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Research Article

Students' ways of thinking about rate of change

Öğrencilerin değişim oranına ilişkin düşünme yolları

Araştırma Makalesi

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Abstract

Rate of change is an important subject for understanding functions. In fact rate of change can be thought as a function itself. Because reasoning about rate of change of one quantity with respect to another quantity is the basis of the function concept. The aim of this research is to determine the 8th graders' ways of thinking about rate of change. The data were gathered using a teaching experiment methodology. The tasks used within teaching experiment were formed to explain students understanding rate of change. The results show that students' ways of thinking can be categorized as non-quantitative rate of change and quantitative rate of change. After completing the teaching experiment, the students moved from non-quantitative to quantitative rate of change.

Keywords: rate of change, teaching experiment, ways of thinking

Özet

Değişim oranı, fonksiyonların anlaşılmasında önemli bir konudur. Aslında değişim oranı da bir fonksiyon gibi düşünülebilir. Çünkü bir niceliğin diğerine göre değişim oranı hakkında muhakeme etmek fonksiyon kavramının temelini oluşturmaktadır. Bu çalışmanın amacı 8. sınıf öğrencilerinin değişimin oranı konusunda sahip oldukları düşünme yollarını belirlemektir. Çalışmanın verileri uygulanan öğretim deneyi süresince elde edilmiştir. Öğretim deneyinde uygulanan görevler ise öğrencilerin değişimin oranı konusunda anlayışlarını ortaya koyacak şekilde oluşturulmuştur. Çalışmanın sonuçlarına göre öğrencilerin düşünme yolları niceliksel olmayan değişim oranı ve niceliksel değişim oranı şeklinde kategorize edilebilir. Öğretim deneyinin sonucunda öğrencilerin düşünme yolları, niceliksel olmayan değişim oranından niceliksel değişim oranına doğru değişim göstermiştir.

Anahtar Kelimeler: değişim oranı, öğretim deneyi, düşünme yolları

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INTRODUCTION

For students to understand functions, they must be able to understand and comment on how quantities change according to each other (Kalchman&Koedinger, 2005). Accordingly, rate of change should be understood to develop covariational thinking (Carlson et al., 2002). Rate of change is a concept used to qualitatively and quantitatively interpret how two different variables relate to each other (Thompson, 1994b). A change of one variable according to other variables, i.e., rate of change, is the basis for interpreting variables of functions in terms of covariation.

The concept of rate of change is fundamental to understanding functions and differential equations. Additionally, this concept is important to understand biology, physics, chemistry, and relationships between changing variables in economy (Hauger, 1995; Thompson, 1994b). Since this concept is important, researchers identified ways of thinking on rate of change. Thompson and Thompson (1992), and Thompson (1994a) interpreted rate of change as multiplicative relationship that compares changes between functional relations and variables. Rate of change is highly related with ratio and proportion. Therefore, to understand rate of change, ratio should be understood. Thompson and Thompson (1992) stated four levels that shows development of ratio/proportion concepts among students. These levels are *ratio, internalised ratio, interiorised ratio,* and *rate*. According to Confrey and Smith (1994), rate of change can be interpreted in two different ways, namely, multiplicative and additive. Additive interpretation of rate of change is proportioning change of function to change of independent variables. Multiplicative interpretation is related with how much a dependent variable increases based on a unit change in an independent variable. Based on this interpretation of rate of change, ways of thinking for exponential rate of change of students was determined. Accordingly, ways of thinking of students regarding rate of change can be classified as *additive rate of change*, *multiplicative rate of change*, and *proportion new tool rate of change* (Confrey & Smith, 1994). Hauger (1995) defined a framework for students' understanding of the concept of rate of change. Accordingly, the rate of change development of students can be observed as *global, interval* and *point-wise stages*. Global rate of change handles increasing, decreasing and constant, faster/slower increasing or decreasing function properties. Interval rate of change indicates rate of change and average change on dependent variables between independent variable ranges. Point-wise rate of change indicates the speed of how independent variables effect dependent variables (Hauger, 1995). Weber (2012) worked on determining ways of thinking of students on rate of change. Results of this study showed that ways of thinking of students regarding rate of change were identified as *non-quantitative rate of change*, and *quantitative rate of change*. Non-quantitative rate of change ignores comparison between changing quantities and interprets rate of change as a number. On the other hand, quantitative rate of change is the covariation that defines the speed of change of a variable based on change of another variable (Weber, 2012).

