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EXAMPLES OF DEFORMATIONS OF NEARLY PARALLEL G_2 STRUCTURES ON 7-DIMENSIONAL 3-SASAKIAN MANIFOLDS BY CHARACTERISTIC VECTOR FIELDS

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Abstract. It is known that 7-dimensional 3-Sasakian manifolds admit nearly parallel G_2 structures [6, 1]. In this paper, we consider three of these nearly parallel G_2 structures on a 7-dimensional 3-Sasakian manifold given in [1]. We deform the fundamental 3-forms of the manifold by three characteristic vector fields of the Sasakian structure separately. Then we determine how classes of G_2 structures change.

1. Introduction

A G_2 structure on a 7-dimensional Riemannian manifold (M, g) is a reduction of the structure group of the frame bundle of M from $SO(7)$ to G_2. The Riemannian manifolds with G_2 structures are classified by Fernández and Gray. There are 16 classes with different defining relations [5]. An equivalent characterization of each class was done by Cabrera by using $d\varphi$ and $d * \varphi$ in [4].

There are several ways of obtaining new G_2 structures from a fixed G_2 structure. Consider the space of 3-forms on M, denoted by $\Lambda^3 M$. If M has G_2 structure φ, then this space may be written as $\Lambda^3 M = \Lambda^3_1 \oplus \Lambda^3_7 \oplus \Lambda^3_{27}$, where

$$\Lambda^1_3 = \{t\varphi | t \in \mathbb{R}\},$$
$$\Lambda^3_7 = \{*(\beta \wedge \varphi) | \beta \in \Lambda^1 M\} = \{\omega \lrcorner * \varphi \mid \omega \in \Gamma(TM)\},$$
$$\Lambda^3_{27} = \{\gamma \in \Lambda^3 M | \gamma \wedge \varphi = 0, \gamma \wedge *\varphi = 0\}$$

and Λ^l_k denotes a k-dimensional G_2-irreducible subspace of $\Lambda^l M$ and $\Gamma(TM)$ is the set of smooth vector fields on M. One way of constructing a new G_2 structure is to add an element of a subspace Λ^l_k to the fundamental 3-form φ of the manifold. Adding an element of Λ^1_3 to φ means conformally deforming the 3-form. Conformal deformations were studied by Fernández and Gray. It was observed how conformally changing the fundamental 3-form changes the class the manifold belongs to. In

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TABLE 1. Defining relations for classes of G_2 structures

$\mathcal{P}$	$d\varphi = 0$ and $d * \varphi = 0$
$\mathcal{W}_1$	$d\varphi = k * \varphi$ and $d * \varphi = 0$
$\mathcal{W}_2$	$d\varphi = 0$
$\mathcal{W}_3$	$d * \varphi = 0$ and $d\varphi \wedge \varphi = 0$
$\mathcal{W}_4$	$d\varphi = \alpha \wedge \varphi$ and $d * \varphi = \beta \wedge *\varphi$
$\mathcal{W}_1 \oplus \mathcal{W}_2$	$d\varphi = k * \varphi$ and $*d * \varphi \wedge *\varphi = 0$
$\mathcal{W}_1 \oplus \mathcal{W}_3$	$d * \varphi = 0$
$\mathcal{W}_2 \oplus \mathcal{W}_3$	$d\varphi \wedge \varphi = 0$ and $*d\varphi \wedge \varphi = 0$
$\mathcal{W}_1 \oplus \mathcal{W}_4$	$d\varphi = \alpha \wedge \varphi + f * \varphi$ and $d * \varphi = \beta \wedge *\varphi$
$\mathcal{W}_2 \oplus \mathcal{W}_4$	$d\varphi = \alpha \wedge \varphi$
$\mathcal{W}_3 \oplus \mathcal{W}_4$	$d\varphi \wedge \varphi = 0$ and $d * \varphi = \beta \wedge *\varphi$
$\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3$	$*d\varphi \wedge \varphi = 0$ or $*d * \varphi \wedge *\varphi = 0$
$\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_4$	$d\varphi = \alpha \wedge \varphi + f * \varphi$
$\mathcal{W}_1 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$	$d * \varphi = \beta \wedge *\varphi$
$\mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$	$d\varphi \wedge \varphi = 0$
$\mathcal{W}$	no relation

