ON RANDERS CHANGE OF $m$-TH ROOT FINSLER METRICS

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Abstract. In this paper, we consider Randers change of $m$-th root Finsler metrics. We find necessary and sufficient condition under which a Randers change of an $m$-th root metric be locally dually flat. Then we prove that the Randers change of an $m$-th root Finsler metric is locally projectively flat if and only if it is locally Minkowskian.

1. Introduction

A change of Finsler metric $F \rightarrow \bar{F}$ is called a Randers change of $F$, if

$$\bar{F}(x, y) = F(x, y) + \beta(x, y),$$

where $\beta(x, y) = b_i(x)y^i$ is a 1-form on a smooth manifold $M$. It is easy to see that, if $\sup_{F(x, y)} |b_i(x)y^i| < 1$, then $\bar{F}$ is again a Finsler metric. Hashiguchi-Ichijyo showed that if $\beta$ is closed, then $\bar{F}$ is pointwise projective to $F$. The notion of a Randers change has been proposed by Matsumoto, named by Hashiguchi-Ichijyo and studied in detail by Shibata [7][9][12]. If $F$ reduces to a Riemannian metric then $\bar{F}$ reduces to a Randers metric. Due to this reason the transformation (1.1) has been called the Randers change of Finsler metric. For other Finslerian transformations see [12][17].

The Randers change is projective if and only if $b_i(x)$ is locally a gradient vector field. According to Hashiguchi-Ichijyo, a Randers change is projective, if and only if $b_{ij} = b_{ji}$, that is $b_i(x)$ is locally a gradient vector field and symbols “$|$” mean the covariant derivatives in $\bar{F}$ with respect to Berwald connection [7]. It is remarkable that, if $\bar{F}$ is absolutely homogeneous then the necessary and sufficient condition for $\bar{F}$ to have reversible geodesics is that $\beta$ is closed and it is a first integral of the geodesic flow of $\bar{F}$ [6]. Consider the Randers metric $F = \alpha + \beta$, where $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ and $||\beta|| := |a_{ij}b^ib^j| < 1$. If $\beta$ is a closed 1-form, then $F$ has reversible geodesics and if it is parallel with respect to $\alpha$ (i.e., $b_{ij} = 0$) then $F$ has strictly reversible geodesics.

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In [2], Amari-Nagaoka introduced the notion of dually flat Riemannian metrics when they study the information geometry on Riemannian manifolds. Information geometry has emerged from investigating the geometrical structure of a family of probability distributions and has been applied successfully to various areas including statistical inference, control system theory and multi-terminal information theory [1]. In Finsler geometry, Shen extends the notion of locally dually flatness for Finsler metrics [11]. A Finsler metric $F$ on a manifold $M$ is said to be locally dually flat if at any point there is a coordinate system $(x^i, y^i)$ in which the spray coefficients are in the following form $G^i = -\frac{1}{2} g^{ij} H_{ij}$ where $H = H(x,y)$ is a $C^\infty$ homogeneous scalar function on $TM_0$. Such a coordinate system is called an adapted coordinate system [14]. Indeed, a Finsler metric $F$ on an open subset $U \subset \mathbb{R}^n$ is called dually flat if it satisfies
\[
\frac{\partial^2 F^2}{\partial x^i \partial y^j} y^k = 2 \frac{\partial F^2}{\partial x^i}.
\]
Let $(M, F)$ be a Finsler manifold of dimension $n$, $TM$ its tangent bundle and $(x^i, y^i)$ the coordinates in a local chart on $TM$. Let $F$ be the following function on $M$, by $F = \sqrt{A}$, where $A$ is given by $A := a_{1 \ldots n}(x) y^1 \ldots y^n$ with $a_{1 \ldots n}$ symmetric in all its indices (for example see [3][4][5][10][13][14][15]). Then $F$ is called an $m$-th root Finsler metric. Suppose that $A_{ij}$ define a positive definite tensor and $A^{ij}$ denotes its inverse. For an $m$-th root metric $F$, put
\[
A_i = \frac{\partial A}{\partial y^i}, \quad A_{ij} = \frac{\partial^2 A}{\partial y^i \partial y^j}, \quad A_{ix} = \frac{\partial A}{\partial x^i}, \quad A_0 = A_{xx}.\]
In this paper, we consider Randers change of an $m$-th root Finsler metric and find necessary and sufficient condition under which a Randers change of an $m$-th root metric be locally dually flat. More precisely, we prove the following.