Although the importance of rate of change was emphasised, it is known that students face challenges (Bezuidenhout, 1998; Carlson, Larsen, & Jacobs, 2001; Carlson et al., 2003; Ellis, 2009). As researchers and teachers, it is believed that ways of thinking of students should be identified to help students with their problems and forming knowledge on this subject. The aim of this study is to determine ways of thinking of students regarding rate of change. In this study, ways of thinking that was presented by Weber (2012) about rate of change was adopted.

METHOD

This study consists of one section of a more comprehensive study to determine ways of thinking of students about generalising act. Under the scope of this study, ways of thinking of students based on rate of change were reported. A teaching experiment was designed and applied to students to identify ways of thinking. Teaching experiments are conceptual tools to design teaching applications. How mathematical knowledge is formed and the development of knowledge in the process are researched in the teaching experiments (Steffe & Thompson, 2000). We can not have any information about students' knowledge unless investigating their mathematical actions at the instruction process. Therefore, teaching experiments should include teaching (Steffe & Ulrich, 2013).

Participants

The teaching experiment was conducted on 9 eighth graders who were at a state secondary school at 2016. Participants were selected using criterion sampling, which is studying the situations that meet the criteria set by the researcher according to the purpose of the study (Yıldırım ve Şimşek, 2005). To select the participants, operational and conceptual algebra tests were developed and applied to 167 eighth graders. Participants were selected among these students who had high scores in operational and conceptual algebra tests. Besides, the ideas of their mathematics teacher were taken in order to determine the students who can express their thoughts about the solutions of the problems easily. So 4 boys and 5 girls were selected as participants. Pseudonyms were used for these students.

Data collection tools

The teaching experiment was developed (Author, 2017) and applied to students to determine their ways of thinking about generalising act. This study contains the findings of two of those tasks. The aim of this study is to determine reasoning of students about rate of change and how they interpret the changing quantities with respect to each other. So the tasks, which were analysed under this study, were related with reasoning of quantities given on the table in the context of a burning candle and distance from home versus time.

TASK 1: How Many Candles Do You Need?

Zaman (dk)	Boy (cm)
0	20
1	18
2	16
3	14
3,5	13
4,5	11
6	8

A utility company in your city announced that tomorrow there will be a 40-minute power cut at 09:00. You have 3 identical candles and the cm of each candle and how long each will burn is given on table.

Accordingly:

1. Which patterns did you recognise on the table? For example, what is the relationship between the length of the candle and how long it is burnt, what is the length of the candle before it was burnt?
2. Based on the timeframe, do you think the candle should be shortened with equal spacing? Explain your reasoning...
3. When will a candle completely burn down? Find a way to benefit from candles during 40-minutes power cut based on the fact that you have 3 candles.
4. Try to draw a length-time graph of candle. Determine dependent and independent variables of the graph and name the graph.
5. Draw a length-time graph of a candle with 30 cm length and burns with equal timeframe of the candle above. Explain your reasoning...
6. Draw length-time graph of a candle with 30 cm length and burns two times faster than the candle above. Explain your reasoning...
7. Let's assume that in the 4th question, you decided to buy a candle to benefit from candle light during the power-cut. The length-time graph of this candle is given in the table. Accordingly,
    a. Based on the timeframe, do you think candle shortens with equal spacing (in unit time)? Explain your reasoning...
    b. How long will it take for the new candle to burn down, please explain?
    c. Are shortening amounts of the new candle and current candle are the same for same timeframe (unit time)? Please explain...