addition conformally invariant classes were determined [5]. Adding an element of $\Lambda^3_7 \simeq \{\omega \lrcorner * \varphi \mid \omega \in \Gamma(TM)\}$ gives a new G_2 structure on M with the fundamental 3-form $\widetilde{\varphi} = \varphi + \omega \lrcorner * \varphi$ for each vector field ω [7]. These deformations were studied in [7]. The new metric $\widetilde{g}$ and the new Hodge-star operator in terms of old ones were obtained as:

$$\widetilde{g}(u,v) = (1 + g(\omega,\omega))^{-2/3} \left(g(u,v) + g(u \times \omega, v \times \omega)\right),$$

where $\times$ is the cross product associated to the first G_2 structure and u, v are any vector fields,

$$\widetilde{*}\alpha = (1 + g(\omega,\omega))^{\frac{2-k}{3}} (*\alpha + (-1)^{k-1} \omega \lrcorner (*(\omega \lrcorner \alpha)))$$

where α is a k-form. Using this formula $\widetilde{*}\widetilde{\varphi}$ was written as:

$$\widetilde{*}\widetilde{\varphi} = (1 + g(\omega,\omega)^{-1/3})(*\varphi + *(\omega \lrcorner * \varphi) + \omega \lrcorner * (\omega \lrcorner \varphi)).$$

It was not studied, however, how deforming φ by an element of Λ^3_7 changes the class the manifold belongs to. In this paper our aim is to investigate how the class the manifold belongs to changes after deforming the fundamental 3-form by a vector field on specific examples.

2. Preliminaries

Let (M, g) be a 7-dimensional Riemannian manifold with G_2 structure φ. It is known that for each $p \in M$, $\nabla\varphi$ belongs to the space

$$W = \{\alpha \in T^*_p M \otimes \Lambda^3 T^*_p M \mid \alpha(x, y \wedge z \wedge (y \times z)) = 0 \text{ for all } x, y, z \in T_p M\}.$$

The space W can be decomposed as $W = W_1 \oplus W_2 \oplus W_3 \oplus W_4$ where W_i are G_2 irreducible subspaces [5]. A G_2 structure is said to be of type $\mathcal{P}$, $\mathcal{W}_i$, $\mathcal{W}_i \oplus \mathcal{W}_j$, $\mathcal{W}_i \oplus \mathcal{W}_j \oplus \mathcal{W}_k$ or $\mathcal{W}$, if the covariant derivative $\nabla\varphi$ lies in $\{0\}$, W_i, $W_i \oplus W_j$, $W_i \oplus W_j \oplus W_k$ or W, respectively, for $i, j, k = 1, 2, 3, 4$ [4]. Characterization of Cabrera is written in Table1 above. Note that $*d\varphi \wedge \varphi = - * d * \varphi \wedge *\varphi$, $\alpha = -\frac{1}{4} * (*d\varphi \wedge \varphi)$,

$\beta = -\frac{1}{3} * (*d\varphi \wedge \varphi)$ and $f = \frac{1}{7} * (\varphi \wedge d\varphi)$ [4]. Following definitions and properties can be found in [2, 3].

Definition 2.1. Let (M, g) be a Riemannan manifold with Levi-Civita covariant derivative ∇ of g. (M, g) is called a Sasakian manifold if there exists a Killing vector field of unit length with the property that the endomorphism defined by $\Phi := \nabla\xi$ satisfies

$$(\nabla_x \Phi)(y) = g(\xi, y)x - g(x, y)\xi$$

for all vector fields x, y.

The triple (ξ, η, Φ) where η is the metric dual of ξ is called a Sasakian structure on (M, g). The Killing vector field ξ and the one-form η are respectively called the characteristic vector field and the characteristic one-form of the Sasakian structure.

Any Sasakian manifold (M, g) with the Sasakian structure (ξ, η, Φ) satisfies the following relations:

$$\Phi \circ \Phi(y) = -y + \eta(y)\xi,$$
$$\Phi(\xi) = 0, \ \eta(\Phi(y)) = 0,$$
$$g(x, \Phi(y)) + g(\Phi(x), y) = 0,$$
$$g(\Phi(y), \Phi(x)) = g(y, x) - \eta(y)\eta(x),$$
$$d\eta(x, y) = 2g(\Phi(x), y)$$

for all vector fields x, y.

