

**AN ESTIMATE FOR THE GAUSS-KRONECKER CURVATURE
OF 3-DIMENSIONAL SMOOTH HYPERSURFACES IN \mathbb{E}^4**

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ABSTRACT. We give two alternate proofs of B.-Y. Chen's inequality with classical curvature invariants for 3-dimensional hypersurfaces in \mathbb{E}^4 . Then, by using an idea described by Cvetkovski, we obtain an estimate of the Gauss-Kronecker curvature of a three-dimensional smooth hypersurface in the four dimensional Euclidean space in function of its mean curvature and its scalar curvature.

1. INTRODUCTION

There are many fundamental properties of real numbers that contribute essentially to the special geometric properties that the dimension 3 objects have. In the present paper we use the fact that the derivative of a cubic equation is a quadratic function, whose properties can be controlled. By using this fundamental fact, we prove an inequality in terms of curvature invariants of a hypersurface in a four dimensional Euclidean ambient.

First, we remind some classical notations in the differential geometry of smooth hypersurfaces. Let $\sigma : U \subset \mathbb{R}^n \rightarrow \mathbb{R}^{n+1}$ be a hypersurface given by the smooth map σ . Let p be a point on the hypersurface. Denote $\sigma_k(p) = \frac{\partial \sigma}{\partial x_k}$, for all k from 1 to n . Consider $\{\sigma_1(p), \sigma_2(p), \dots, \sigma_n(p), N(p)\}$, the Gauss frame of the hypersurface, where N denotes the normal vector field. We denote by $g_{ij}(p)$ the coefficients of the first fundamental form and by $h_{ij}(p)$ the coefficients of the second fundamental form. Then we have

$$g_{ij}(p) = \langle \sigma_i(p), \sigma_j(p) \rangle, \quad h_{ij}(p) = \langle N(p), \sigma_{ij}(p) \rangle.$$

The Weingarten map $L_p = -dN_p \circ d\sigma_p^{-1} : T_{\sigma(p)}\sigma \rightarrow T_{\sigma(p)}\sigma$ is linear. Denote by $(h_j^i(p))_{1 \leq i, j \leq n}$ the matrix associated to Weingarten's map, that is:

$$L_p(\sigma_i(p)) = h_i^k(p)\sigma_k(p),$$

where the repeated index and upper script above indicates Einstein's summation convention. Weingarten's operator is self-adjoint, which implies that the roots of

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the algebraic equation

$$\det(h_j^i(p) - \lambda(p)\delta_j^i) = 0$$

are real. The eigenvalues of Weingarten's linear map are called principal curvatures of the hypersurface. They are the roots $\lambda_1(p), \lambda_2(p), \dots, \lambda_n(p)$ of this algebraic equation. The mean curvature at the point p is

$$H(p) = \frac{1}{n}[\lambda_1(p) + \dots + \lambda_n(p)],$$

and the Gauss-Kronecker curvature is

$$K(p) = \lambda_1(p)\lambda_2(p)\dots\lambda_n(p).$$

If M is a submanifold of a Riemannian manifold \bar{M} , and if their sectional curvatures are \sec and $\overline{\sec}$, respectively, then from the Gauss equation (see e.g. [9], pg.131) we have:

$$\sec(e_i \wedge e_j) - \overline{\sec}(e_i \wedge e_j) = \lambda_i \lambda_j.$$

If the ambient space is Euclidean, then $\overline{\sec}(e_i \wedge e_j) = 0$ at every point, in every planar direction. This means that the scalar curvature of a hypersurface in Euclidean ambient space at point $p \in M^n$ is

$$scal(p) = \sum_{i < j} \sec(e_i \wedge e_j) = \sum_{i < j} \lambda_i \lambda_j.$$

The normalized scalar curvature is $\rho = \binom{n}{2}^{-1} scal$.

In [13] it was proved that for a hypersurface M^n in a Riemannian $(n+1)$ -manifold \bar{M}^{n+1} , at every point $p \in M$ the following inequality holds:

$$(1.1) \quad scal(p) \leq \frac{n(n-1)}{2} H^2 + \sum_{i < j} \overline{\sec}(e_i \wedge e_j),$$

where $scal$ is the scalar curvature of M at p , H is the mean curvature at p , and $\overline{\sec}(e_i \wedge e_j)$ is the sectional curvature on the plane generated by vectors e_i and e_j tangent to the ambient space \bar{M} .