Zaman (dk)	Boy (cm)
0	24
1,5	21
3	18
4	16
4,5	15

TASK 2: Has the distance made in time been increased or decreased?

The change of the distance made from home according to the time as Feyza goes to school has been given at the table. So then:

1. Does the distance Feyza is making change, increase or decrease according to the time? Give a brief explanation to your answer...

2. Has Feyza's speed changed, increased or decreased? Give a brief explanation to your answer...

3. Let's assume that Feyza is making 16 meters in 10 seconds. And Melike, who is going the same distance as Feyza at the same time gaps, goes .... meters in ...... seconds. Give values to the blanks and explain why wrote down those numbers.

Form 3 statements about the distance that Melike has made in certain seconds.

Zaman (sn)	Uzaklık (m)
0	2
30	20.75
10	8.25
4	4.5
24	17

The sessions were carried out by the researcher at a mathematics classroom in the school. The researcher was in the role of the teacher in order to see the development of the mathematical knowledge of the students (Cobb & Steffe, 1983). Participants of the teaching experiments worked in three groups with three students in each group. To analyse the work of these students in detail, one camera for each group and one camera for the class were positioned. A total of four cameras were used for recording the sessions. The tasks were given to the students as work sheets in each session. Students were asked to show their work on these work sheets. Thus, the data of the study consists of work sheets and camera records. A video camera image belonging to one of the groups is given below to give an idea of how the sessions took place.

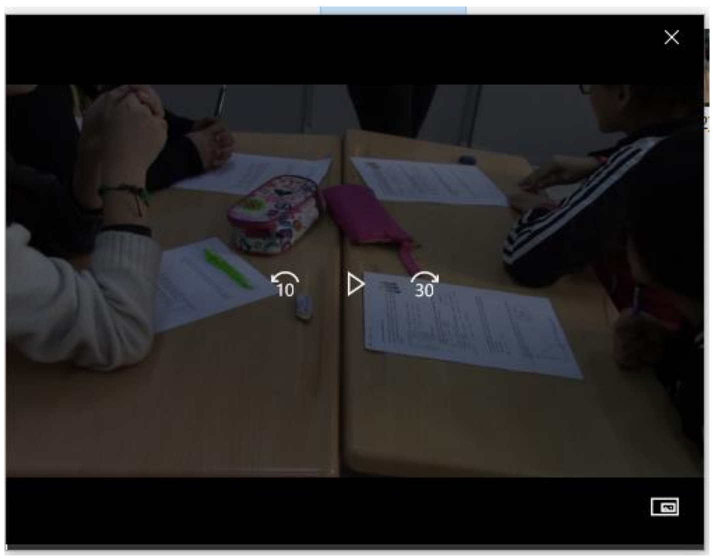


Figure 1: A section of one of the sessions.

Data Analysis

Data was described using descriptive analysis. In analysing the data, the framework determined by Weber (2012) was used. Work sheets of students were analysed with NVivo 8. Camtasia Studio 8 was used for analysing camera recordings. Powell, Francisco and Maher (2003) provided a model to analyse the video recordings to identify development of mathematical thinking among students. This model was created to analyse development of mathematical thinking and contains 7 interrelated stages. The video recording of this study were analysed based on this model.

Validity and Reliability

In teaching experiments design, practice and measurement processes of the draft are carried out over time and support each other (Design Based Research Collective, 2003). The ideas of 2 mathematics teachers and 5 mathematics lecturers were taken into account when forming the tasks. So the validity of the study has been ensured. The data of this study consists of work sheets, camera records and researcher logs. In this way, triangulation is made by data obtained from many different sources supporting each other. Triangulation is an important technique to ensure the reliability and internal validity of the study (Miles & Huberman, 1994).