Definition 2.2. Let (M, g) be a Riemannan manifold. (M, g) is called a 3-Sasakian manifold if there are three Sasakian structures $(\xi_i, \eta_i, \Phi_i)_{i=1,2,3}$ such that $g(\xi_i, \xi_j) = \delta_{ij}$, $[\xi_1, \xi_2] = 2\xi_3$, $[\xi_2, \xi_3] = 2\xi_1$ and $[\xi_3, \xi_1] = 2\xi_2$.

Each 3-Sasakian manifold has the properties below:

$$\eta_i(\xi_j) = \delta_{ij},$$
$$\Phi_i(\xi_j) = -\varepsilon_{ijk}\xi_k,$$
$$\Phi_i \circ \Phi_j - \xi_i \otimes \eta_j = -\varepsilon_{ijk}\Phi_k - \delta_{ij} Id.$$

Let (M, g) be a 7-dimensional 3-Sasakian manifold with Sasakian structures (ξ_i, η_i, Φ_i) for $i = 1, 2, 3$. The tangent bundle TM can be written as $TM = T^v + T^h$, where T^v is the vertical subbundle spanned by $\{\xi_1, \xi_2, \xi_3\}$ and the horizontal subbundle T^h is the orthogonal complement of T^v [1]. For topological reasons, we assume that M is compact and simply-connected. The structure group of (M, g) is $SU(2) \subset G_2 \subset SO(7)$. In [1], a locally orthonormal frame $\{e_1, \cdots, e_7\}$ such that $e_1 = \xi_1$, $e_2 = \xi_2$ and $e_3 = \xi_3$ was given together with endomorphisms Φ_i acting on T^h by the following matrices:

$$\Phi_1 := \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad \Phi_2 := \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \quad \Phi_3 := \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

The corresponding orthonormal coframe is denoted by $\{\eta_1, \cdots, \eta_7\}$. According to this frame, the exterior derivatives of characteristic 1-forms are computed as:

$$d\eta_1 = -2(\eta_{23} + \eta_{45} + \eta_{67}), \quad d\eta_2 = 2(\eta_{13} - \eta_{46} + \eta_{57}), \quad d\eta_3 = -2(\eta_{12} + \eta_{47} + \eta_{56}).$$

The following three nearly parallel G_2 structures (i.e. $d\varphi_i = -4 * \varphi_i$) on 7-dimensional 3-Sasakian manifolds are expressed in [1]:

$$\varphi_1 = \frac{1}{2}\eta_1 \wedge d\eta_1 - \frac{1}{2}\eta_2 \wedge d\eta_2 - \frac{1}{2}\eta_3 \wedge d\eta_3,$$

$$\varphi_2 = -\frac{1}{2}\eta_1 \wedge d\eta_1 + \frac{1}{2}\eta_2 \wedge d\eta_2 - \frac{1}{2}\eta_3 \wedge d\eta_3,$$

$$\varphi_3 = -\frac{1}{2}\eta_1 \wedge d\eta_1 - \frac{1}{2}\eta_2 \wedge d\eta_2 + \frac{1}{2}\eta_3 \wedge d\eta_3.$$

3. Deformations of one of the nearly parallel G_2 Structures by Characteristic Vector Fields

Let (M, g) be a 7-dimensional 3-Sasakian manifold equipped with Sasakian structures $(\xi_i, \eta_i, \phi_i)_{i=1,2,3}$, having the nearly parallel G_2 structure φ_1 given in the previous section. On an open subset of M, we choose the local orthonormal frame given in [1] with the properties mentioned in the previous section. We denote the corresponding coframe via the Riemannian metric by $\{\eta_1, \cdots, \eta_7\}$. Now we deform φ_1 by three characteristic vector fields. We begin by deforming φ_1 by ξ_1. We write the new deformed 3-form $\widetilde{\varphi}_1 = \varphi_1 + \xi_1 \lrcorner * \varphi_1$ by