**Theorem 1.1.** Let $F = \sqrt{A}$ be an $m$-th root Finsler metric on an open subset $U \subset \mathbb{R}^n$, where $A$ is irreducible. Suppose that $\tilde{F} = F + \beta$ be Randers change of $F$ where $\beta = b_1(x) y^i$. Then $\tilde{F}$ is locally dually flat if and only if there exists a $1$-form $\theta = \theta_1(x) y^i$ on $U$ such that the following hold
\[
\begin{align*}
\beta_{0l} &+ \beta_l \beta_0 = 2 \beta_{xl}, \\
A_{xl} &= \frac{1}{3m} [mA_{0l} + 2\theta A_1], \\
(\frac{1}{m} - 1) A_{i1} A^{-1} A_{0l} &+ (A_{0l} \beta_0) - 3 A_{xl} \beta + A_l \beta_0 = A(2 \beta_{xl} - \beta_{0l}),
\end{align*}
\]
where $\beta_{0l} = \beta_{xk} y^k$, $\beta_{xl} = (b_1)_{xl} y^i$, $\beta_0 = \beta_{x1} y^1$ and $\beta_{0l} = (b_l)_{0}$.

A Finsler metric is said to be locally projectively flat if at any point there is a local coordinate system in which the geodesics are straight lines as point sets. It is known that a Finsler metric $F(x, y)$ on an open domain $U \subset \mathbb{R}^n$ is locally projectively flat if and only if
\[
G^i = P y^i,
\]
where $P(x, \lambda y) = \lambda P(x, y)$, $\lambda > 0$ [8]. Projectively flat Finsler metrics on a convex domain in $\mathbb{R}^n$ are regular solutions to Hilbert’s Fourth Problem: determine the metrics on an open subset in $\mathbb{R}^n$, whose geodesics are straight lines.
Theorem 1.2. Let $F = \sqrt[2]{A}$ be an $m$-th root Finsler metric on an open subset $U \subset \mathbb{R}^n$, where $A$ is irreducible. Suppose that $\widetilde{F} = F + \beta$ be Randers change of $F$ where $\beta = b_i(x)y^i$. Then $\widetilde{F}$ is locally projectively flat if and only if it is locally Minkowskian.

2. Proof of the Theorem 1.1

In this section, we will prove a generalized version of Theorem 1.1. Indeed we find necessary and sufficient condition under which a Randers change of an generalized $m$-th root metric be locally dually flat. Let $F$ be a scalar function on $TM$ defined by following

$$F = \sqrt{A^{2/m} + B},$$

where $A$ and $B$ are given by

$$(2.1) \quad A := a_{i_1...i_m}(x)y^{i_1}...y^{i_m}, \quad B := b_{ij}(x)y^iy^j.$$  

Then $F$ is called generalized $m$-th root Finsler metric. Suppose that the matrix $(A_{ij})$ defines a positive definite tensor and $(A^V)$ denotes its inverse. Then the following hold

$$g_{ij} = \frac{A^{2/m} - 2}{m^2}[mAA_{ij} + (2 - m)A_iA_j] + b_{ij},$$

$$A_i = \frac{\partial A}{\partial y^i}, \quad A_{ij} = \frac{\partial^2 A}{\partial y^i \partial y^j}, \quad B_i = \frac{\partial B}{\partial y^i}, \quad B_{ij} = \frac{\partial^2 B}{\partial y^i \partial y^j},$$

$$A_{x^i} = \frac{\partial A}{\partial x^i}, \quad A_0 = A_{x^i}y^i, \quad B_{x^i} = \frac{\partial B}{\partial x^i}, \quad B_0 = B_{x^i}y^i.$$  

Now, we are going to prove the following.

Theorem 2.1. Let $F = \sqrt{A^{2/m} + B}$ be an generalized $m$-th root Finsler metric on an open subset $U \subset \mathbb{R}^n$, where $A$ is irreducible. Suppose that $\widetilde{F} = F + \beta$ be Randers change of $F$ where $\beta = b_i(x)y^i$. Then $\widetilde{F}$ is locally dually flat if and only if there exists a 1-form $\theta = \theta_i(x)y^i$ on $U$ such that the following holds

$$(2.2) \quad \beta_{i0} + \beta_{i0} + B_{0l} = 2[\beta_{x^i} + B_{x^i}],$$

$$(2.3) \quad A_{x^i} = \frac{1}{3m}[mA\theta_l + 2\theta A],$$

$$(2.4) \quad Y_0\beta = 2\mathcal{Y} \left[ (Y_0\beta + Y_0\beta_l + Y_{l0}\beta_0 - 2Y_{x^i}\beta) + 2\mathcal{Y}(\beta_{0l} - 2\beta_{x^i}) \right],$$

where $\beta_{0l} = \beta_{x^i}y^k$, $\beta_{x^i} = (b_i)_x y^i$, $\beta_0 = (b_i)_0 y^i$, $\beta_{0l} = (b_i)_0$, $\mathcal{Y} := A^{2/m} + B$ and

$$Y_p := \frac{2}{m} A^{2/m - 1} A_p + B_p,$$

$$Y_{0p} := \frac{2}{m} A^{2/m - 2} \left[ \frac{2}{m} - 1 \right] A_p A_0 + AA_{0p} + B_{0p}.$$  

To prove Theorem 2.1, we need the following.