FINDINGS

During the teaching experiment, tasks to analyse change were applied to the students. Under the scope of the study, the task that questions the relationship between time and length of a burning candle was investigated. Students tried to find patterns of change of length based on time by analysing information on the table.

Researcher (R): What kind of patterns did you see on the table?

Gül: As minutes increased, length decreased. Candle was burnt.

Bartu: Candle was 20 cm at the beginning, after 1 minute it was 18 cm. It burnt 2 cm. After 2 minute it was 16 cm. It burnt 2 cm again. This is happened all the time. Numbers on x and y columns of the table show this.

Oğuz: Between minute 3 and 3.5, the candle burnt 1 cm. If it burns 1 cm in half a minute, it will burn 2 cm in 1 minute. This rules applies here.

Students analysed values on time column and length column separately and together. Accordingly, they stated that candle burns 1 cm in half a minute, and 2 cm in one minute.

R: Can you find a rule between time and length of candle?

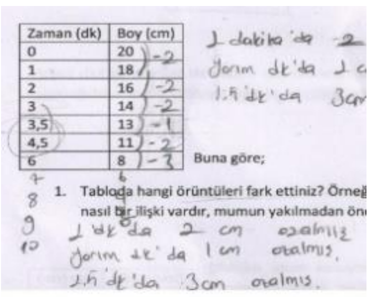
Oğuz: $(2n-4)/2$. *If we place 20 to n, we have 18.*

R: What is n?

Oğuz: Well...sorry, I mixed it. There should be a rule between length and time, I only found length.

Oğuz found n-2 rule by considering numbers on length column. When he placed 20 on n, he obtained the next number 18. In the previous tasks of teaching experiment, relationships that should be looked for on a table were mentioned. On a table, relationships between x values, y values, x and y values should be analysed. However, Oğuz only analysed one variable. However, since this operation is done in general to find a rule in mathematics class, this could be normal.

For how long it will take the candle to burn, students continue the operations by considering the candle burns down 2 cm per minute and found how long it will take.



Zaman (dk)	Boy (cm)
0	20
1	18
2	16
3	14
3,5	13
4,5	11
6	8

Buna göre;

1. Tabloda hangi örüntüleri fark ettiniz? Örneğ
nasıl bir ilişki vardır, mumun yakılmadan ön

Figure 2.1 Work sheet of Sezen

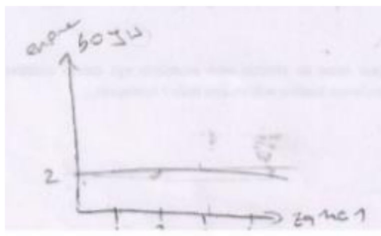


Figure 3. Work sheet of Gül

Gül had drawn time vs burning rate graph on contrary to her friends when the question to draw the graph of time and length of candle was asked. Sezen stated that this graph of Gül is wrong. She explained this situation as "at your graph melting is constant with 2 cm. At not all of them melting is 2 cm. It melted 2 cm in first three minutes. But later it melted 1 cm in half a minute. It melted 3 cm in 1.5 minutes". Sezen didnot see the information that suggests the candle melts 2 cm per minute in Gül's graph and drew the following:

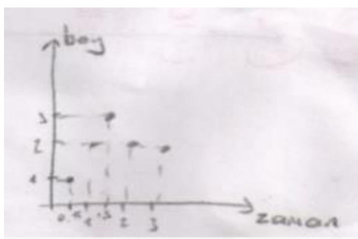


Figure 4.2 Graph of Sezen

Sezen marked relationship between time and length values and showed as dot chart. This could be interpreted as Sezen struggled to interpret the oral statements in graph form.