The equality holds at p if and only if p is an umbilical point.

This inequality was also studied in [12]. For a general formulation in pseudo-Riemannian ambient spaces, see e.g. [6] pp.36-37, where the same fact is derived as a consequence of Gauss' equation and Cauchy-Schwarz inequality. In [13] it is shown how from the inequality (1.1) one may derive Chen's inequality with classical curvature invariants, proved originally by B.-Y. Chen in 1996.

Theorem 1.1. [3] *Let $\rho = 2 scal(p)/n(n-1)$ denote the normalized scalar curvature of a hypersurface M^n isometrically immersed in a Riemannian space form $R^{n+1}(\epsilon)$. Then we have the inequality:*

$$(1.2) \quad \rho \leq H^2 + \epsilon,$$

at every point $p \in M$.

The equality holds if and only if p is an umbilical point.

The general form of this result, for submanifolds of arbitrary codimension, is Proposition 2.5 in [6].

In the particular case when the ambient space is Euclidean, we have $\epsilon = 0$. We provide below two alternate proof for this inequality for three-dimensional hypersurfaces in four-dimensional Euclidean ambient space. For further discussion see also [15].

This present development is a natural continuation of the study of new curvature invariants, as it was inspired by the ideas considered by B.-Y. Chen in [2, 4, 5]. For the whole vision of this research direction, the most comprehensive reference is the recent monograph [6]. For recent explorations of various inequalities of algebraic or analytic inspiration that yield geometric consequences in terms of curvature invariants, see [1, 7, 14], all inspired by the philosophy developed in [6].

2. CHEN'S FUNDAMENTAL INEQUALITY WITH CLASSICAL CURVATURE INVARIANTS FOR HYPERSURFACES OF DIMENSION THREE

We focus now on a particular case of Theorem 1.1, namely the following.

Corollary 2.1. *Let $M^3 \subset \mathbb{R}^4$ be a smooth hypersurface and $\lambda_1, \lambda_2, \lambda_3$ be its principal curvatures in the ambient space \mathbb{R}^4 endowed with the canonical metric. Let $p \in M$ be an arbitrary point. Denote by $H(p) = \frac{1}{3}(\lambda_1 + \lambda_2 + \lambda_3)$ the mean curvature, by $scal(p) = \sec(e_1 \wedge e_2) + \sec(e_3 \wedge e_1) + \sec(e_2 \wedge e_3) = \lambda_1(p)\lambda_2(p) + \lambda_3(p)\lambda_1(p) + \lambda_2(p)\lambda_3(p)$ the scalar curvature at the point $p \in M^3$. Then*

$$H^2(p) \geq \rho(p),$$

with equality if and only if the point is umbilical, i.e. when $\lambda_1(p) = \lambda_2(p) = \lambda_3(p)$.

Alternate proof 1: We have

$$\begin{aligned} 9H^2 &= (3H)^2 = (\lambda_1 + \lambda_2 + \lambda_3)^2 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 + 2(\lambda_1\lambda_2 + \lambda_3\lambda_1 + \lambda_2\lambda_3) \geq \\ &\geq 3(\lambda_1\lambda_2 + \lambda_3\lambda_1 + \lambda_2\lambda_3), \end{aligned}$$

this last inequality holds true since

$$\lambda_1^2 + \lambda_2^2 + \lambda_3^2 \geq \lambda_1\lambda_2 + \lambda_3\lambda_1 + \lambda_2\lambda_3,$$

and in this inequality equality holds if and only if $\lambda_1 = \lambda_2 = \lambda_3$. From Gauss's equation, we express the scalar curvature of the hypersurface M^3 as

$$scal = \lambda_1\lambda_2 + \lambda_3\lambda_1 + \lambda_2\lambda_3.$$

This yields $9H^2 \geq 3scal$. The normalized scalar curvature in dimension three is $\rho = \frac{1}{3}scal$, therefore $H^2 \geq \rho$, with equality if and only if the point is an umbilic. \square

