starting at $\gamma(0) = (x_0, y_0, z_0)$ is a $\widetilde{\nabla}^t$-geodesic, then γ is given explicitly by*

$$(10.28) \qquad x(s) = (e^{c_1 z_0} T_1)s + x_0, \;\; y(s) = (e^{c_2 z_0} T_2)s + y_0, \;\; z(s) = z_0,$$

where T_1 and T_2 are constants such that $T_1^2 + T_2^2 = 1$.

Here we give a group theoretic interpretation of (10.28). Let us define a 1-parameter subgroup $\{a(s) \mid s \in \mathbb{R}\}$ of $G(c_1, c_2)$ by

$$a(s) = (T_1 s, T_2 s, 0) = \exp \left\{ s \begin{pmatrix} 0 & 0 & T_1 \\ 0 & 0 & T_2 \\ 0 & 0 & 0 \end{pmatrix} \right\}.$$

Then one can see that $a(s)$ is an almost Legendre curve with initial condition $a(0) = (0, 0, 0)$ and $a'(0) = T_1 e_1 + T_2 e_2$. Moreover $a(s)$ is a $\widetilde{\nabla}^t$-geodesic. The formula (10.28) can be rewritten as $\gamma(s) = (x_0, y_0, z_0) * a(s)$ by using the group multiplication of $G(c_1, c_2)$.

Corollary 10.1. *Every almost Legendre $\widetilde{\nabla}^t$-geodesic in $G(c_1, c_2)$ is obtained by as a left translation of an almost Legendre $\widetilde{\nabla}^t$-geodesic starting at the origin.*

These results were obtained for the case $\mathbb{H}^3(-1) = G(-1, -1)$ in [40].
Next we study $\widetilde{\nabla}^t$-biharmonic almost Legendre curves in $G(c_1, c_2)$.
By using (8.1) we calculate

$$(10.29) \qquad \widetilde{\nabla}^t \widetilde{\nabla}^t \widetilde{H} = -3\widetilde{\kappa}(\widetilde{\kappa})' \gamma' + \{(\widetilde{\kappa})'' - (\widetilde{\kappa})^3\} \widetilde{N}.$$

From (10.25) we get $\widetilde{\nabla}^t_{\gamma'} \widetilde{\mathfrak{T}}^t(\widetilde{H}, \gamma') = 0$. Therefore we obtain

Theorem 10.5. *If γ is an almost Legendre curve in $G(c_1, c_2)$, then γ is $\widetilde{\nabla}^t$-biharmonic if and only if γ is a $\widetilde{\nabla}^t$-geodesic.*

We have studied slant curves in the model space Sol_3 of solv-geometry in the sense of Thurston equipped with left invariant *almost cosymplectic* structure. On the other hand, Sol_3 admits also a natural left invariant *contact metric* structure. In a separate publication, we shall study slant curves in Sol_3 equipped with natural contact metric structure. Moreover in [42], slant curves in 3-dimensional f-Kenmotsu manifolds are studied.

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