Lemma 2.1. Suppose that the equation $\Phi A^{2/m - 2} + \Psi A^{1/m - 1} + \Theta = 0$ holds, where $\Phi, \Psi, \Theta$ are polynomials in $y$ and $m > 2$. Then $\Phi = \Psi = \Theta = 0$.  

Proof of Theorem 2.1: Let $\bar{F}$ be a locally dually flat metric. We have

$$F^2 = A\hat{\bar{F}} + B + 2\beta(A\hat{\bar{F}} + B)^{1/2} + \beta^2,$$

$$(\bar{F}^2)_{x^k} = \frac{2}{m}A\hat{\bar{F}}^{-1}A_{x^k} + B_{x^k} + (A\hat{\bar{F}} + B)^{-1/2}(\frac{2}{m}A\hat{\bar{F}}^{-1}A_{x^k} + B_{x^k})\beta + 2(A\hat{\bar{F}} + B)^{1/2}\beta_{x^k} + 2\beta_{x^k}\beta.$$  

Then

$$[\bar{F}^2]_{x^k'y^k} = \frac{2}{m}A\hat{\bar{F}}^{-2}\left[(\frac{2}{m} - 1)A_0 A_0 + AA_0l - 2AA_{x^k}\right]$$

$$+ (A\hat{\bar{F}} + B)^{-3/2}\left[\cdot\right] + 2(\beta_{0l}\beta + \beta_0\beta_l - 2\beta_{x^k}\beta) + 2(\beta_{0l}\beta + \beta_0\beta_l - 2\beta_{x^k}\beta) + B_{0l} - 2B_{x^k} = 0.$$  

By Lemma 2.1, we have

$$(2.5) \quad (\frac{2}{m} - 1)A_0 A_0 + AA_0l = 2AA_{x^k},$$

$$(2.6) \quad -\frac{1}{2}Y_0 Y_0\beta + C[Y_0\beta + Y_0\beta_l + Y_l\beta_0 - 2Y_{x^k}\beta] = 2C^2(2\beta_{x^k} - \beta_{0l}),$$

$$(2.7) \quad 2(\beta_{0l}\beta + \beta_0\beta_l - 2\beta_{x^k}\beta) = 2B_{x^k} - B_{0l},$$

One can rewrite (2.5) as follows

$$(2.8) \quad A(2A_{x^k} - A_{0l}) = (\frac{2}{m} - 1)A_0 A_0.$$  

Irreducibility of $A$ and

$$deg(A_0) = m - 1$$

imply that there exists a 1-form $\theta = \theta_{y^k}$ on $U$ such that

$$(2.9) \quad A_0 = \theta A.$$  

Plugging (2.9) into (2.8), we get

$$(2.10) \quad A_{0l} = A\theta_l + \theta A_l - A_{x^k}.$$  

Substituting (2.9) and (2.10) into (2.8) yields (2.3). The converse is a direct computation. This completes the proof. $\square$
3. Proof of the Theorem 1.2

In this section, we will prove a generalized version of Theorem 1.2. Indeed we study the Randers change of an generalized $m$-th root metric

$$F = \sqrt{A^m + B},$$

where $A$ and $B$ are given by

$$A := a_{i_1 \ldots i_m}(x)y^{i_1} \ldots y^{i_m}, \quad B := b_{ij}(x)y^iy^j$$

and $A$ is irreducible. More precisely, we prove the following.

**Theorem 3.1.** Let $F = \sqrt{A^m + B}$ be an generalized $m$-th root Finsler metric on an open subset $U \subset \mathbb{R}^n$, where $A$ is irreducible. Suppose that $\bar{F} = F + \beta$ be Randers change of $F$ where $\beta = b_i(x)y^i$. Then $\bar{F}$ is locally projectively flat if and only if it is locally Minkowskian.

To prove Theorem 3.1, we need the following.