Alternate proof 2: For some arbitrary point $p \in M^3$, consider the cubic equation that has as roots the real numbers $\lambda_1, \lambda_2, \lambda_3$. Namely, by using Viète's formulae, consider:

$$f(x) = x^3 - 3Hx^2 + scal \cdot x - K = 0.$$

This is a differentiable function in the variable x , and its derivative is

$$f'(x) = 3x^2 - 6Hx + scal = 0.$$

Since the variable x is real, the discriminant of this quadratic function must be nonnegative, thus

$$36H^2 - 4 \cdot 3 \cdot scal \geq 0,$$

thus $3H^2 - scal \geq 0$, which means we obtain again $H^2 \geq \rho$. \square

3. AN ESTIMATE FOR GAUSS-KRONECKER CURVATURE

In this section we apply the technique described by Cvetkovski in Chapter 15 in his recent interesting volume [8] to prove the following.

Theorem 3.1. *Let $M^3 \subset \mathbb{R}^4$ be a smooth hypersurface and $\lambda_1, \lambda_2, \lambda_3$ be its principal curvatures in the ambient space \mathbb{R}^4 endowed with the canonical metric. Let $p \in M$ be an arbitrary point. Denote by $H(p) = \frac{1}{3}(\lambda_1 + \lambda_2 + \lambda_3)$ the mean curvature, by $scal(p) = \sec(e_1 \wedge e_2) + \sec(e_3 \wedge e_1) + \sec(e_2 \wedge e_3) = \lambda_1(p)\lambda_2(p) + \lambda_3(p)\lambda_1(p) + \lambda_2(p)\lambda_3(p)$ the scalar curvature at the point $p \in M^3$, and by $K(p) = \lambda_1(p)\lambda_2(p)\lambda_3(p)$ the Gauss-Kronecker curvature of the hypersurface. Then*

$$\left(H + \frac{q}{3}\right) \left(-H^2 - \frac{1}{3}Hq + \frac{2}{3}scal\right) \leq K \leq \left(H - \frac{q}{3}\right) \left(-H^2 + \frac{1}{3}Hq + \frac{2}{3}scal\right),$$

where $q = \sqrt{9H^2 - 9\rho} = \sqrt{9H^2 - 3scal}$. The equality holds if and only if the point is semi-umbilical, i.e. if the following holds

$$(\lambda_1(p) - \lambda_2(p))(\lambda_3(p) - \lambda_1(p))(\lambda_2(p) - \lambda_3(p)) = 0.$$

Proof: Note first that $9H^2 - 9\rho \geq 0$ is a consequence of Chen's fundamental inequality with classical curvature invariants, explored in the previous section, so it makes sense to take $q^2 = 9H^2 - 9\rho$.

If $q = 0$ at $p \in M$, then, by the equality case in Chen's fundamental inequality with classical curvature invariants, p is an umbilical point. In this case, the double inequality in the statement turns out to be an identity:

$$K = -H^3 + \frac{2}{3}scal \cdot H,$$

which reduces immediately to $2\lambda^3 = 2\lambda^3$.

Now assume that p is not an umbilical point, i.e. $q > 0$ at $p \in M$.

Consider the function given by the real polynomial of degree three whose roots at every point at the principal curvatures of the hypersurface:

$$f(x) = (x - \lambda_1)(x - \lambda_2)(x - \lambda_3) = x^3 - 3Hx^2 + scal \cdot x - K.$$

Then the derivative of f is the quadratic function

$$f'(x) = 3x^2 - 6Hx + scal.$$

The equation $f'(x) = 0$ has the real roots:

$$x_{1,2} = \frac{6H \pm \sqrt{36H^2 - 12 \cdot scal}}{6} = H \pm \frac{q}{3}.$$

Denote by $x_1 = H + \frac{q}{3}$, and by $x_2 = H - \frac{q}{3}$, that is $x_2 \leq x_1$. Remark that the second derivative is

$$f''(x) = 6x - 6H,$$

and this yields

$$f''(x_1) = 6\left(H + \frac{q}{3}\right) - 6H = 2q > 0,$$

which means that f has a local minimum at x_1 . Similarly,

$$f''(x_2) = 6\left(H - \frac{q}{3}\right) - 6H = -2q < 0,$$

and this means that the original function f has a local maximum at x_2 . So, f is a polynomial of degree three, with three real roots. This means the local minimum

must have a nonpositive value, while the local maximum must have a nonnegative value:

$$f\left(\frac{3H-q}{3}\right) \geq 0, \quad f\left(\frac{3H+q}{3}\right) \leq 0.$$