$$\widetilde{\varphi}_1 = \frac{1}{2}\eta_1 \wedge d\eta_1 - \frac{1}{2}\eta_2 \wedge d\eta_2 - \frac{1}{2}\eta_3 \wedge d\eta_3 + \frac{1}{2}\eta_3 \wedge d\eta_2 - \frac{1}{2}\eta_2 \wedge d\eta_3,$$

locally

$$\begin{aligned} \widetilde{\varphi}_1 \quad = \quad & \eta_{123} - \eta_{145} - \eta_{167} + \eta_{246} - \eta_{257} + \eta_{347} \\ & + \eta_{356} + \eta_{357} - \eta_{346} + \eta_{256} + \eta_{247}. \end{aligned}$$

Now we investigate the class the Riemannian manifold $\widetilde{M} := (M, \widetilde{g})$ belongs to. We compute

$$d\widetilde{\varphi}_1 = d\varphi_1 - d(*(\eta_1 \wedge \varphi_1)).$$

Since $*(\eta_1 \wedge \varphi_1) = -\frac{1}{2}\eta_3 \wedge d\eta_2 + \frac{1}{2}\eta_2 \wedge d\eta_3$, $d(*(\eta_1 \wedge \varphi)) = 0$ and thus $d\widetilde{\varphi}_1 = d\varphi_1$. This may locally be written as

$$d\widetilde{\varphi}_1 = 4\{-\eta_{1247} - \eta_{1256} + \eta_{1346} - \eta_{1357} + \eta_{2345} + \eta_{2367} - \eta_{4567}\}.$$

Since $d\widetilde{\varphi}_1 \neq 0$, the defining relations of G_2 structures imply $\widetilde{M} \notin \mathcal{P}$ and $\widetilde{M} \notin \mathcal{W}_2$.

Assume α is a 1-form on M such that $d\widetilde{\varphi}_1 = \alpha \wedge \widetilde{\varphi}_1$. This 1-form may be locally written as $\alpha = \sum \alpha_i \eta_i$ for smooth functions α_i. Then

$$\begin{aligned} \alpha \wedge \widetilde{\varphi}_1 \quad = \quad & -\alpha_4\eta_{1234} - \alpha_5\eta_{1235} - \alpha_6\eta_{1236} - \alpha_7\eta_{1237} + \alpha_2\eta_{1245} + \alpha_1\eta_{1246} \\ & + \alpha_1\eta_{1247} + \alpha_1\eta_{1256} - \alpha_1\eta_{1257} + \alpha_2\eta_{1267} + \alpha_3\eta_{1345} - \alpha_1\eta_{1346} \\ & + \alpha_1\eta_{1347} + \alpha_1\eta_{1356} + \alpha_1\eta_{1357} + \alpha_3\eta_{1367} + \alpha_6\eta_{1456} + \alpha_7\eta_{1457} \\ & + \alpha_4\eta_{1467} + \alpha_5\eta_{1567} - (\alpha_2 + \alpha_3)\eta_{2346} + (\alpha_2 - \alpha_3)\eta_{2347} \\ & + (\alpha_2 - \alpha_3)\eta_{2356} + (\alpha_2 + \alpha_3)\eta_{2357} + (\alpha_5 - \alpha_4)\eta_{2456} + (\alpha_4 + \alpha_5)\eta_{2457} \\ & + (\alpha_6 - \alpha_7)\eta_{2467} - (\alpha_6 + \alpha_7)\eta_{2567} - (\alpha_4 + \alpha_5)\eta_{3456} + (\alpha_5 - \alpha_4)\eta_{3457} \\ & + (\alpha_6 + \alpha_7)\eta_{3467} + (\alpha_6 - \alpha_7)\eta_{3567}. \end{aligned}$$

The coefficient of η_{2345} in $d\widetilde{\varphi}_1$ is 4, while in $\alpha \wedge \widetilde{\varphi}_1$ there does not exist an η_{2345} term. Thus there is no such 1-form α with the property $d\widetilde{\varphi}_1 = \alpha \wedge \widetilde{\varphi}_1$ even locally. Hence $\widetilde{M} \notin \mathcal{W}_4$ and $\widetilde{M} \notin \mathcal{W}_2 \oplus \mathcal{W}_4$.