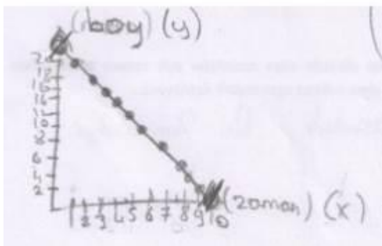


Figure 5.3 Work sheet of Koray

Koray: I drew it like this, and circled the end points like yesterday. Because it ends here, length is 0. Since it completely burnt on 10th minute and it can no longer burn, I circled it and ended the line.

Koray drew the graph by relating to previous question. Since the candle can no longer burn down after it is completely burned down, he circled the end points of the line. He cut the line and excluded these points. However, this question was different than previous one. In this question, (10, 0) and (0, 20) on each axis should be included to line. Because length of the candle was 20 cm on 0th second, and 0 cm on 10th second. However, Koray who formed an incorrect reasoning by relating to the previous question, excluded the information of 0 cm on 10th minute.

R: Do you think we can match these points on a line?

Gül: Of course. They all decreased in the same way. (She draws a declining line)

Ali: No, we can't. Because, the length cannot be minus, it can only be 0.

Oğuz: No teacher, the line will go to infinity. It will only be line segment. We will draw a line segment and cut it there.

Gül: We did this in the previous question, since the slope is 2, we can connect all points.

Koray: For example, yesterday we said that there is no such thing as half-circumference, but today half minutes are okay. Quarter minutes can happen as well. Each point on this line will fit our rule.

Koray and Oğuz stated the ending points of line as "cutting the line" and indicated these points by drawing circles. Students related back this question with a similar one that they studied on previous days and stated that for all x values, all y values would fit the rule. Additionally, Gül stated with "they all decreased in the same manner" expression that dots were linear, had same slope and thus, could be connected with line. Based on the fact that length of candle cannot have negative value, students stated that this should be line segment.

R: Consider that you are drawing length-time graph of a candle with 30 cm length and burns with equal timeframe of the candle. What can you say about slope without drawing graphic?

…

Ali: 2. Because when vertical (y column) is 30, and horizontal (burning time) is 15, it would be 30/15=2

…

Oğuz: We said dependent and independent on the graph, our dependent is y-length, and independent x-time. This is the same graphic as the one before. y decreases 2 and time increases 1. So, the slope is 2/1 = 2.

Ali drew the graph of the line and found intersection points to be (0, 30) and (15, 0). With knowledge of intersection point of x axis which is (15, 0) and candle that burned equally in equal timeframe, he calculated 30/2. This means, by accepting that a 30 cm candle would burn 2 cm per unit time, he divided the length to time, and found 2 cm burn rate per unit time again. Slope was found in similar manner. Since there was same length decrease in same timeframe, Oğuz formed a proportion between 2-unit change on y axis and 1-unit change on x axis.

When it comes to the task which is about the change of Feyza's distance to school depending on time, students have started to study on relations between quantities on the table. First of all, they have suggested to put the numbers in order at the column that are given randomly.

R: You have implied that the distance to home increases as time passes. What can you say about the distance that Feyza makes per unit?

...

Bartu: The distance she makes in a second is, ummm, we need to consider the difference at each column.

Oğuz: Distance has increased 2,5 every 4 seconds and 3,75 every 6 seconds. How much does the distance increase or decrease in how much seconds, I'm trying to sort out what I can do to find that out.

Ali: I think the distance is stable. Because at the zero beginning she was at 2nd meter. If we subtract 2 from the numbers, we see that she's at less than 4 m at 4th second or less than 8,25 m at 10th second for instance. I mean, it has to be 2 m less. Since she's made 2 m at the very beginning, there's no change, it goes in a stable speed.

R: Which data made you infer that the speed is stable?