**Lemma 3.1.** Let $(M, F)$ be a Finsler manifold. Suppose that $\bar{F} = F + \beta$ be a Randers change of $F$. Then $\bar{F}$ is a projectively flat Finsler metric if and only if the following holds

$$(3.1) \quad F_{0l} - F_{xl} = (b_l)_x y^i - (b_l)_0.$$

In local coordinates $(x^i, y^i)$, the vector field

$$G = y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i}$$

is a global vector field on $TM_0$, where $G^i = G^i(x, y)$ are local functions on $TM_0$ given by following

$$G^i := \frac{1}{4} g^{ij} \left\{ \frac{\partial^2 F^2}{\partial x^k \partial y^j} y^k - \frac{\partial F^2}{\partial x^j} \right\}, \quad y \in T_x M.$$

A Finsler metric $F$ is called a Berwald metric if

$$G^i = \frac{1}{2} \Gamma^i_{jk}(x)y^j y^k$$

is quadratic in $y \in T_x M$ for any $x \in M$. The projection of an integral curve of $G$ is a geodesic in $M$. In local coordinates, a curve $c(t)$ is a geodesic if and only if its coordinates $(c^i(t))$ satisfy $c^i + 2G^i(c) = 0$ [18].

Now, by using Lemma 3.1, we are going to prove the following.

**Proposition 3.1.** Let $F = \sqrt{A^m + B}$ be an generalized $m$-th root Finsler metric on an open subset $U \subset \mathbb{R}^n$, where $A$ is irreducible, $m > 4$ and $B \neq 0$. Suppose that $\bar{F} = F + \beta$ be Randers change of $F$ where $\beta = b_i(x)y^i$. In that case, if $\bar{F}$ is projectively flat metric then $F$ reduces to a Berwald metric.
Proof. By Lemma 3.1, we get

\[ F_{x^l} = \frac{2A^{2/m}A_{x^l} + mAB_{x^l}}{2mA\sqrt{A^2 + B}} \]

and

\[ F_{x^k y^l} = -\frac{1}{4}(A^{2/m} + B^{-1/2}) \left[ \frac{2A^{2/m}A_0}{mA} + B_0 \left( \frac{2A^{2/m}A_l}{mA} + B_l \right) \right]^{-1} \]

\[ + \frac{1}{2}(A^{2/m} + B^{-1/2}) \left[ \frac{2A^{2/m}A_0 A_l}{mA^2} + \frac{2A^{2/m}A_{0l}}{mA} - \frac{2A^{2/m}A_0 A_l}{mA^2} + B_{0l} \right]. \]

By (3.1), we obtain the following

\[ \Phi A^A + \Psi A^\pm + \Theta = 0, \]

where

\[
\Phi = -\frac{1}{2}mA \left[ A_0 B_l + B_0 A_l + 2B(A_{x^l} - A_{0l}) + mA(B_{x^l} - B_{0l}) \right] - (m - 2)A_0 A_l B,
\]

\[
\Psi = mA(A_{0l} - A_{x^l}) - (m - 1)A_0 A_l,
\]

\[
\Theta = \frac{1}{4}m^2A^2 \left[ -2B_0 B + B_0 B_l + 2B_{x^l} B \right] + (b_l) - (b_{x^l}) y^l.
\]

By Lemma 2.1, we have

(3.2) \quad \Phi = 0,

(3.3) \quad \Psi = 0,

(3.4) \quad \Theta = 0.

By (3.3), we result that

(3.5) \quad mA(A_{0l} - A_{x^l}) = (m - 1)A_0 A_l.

Then irreducibility of \( A \) and \( \text{deg}(A_l) = m - 1 < \text{deg}(A) \) implies that \( A_0 \) is divisible by \( A \). This means that, there is a 1-form \( \theta = \theta_l y^l \) on \( U \) such that,

(3.6) \quad A_0 = 2mA\theta.

Substituting (3.6) into (3.5), yields

(3.7) \quad A_{0l} = A_{x^l} + 2(m - 1)\theta A_l.

Plugging (3.6) and (3.7) into (3.2), we get

(3.8) \quad mA(2\theta B_l - B_{0l} + B_{x^l}) = A_l(4B\theta - B_0).

Clearly, the right side of (3.8) is divisible by \( A \). Since \( A \) is irreducible, \( \text{deg}(A_l) \) and \( \text{deg}(2\theta B - \frac{1}{2}B) \) are both less than \( \text{deg}(A) \), then we have have

(3.9) \quad B_0 = 4B\theta.

By (3.6) and (3.9), we get the spray coefficients \( G^i = Py^i \) with \( P = \theta \). Then \( F \) is a Berwald metric. \( \square \)

**Proof of Theorem 3.1:** By Proposition 3.1, if \( F \) is projectively flat then it reduces to a Berwald metric. Now, if \( m > 4 \) then by Numata’s Theorem every Berwald metric of non-zero scalar flag curvature \( K \) must be Riemannian. This is contradicts with our assumption. Then \( K = 0 \), and in this case \( F \) reduces to a locally Minkowskian metric. \( \square \)
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