We apply the Hodge-star formula for a k-form given in the introduction to the 3-form $\widetilde{\varphi}_1$ and also the identities given in the appendix of [7] to obtain $\widetilde{*}\widetilde{\varphi}_1$:

$$\begin{aligned}\widetilde{*}\widetilde{\varphi}_1 &= 2^{-1/3}\{*\varphi_1 + *(\xi_1\lrcorner * \varphi_1) + \xi_1\lrcorner * (\xi_1\lrcorner\varphi_1)\}\\ &= 2^{-1/3}\{*\varphi - \eta_1\wedge\varphi_1 + *(\eta_1\wedge *(\eta_1\wedge *\varphi_1))\}\\ &= 2^{-1/3}\{-\tfrac{1}{4}d\eta_1\wedge d\eta_1 + \tfrac{1}{8}d\eta_2\wedge d\eta_2 + \tfrac{1}{8}d\eta_3\wedge d\eta_3\\ &\quad +2*\eta_{123} + \tfrac{1}{2}\eta_{12}\wedge d\eta_2 + \tfrac{1}{2}\eta_{13}\wedge d\eta_3\}.\end{aligned}$$

Local expression of $\widetilde{*}\widetilde{\varphi}_1$ is

$$\begin{aligned}\widetilde{*}\widetilde{\varphi}_1 &= 2^{-1/3}\{-\eta_{1246} + \eta_{1247} + \eta_{1256} + \eta_{1257} - \eta_{1346} - \eta_{1347}\\ &\quad -\eta_{1356} + \eta_{1357} - 2\eta_{2345} - 2\eta_{2367} + 2\eta_{4567}\}.\end{aligned}$$

Assume $d\widetilde{\varphi}_1 = k\widetilde{*}\widetilde{\varphi}_1$ for a non-zero constant k. Comparing the coefficients of η_{1256} and η_{4567} in $d\widetilde{\varphi}_1$ and $k\widetilde{*}\widetilde{\varphi}_1$ respectively gives $-4 = 2^{-1/3}k$ and $-4 = 2^{2/3}k$, a contradiction. Thus there is no such non-zero constant. This eliminates the classes $\mathcal{W}_1$ and $\mathcal{W}_1\oplus\mathcal{W}_2$.

The exterior derivative of $\widetilde{*}\widetilde{\varphi}_1$ is

$$\begin{aligned}d\widetilde{*}\widetilde{\varphi}_1 &= 2^{-1/3}\left\{\tfrac{1}{2}\eta_2\wedge d\eta_1\wedge d\eta_2 - \tfrac{1}{2}\eta_1\wedge d\eta_2\wedge d\eta_2\right.\\ &\quad \left.+\tfrac{1}{2}\eta_3\wedge d\eta_1\wedge d\eta_3 - \tfrac{1}{2}\eta_1\wedge d\eta_3\wedge d\eta_3\right\}.\end{aligned}$$

This is locally equivalent to $d\widetilde{*}\widetilde{\varphi}_1 = 2^{5/3}\eta_{12345} + 2^{5/3}\eta_{12367} - 2^{8/3}\eta_{14567}$ which is nonzero.

Let β be a smooth 1-form on M satisfying $d\widetilde{*}\widetilde{\varphi}_1 = \beta\wedge\widetilde{*}\widetilde{\varphi}_1$. Then β may be locally written as $\beta = \sum\beta_i\eta_i$ for smooth functions β_i.

$$\begin{aligned}2^{1/3}\beta\wedge\widetilde{*}\widetilde{\varphi} &= -2\beta_1\eta_{12345} + (\beta_2-\beta_3)\eta_{12346} + (\beta_2+\beta_3)\eta_{12347}\\ &\quad +(\beta_2+\beta_3)\eta_{12356} + (\beta_3-\beta_2)\eta_{12357} - 2\beta_1\eta_{12367}\\ &\quad +(\beta_4+\beta_5)\eta_{12456} + (\beta_4-\beta_5)\eta_{12457} - (\beta_6+\beta_7)\eta_{12467}\\ &\quad +(\beta_7-\beta_6)\eta_{12567} + (\beta_5-\beta_4)\eta_{13456} + (\beta_4+\beta_5)\eta_{13457}\\ &\quad +(\beta_6-\beta_7)\eta_{13467} - (\beta_6+\beta_7)\eta_{13567} + 2\beta_1\eta_{14567}\\ &\quad -2\beta_6\eta_{23456} - 2\beta_7\eta_{23457} - 2\beta_4\eta_{23467}\\ &\quad -2\beta_5\eta_{23567} + 2\beta_2\eta_{24567} + 2\beta_3\eta_{34567}.\end{aligned}$$