*Ali: I subtracted 2 from each column. Then I checked the amount of increase (*He subtracted 2 from the distance value on the table*), it increases 1,25 m every 2 seconds. This way, the distance to home increases.*

Students have discussed what they need to do to examine whether the changes are the same or not after subtracting the serial differences of the numbers at the column. Ali found out that the distance was stable intuitively but he could not explain the reason. Afterwards Oğuz indicated that he was sure the *speed* was stable but he was not sure about the *distance over time.* Oğuz is not aware that "speed" and "distance over time" is the same thing yet. Oğuz has made his reasoning, using proportional way of thinking as "*I found it from the amount of increase. If it increases 2,5 meters in 4 seconds, it will increase 1,25 in 2 seconds, 3,75 in 6 seconds. So the speed is really stable*". Thus, Oğuz has expressed that speed is stable by inferring this from the values at the table. But he could not think he needs to compare the rate of changes to comment the relations.

Elif: We'll check the rates between the variables. We'll check the rates between 4 and 4,5 or 10 and 8,25… I suppose it decreases. Because the distance has been 4,5 in 4 seconds, it increases half value. But the distance has been 8,25 m. at the 10th second, it decreases 1,75. So the first one increases but the second one decreases. So I guess it has decreased.

This means Elif has searched for a pattern among the distance values that confronts the time values. According to Elif's way of thinking, if 4,5 m has been made in 4 seconds, the distance to home should have been more than 10 m in 10 seconds. However, since the distance has been given as 8,25 m at 10th second, Elif inferred that the speed has decreased. Likewise, Ezgi said "*If we take the 4th and the 24th second for granted; 24 is 6 times as 4 (the distance at 24th seconds). So, I divided 17 to 6. That made something like 2,8. I mean, it (the distance at 4th seconds) results less than 4,5 seconds. So, it decreases*". Ezgi rated 4 to 24, then rated the distance values that confronts these seconds. Since the second rate is lower than the first rate, the speed decreases. After all the debates, students have interpreted the differences between the values correctly and have come to an agreement that rate of change was stable.

R: What does 'the rate of change is stable' mean?

Gül: The rate of change was the slope. The slope is the same in every point.

R: …. We found that the rates of changes of these points are 0,625… The rate of the changes at distance and changes at time is 0,625. What if I asked it this way, if I made 2,5 m. in 4 seconds, how long can I make in 1 seconds? What operation would you do?

Bartu: I'll divide 2,5 to 4. Let me do it right now…0,625. It's the same result.

Melike: So 0,625 is the distance made in time value. Then the distance made in 1 second is….

Oğuz: If the distance made in every second is the same, it means she's walking at the same speed.

….

Melike: So, does this mean slope and the speed are the same thing?

By inferring that the rates of change of the values are stable, they related this with slope's definition. The researcher has made the students create a simple direct proportion in order to lead their thoughts. By this way the students have realized that *change in distance over change in time* was actually the distance made in every 1 second. If the distance made is the same in every 1 second, since the rates of changes at the table are same, then they have concluded that Feyza has walked at a stable speed. They have reached to the description of "speed" too, by this way. It is thought that the students, who related speed with slope, have made relational learning.

Students have stated some possibilities about "the distances at some of the time values that Melike could have made since she has made the same distance with Feyza at the same time values"; such as 8 meters in 5 seconds, 32 meters in 20 seconds, 64 meters in 40 seconds. When they were asked how they inferred these results, they have stated that they have the possibility of multiplying or dividing these numbers with the same number since the slope is equal when she made 16 meters in 10 seconds.

Gül: I think, they are parallels. If we draw their lines (she draws it on the air), since she walked 16 meters in 10 seconds, she must have walked 8 meters in 5 seconds. It makes: 16/10 = 8/5. First one is 2 times as the second one. As their slopes are same, they are parallel lines.

As a different point of view, Gül has abbreviated the numbers and inferred that (5, 8) is the smallest pair of numbers when considering 16 meters of distance taken in 10 seconds. She stated the rates will remain the same in case of multiplying or dividing these numbers to any number. She also expressed that the drawn lines will be parallel since the rates are the same and the slopes will not change.