By evaluating f at these two values corresponding to x_2 and x_1 , respectively, we have:

$$\begin{aligned} \left(\frac{3H-q}{3}\right)^3 - 3H\left(\frac{3H-q}{3}\right)^2 + scal \cdot \left(\frac{3H-q}{3}\right) - K &\geq 0, \\ \left(\frac{3H+q}{3}\right)^3 - 3H\left(\frac{3H+q}{3}\right)^2 + scal \cdot \left(\frac{3H+q}{3}\right) - K &\leq 0. \end{aligned}$$

By joining these two inequalities, we have:

$$\begin{aligned} \left(\frac{3H+q}{3}\right)^3 - 3H\left(\frac{3H+q}{3}\right)^2 + scal \cdot \left(\frac{3H+q}{3}\right) &\leq K \leq \\ &\leq \left(\frac{3H-q}{3}\right)^3 - 3H\left(\frac{3H-q}{3}\right)^2 + scal \cdot \left(\frac{3H-q}{3}\right). \end{aligned}$$

A direct computation shows that the upper bound is:

$$\begin{aligned} \left(H - \frac{q}{3}\right) \left[\left(H - \frac{q}{3}\right)^2 - 3H\left(H - \frac{q}{3}\right) + scal \right] &= \\ = \left(H - \frac{q}{3}\right) \left[H^2 - \frac{2}{3}Hq + \frac{q^2}{9} - 3H^2 + Hq + scal \right] &= \\ = \left(H - \frac{q}{3}\right) \left[-2H^2 + \frac{1}{3}Hq + \frac{9H^2 - 3 \cdot scal}{9} + scal \right]. \end{aligned}$$

This last expression reduces to the right hand side term in the double inequality stated in the Theorem. The expression in the lower bound is obtained by a similar computation.

To complete the proof, we need to discuss now when the equality holds in the two stated inequalities. Equality holds in either of the two inequalities when we have either

$$f\left(\frac{3H-q}{3}\right) = 0,$$

or

$$f\left(\frac{3H+q}{3}\right) = 0.$$

Suppose $f\left(\frac{3H-q}{3}\right) = 0$. Then $f(x) = (x - \lambda_1)(x - \lambda_2)(x - \lambda_3)$, which means that $\frac{3H-q}{3}$ should be one of the roots. Suppose $\frac{3H-q}{3} = \lambda_1$. Then we get $\lambda_2 + \lambda_3 - q = 2\lambda_1$. By squaring both sides in

$$\lambda_2 + \lambda_3 - 2\lambda_1 = \sqrt{9H^2 - 3 \cdot scal}$$

we obtain

$$3\lambda_1^2 + 3\lambda_2\lambda_3 - 3\lambda_1\lambda_2 - 3\lambda_1\lambda_3 = 0,$$

which factors as $(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3) = 0$, which means that at least a second principal curvature λ_2 or λ_3 must equal λ_1 , and this can be written as

$$(\lambda_1(p) - \lambda_2(p))(\lambda_3(p) - \lambda_1(p))(\lambda_2(p) - \lambda_3(p)) = 0.$$

The case $f\left(\frac{3H+q}{3}\right) = 0$ yields a similar conclusion. \square

There are many examples of hypersurfaces satisfying the semi-umbilicity condition at every point, e.g. $S^2 \times \mathbb{R}$, or $S^1 \times \mathbb{R}^2$.

A nontrivial example is mentioned by Hasanis and Vlachos in [11], in Proposition 4.1. In their example, for that class of hypersurfaces in \mathbb{R}^4 , the principal curvatures are $\lambda_1 = \lambda_2 = \frac{9H}{4}$, and $\lambda_3 = -\frac{3H}{2}$.

It is possible that some techniques in the classical theory of inequalities, as it is presented in classical references e.g. [10], may yield inequalities whose geometric interpretation encodes information in terms of curvature invariants. Some of these connections are still to be explored.

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