Comparing the coefficients of η_{12345} and η_{14567} in $2^{1/3}d\widetilde{*}\widetilde{\varphi}_1$ and $2^{1/3}\beta\wedge\widetilde{*}\widetilde{\varphi}_1$ respectively gives $\beta_1 = -2$, $\beta_1 = -4$. This is a contradiction. Hence there does not exist a 1-form β satisfying the defining relation $d\widetilde{*}\widetilde{\varphi}_1 = \beta\wedge\widetilde{*}\widetilde{\varphi}_1$. This means that $\widetilde{M}\notin\mathcal{W}_1\oplus\mathcal{W}_4$.

If we take $\alpha = -2\eta_1$ and $f = -2^{4/3}$, then direct computation yields the equality $d\widetilde{\varphi}_1 = \alpha\wedge\widetilde{\varphi}_1 + f\widetilde{*}\widetilde{\varphi}_1$. Hence $\widetilde{M}\in\mathcal{W}_1\oplus\mathcal{W}_2\oplus\mathcal{W}_4$.

Next we deform the 3-form φ_1 by ξ_2. The new deformed 3-form $\widetilde{\varphi}_1 = \varphi_1 + \xi_2\lrcorner*\varphi_1$ is

$$\widetilde{\varphi}_1 = \frac{1}{2}\eta_1\wedge d\eta_1 - \frac{1}{2}\eta_2\wedge d\eta_2 - \frac{1}{2}\eta_3\wedge d\eta_3 + \frac{1}{2}\eta_3\wedge d\eta_1 + \frac{1}{2}\eta_1\wedge d\eta_3$$

which is locally

$$\begin{aligned}\widetilde{\varphi}_1 &= \eta_{123} - \eta_{145} - \eta_{167} + \eta_{246} - \eta_{257} + \eta_{347}\\ &\quad +\eta_{356} - \eta_{345} - \eta_{367} - \eta_{147} - \eta_{156}.\end{aligned}$$

We compute $\widetilde{*}\widetilde{\varphi}_1$:

$$\begin{aligned}\widetilde{*}\widetilde{\varphi}_1 &= 2^{-1/3}\{*\varphi_1 + *(\xi_2\lrcorner * \varphi_1) + \xi_2\lrcorner * (\xi_2\lrcorner\varphi_1)\}\\ &= 2^{-1/3}\{*\varphi - \eta_2\wedge\varphi_1 + *(\eta_2\wedge *(\eta_2\wedge *\varphi_1))\}.\end{aligned}$$

Note that

$$*\varphi_1 = -\frac{1}{8}d\eta_1 \wedge d\eta_1 + \frac{1}{8}d\eta_2 \wedge d\eta_2 + \frac{1}{8}d\eta_3 \wedge d\eta_3,$$

$$\eta_2 \wedge \varphi_1 = \frac{1}{4}d\eta_1 \wedge d\eta_3$$

and

$$*(\eta_1 \wedge *(\eta_1 \wedge *\varphi_1)) = \frac{1}{8}d\eta_2 \wedge d\eta_2.$$

This yields $d\widetilde{*}\widetilde{\varphi}_1 = 0$. The new G_2 structure can be an element of $\mathcal{P}$, $\mathcal{W}_1$, $\mathcal{W}_3$ or $\mathcal{W}_1 \oplus \mathcal{W}_3$. The exterior derivative of $\widetilde{\varphi}_1$ is

$$d\widetilde{\varphi}_1 = \frac{1}{2}d\eta_1 \wedge d\eta_1 - \frac{1}{2}d\eta_2 \wedge d\eta_2 - \frac{1}{2}d\eta_3 \wedge d\eta_3 + d\eta_1 \wedge d\eta_3.$$

This may locally be written as

$$\begin{aligned} d\widetilde{\varphi}_1 &= 4\{\eta_{1245} - \eta_{1247} + \eta_{1267} - \eta_{1256} + \eta_{1346} - \eta_{1357} \\ &\quad +\eta_{2345} + \eta_{2347} + \eta_{2356} + \eta_{2367} - \eta_{4567}\}. \end{aligned}$$

Since $d\widetilde{\varphi}_1 \neq 0$, the class $\mathcal{P}$ is eliminated.