R: Finally, when it's 16 meters in 10 seconds, you say she goes 32 m in 20 seconds but I claim that she can walk 33 meters in 21 seconds. Is it right?

Bartu: No. It's got to be 1,6 metres in each second.

Oğuz: It gives us the distances at 20th second and at 1st second here, we add them and get 21. We get 32 when we add 1,6 to 33,6. So, 0,6 is missing.

The students are aware that the change between variables is proportional. So, we can say that these students are multiplicative thinkers rather than additive thinkers.

DISCUSSION AND CONCLUSION

It was determined that students had non-quantitative rate of change and quantitative rate of change ways of thinking based on findings. Students with non- quantitative rate of change, considered change between two or more variables and indicated this change in terms of algebra or graph. However, students interpreted rate of change from operations independently from quantities. This case was caused by not analysing these quantity and quality changes together rather than not knowing variation. This way of thinking caused by lack of covariational thinking that is thinking two or more variables together can be classified as a way of thinking that needs to be developed (Weber, 2012). When drawing time and length of candle question was asked, Sezen was unable to see 2 cm burn rate information on Gül's graph. So she was unable to interpret relationship between time and length values on her graph. Thus, it could be stated that Sezen has non-quantitative rate of change.

Students with quantitative rate of change both realises the changes between quantities and can interpret the speed of change. Rate of change can be stated by interpreting the relationship between changes of variables that is called measure of covariation. In both ways of thinking, students can determine the rate between quantities. However, students who interpret the rate of change as non-quantitative, are unable to interpret the rate between quantities and the speed of change (Weber, 2012). Students, who investigated change of length of candle based on time, analysed time column and length column on the table separately and together. It could be stated that students who investigated variables and change of variables together have measure of covariation. Related with time graph of candle with same timeframe and candle that burns the same, way of thinking of Oğuz can be interpreted under this way of thinking. Since there is same length decrease in same timeframe, Oğuz formed a proportion between 2-unit change on y axis and 1-unit change on x axis. Thus, it could be stated that Oğuz interpreted rate of change as measure of covariation.

The tasks about rate of change show that the students initially think there's a relation between variables only at x column and y column. So we can say that the students think in a way of non- quantitative rate of change. After then, students have found out the relation between x column and y column after examining the variables in x column and y column. They made their reasoning by considering these three relations. This progress can be commented as changing way of thinking from "non-quantitative rate of change" to "quantitative rate of change". Another relation that the students have realised and found interesting was the relation between speed and slope. As a result of the relation they commented speed in the light of analysis of changes and they reached the description of speed and its formula. This is another noteworthy point to be stated.

When the literature was reviewed, there are studies that comply with these findings. Rate of change is fundamental for students to understand how dependent and independent variables changes covariationally. Therefore, this concept is accepted important to understand various concepts, especially functions (Cooney et al., 2010). It is known that students are facing challenges for rate of change (Bezuidenhout, 1998 ; Carlson, Larsen, & Jacobs, 2001; Carlson et al., 2003; Ellis, 2009). One of these challenges is the misconception of students regarding 'average rate of change', 'average value of a continuous function', and arithmetic average concepts and mixing those concepts (Bezuidenhout, 1998). Similarly, it was reported that students are confusing constant and variable, dependent and independent variable concepts in rate of change (Rowland & Jovanoski, 2004). Herbert and Pierce (2008) investigated how students interpret rate of change. In that study, "ratio", "variable", "relationship between variables" and "nature of related variables" were the focus points. It was indicated that there were misconceptions regarding rate of change and deficiencies about conceptual knowledge. Students are used to additional reasoning. This situation is shown at the tasks in which relations are investigated and the changes are analyzed. So we should present students tasks on which they can reason additively and multiplicatively. Thus, it is important to investigate how rate of change is perceived by students and how the knowledge about this subject is formed. So as researchers and teachers, we should help students to be quantitative thinkers more than non- quantitative thinkers.

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