Assume $d\widetilde{\varphi}_1 = k\widetilde{*}\widetilde{\varphi}_1$ for a on-zero constant k. Since

$$\begin{aligned} \widetilde{*}\widetilde{\varphi}_1 &= 2^{-1/3}\{-\tfrac{1}{8}d\eta_1 \wedge d\eta_1 + \tfrac{1}{8}d\eta_2 \wedge d\eta_2 + \tfrac{1}{8}d\eta_3 \wedge d\eta_3 \\ &\quad -\tfrac{1}{4}d\eta_1 \wedge d\eta_3 + \tfrac{1}{8}d\eta_2 \wedge d\eta_2\} \end{aligned}$$

or, locally

$$\begin{aligned} \widetilde{*}\widetilde{\varphi}_1 &= 2^{-1/3}\{-\eta_{1245} + \eta_{1247} + \eta_{1256} - \eta_{1267} - 2\eta_{1346} + 2\eta_{1357} \\ &\quad -\eta_{2345} - \eta_{2347} - \eta_{2356} - \eta_{2367} + 2\eta_{4567}\}, \end{aligned}$$

comparing the coefficients of η_{1245} and η_{1346} in $d\widetilde{\varphi}_1$ and $k\widetilde{*}\widetilde{\varphi}_1$ respectively gives $4 = -2^{-1/3}k$ and $4 = -2^{2/3}k$, a contradiction. Thus there is no such non-zero constant. This excludes the class $\mathcal{W}_1$.

Computed locally, $d\widetilde{\varphi}_1 \wedge \widetilde{\varphi}_1 = -44\ \eta_{1234567} \neq 0$ and hence $M \notin \mathcal{W}_3$. Therefore M is in the class $\mathcal{W}_1 \oplus \mathcal{W}_3$ with the new G_2 structure $\widetilde{\varphi}_1$.

Finally we deform the 3-form φ_1 by ξ_3. The new deformed 3-form $\widetilde{\varphi}_1 = \varphi_1 + \xi_3 \lrcorner * \varphi_1$ is

$$\widetilde{\varphi}_1 = \frac{1}{2}\eta_1 \wedge d\eta_1 - \frac{1}{2}\eta_2 \wedge d\eta_2 - \frac{1}{2}\eta_3 \wedge d\eta_3 - \frac{1}{2}\eta_2 \wedge d\eta_1 - \frac{1}{2}\eta_1 \wedge d\eta_2$$

which is locally

$$\begin{aligned} \widetilde{\varphi}_1 &= \eta_{123} - \eta_{145} - \eta_{167} + \eta_{246} - \eta_{257} + \eta_{347} \\ &\quad +\eta_{356} + \eta_{267} + \eta_{245} - \eta_{157} + \eta_{146}. \end{aligned}$$

We compute $\widetilde{*}\widetilde{\varphi}_1$:

$$\begin{aligned} \widetilde{*}\widetilde{\varphi}_1 &= 2^{-1/3}\{*\varphi_1 + *(\xi_3 \lrcorner * \varphi_1) + \xi_3 \lrcorner * (\xi_3 \lrcorner \varphi_1)\} \\ &= 2^{-1/3}\{*\varphi_1 - \eta_3 \wedge \varphi_1 + *(\eta_3 \wedge *(\eta_3 \wedge *\varphi_1))\}. \end{aligned}$$

Since

$$*\varphi_1 = -\frac{1}{8}d\eta_1 \wedge d\eta_1 + \frac{1}{8}d\eta_2 \wedge d\eta_2 + \frac{1}{8}d\eta_3 \wedge d\eta_3,$$

$$\eta_3 \wedge \varphi_1 = -\frac{1}{4}d\eta_1 \wedge d\eta_2$$

and

$$*(\eta_3 \wedge *(\eta_3 \wedge *\varphi_1)) = \frac{1}{8}d\eta_3 \wedge d\eta_3,$$

we have $d\widetilde{*\varphi}_1 = 0$. The new G_2 structure can be an element of $\mathcal{P}$, $\mathcal{W}_1$, $\mathcal{W}_3$ or $\mathcal{W}_1 \oplus \mathcal{W}_3$. The exterior derivative of $\widetilde{\varphi}_1$ is

$$d\widetilde{\varphi}_1 = \frac{1}{2}d\eta_1 \wedge d\eta_1 - \frac{1}{2}d\eta_2 \wedge d\eta_2 - \frac{1}{2}d\eta_3 \wedge d\eta_3 - d\eta_1 \wedge d\eta_2.$$

This may be locally written as

$$\begin{aligned} d\widetilde{\varphi}_1 &= 4\{-\eta_{1247} - \eta_{1256} + \eta_{1345} + \eta_{1346} - \eta_{1357} + \eta_{1367} \\ &\quad +\eta_{2345} - \eta_{2346} + \eta_{2357} + \eta_{2367} - \eta_{4567}\}. \end{aligned}$$

Since $d\widetilde{\varphi}_1 \neq 0$, the class $\mathcal{P}$ is eliminated.

Assume $d\widetilde{\varphi}_1 = k\widetilde{*\varphi}_1$ for a non-zero constant k. Since

$$\begin{aligned} \widetilde{*\varphi}_1 &= 2^{-1/3}\{-\tfrac{1}{8}d\eta_1 \wedge d\eta_1 + \tfrac{1}{8}d\eta_2 \wedge d\eta_2 + \tfrac{1}{8}d\eta_3 \wedge d\eta_3 \\ &\quad +\tfrac{1}{4}d\eta_1 \wedge d\eta_2 + \tfrac{1}{8}d\eta_3 \wedge d\eta_3\} \end{aligned}$$

or, locally

$$\begin{aligned} \widetilde{*\varphi}_1 &= 2^{-1/3}\{2\eta_{1247} + 2\eta_{1256} - \eta_{1345} - \eta_{1346} + \eta_{1357} - \eta_{1367} \\ &\quad -\eta_{2345} + \eta_{2346} - \eta_{2357} - \eta_{2367} + 2\eta_{4567}\}, \end{aligned}$$

comparing the coefficients of η_{1247} and η_{1357} in $d\widetilde{\varphi}_1$ and $k\widetilde{*\varphi}_1$ respectively gives $4 = -2^{2/3}k$ and $4 = -2^{-1/3}k$, a contradiction. Thus there is no such non-zero constant. This excludes the class $\mathcal{W}_1$.

Computed locally, $d\widetilde{\varphi}_1 \wedge \widetilde{\varphi}_1 = -44\ \eta_{1234567} \neq 0$ and hence $M \notin \mathcal{W}_3$. Therefore M is in the class $\mathcal{W}_1 \oplus \mathcal{W}_3$ with the new G_2 structure $\widetilde{\varphi}_1$.

To sum up, if we take the nearly parallel structure φ_1 in a 7-dimensional 3-Sasakian manifold and deform this structure by the characteristic vector fields ξ_1, ξ_2, ξ_3 of the Sasakian structure, we get new G_2 structures of types $\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_4$, $\mathcal{W}_1 \oplus \mathcal{W}_3$ and $\mathcal{W}_1 \oplus \mathcal{W}_3$ respectively. Similarly, if we deform φ_2 (respectively φ_3) by ξ_1 and ξ_3 (resp. by ξ_1 and ξ_2), we get G_2 structures of type $\mathcal{W}_1 \oplus \mathcal{W}_3$. If we deform φ_2 (resp. φ_3) by ξ_2 (resp. ξ_3), we get G_2 structures of type $\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_4$ such that $d\widetilde{\varphi}_i = \alpha \wedge \widetilde{\varphi}_i + f\widetilde{*\varphi}_i$ hold for $\alpha = -2\eta_i$ and $f = -2^{4/3}$, for $i = 2, 3$.

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