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SYMMETRIC TENSOR RANK, CACTUS RANK AND RELATED COMPLEXITY MEASURES FOR HOMOGENEOUS POLYNOMIALS

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Dedicated to memory of Proffessor Franki Dillen

Abstract. Let $\nu_d : \mathbb{P}^m \to \mathbb{P}^r$, $r := \binom{m+d}{m} - 1$, be the Veronese embedding. For any $P \in \mathbb{P}^r$ we define its complexity rank (resp. complexity scheme-rank) as the minimal integer $d_1 + \cdots + d_s$ with d_i the degrees of hypersurfaces scheme-theoretically cutting a finite set (resp. a zero-dimensional scheme) $Z \subset \mathbb{P}^m$ with P in the linear span of $\nu_d(Z)$. We study these definitions (and related ones) when either P has border rank ≤ 3 or P is in the linear span of $\nu_d(L)$ for some line $L \subset \mathbb{P}^m$.

For any scheme Z of any projective space $\mathbb{P}^k$, let $\langle Z\rangle \subseteq \mathbb{P}^k$ denote its linear span, i.e. the intersection of all hyperplanes of $\mathbb{P}^k$ containing Z, with the convention $\langle Z\rangle = \mathbb{P}^k$ if there is no such a hyperplane. For any positive integers m, d let $\nu_d : \mathbb{P}^m \to \mathbb{P}^r$, $r := \binom{m+d}{m} - 1$, denote the order d Veronese embedding of $\mathbb{P}^m$. For each $P \in \mathbb{P}^r$ the rank $r_{m,d}(P)$ (resp. the cactus rank or scheme-rank $z_{m,d}(P)$) of P is the minimal cardinality (resp. minimal degree) of a finite set (resp. a zero-dimensional scheme) $Z \subset \mathbb{P}^m$ such that $P \in \langle \nu_d(Z)\rangle$ ([12], [11] (where the cactus rank is called the scheme-rank), [5], [4]). The integers $r_{m,d}(P)$ and $z_{m,d}(P)$ are a measure of the complexity of P with respect to homogeneous polynomials. In this note we study other measures of complexity of finite sets and zero-dimensional schemes $Z \subset \mathbb{P}^m$. Taking these integers instead of the integer $\deg(Z)$ we get various notions of *complexity rank*.

For any finite string $\underline{d} = d_1 \ge \cdots \ge d_s$ of positive integers and any positive real number α set $\|\underline{d}\| = (\sum_{i=1}^{s} d_i^{\alpha})^{1/\alpha}$. Set $\|\underline{d}\| := \|\underline{d}\|_1$. For each zero-dimensional scheme $Z \subset \mathbb{P}^m$ let $cc(Z)$ be the minimal integer $\|\underline{d}\|$, where $d_1 \ge \cdots \ge d_s$ are the degrees of some hypersurfaces $Y_1, \dots, Y_s$ cutting out Z scheme-theoretically, i.e. such that $Z = Y_1 \cap \cdots \cap Y_s$ (scheme-theoretic intersection). Let $\widehat{c}r_{m,d}(P)$ (resp. $\widehat{c}z_{m,d}(P)$) be the minimal integer $cc(Z)$ for some finite set (resp. zero-dimensional scheme) Z such that $P \in \langle \nu_d(Z)\rangle$. We say that $\widehat{c}r_{m,d}(P)$ (resp. $\widehat{c}z_{m,d}(P)$) is the *complexity rank* (resp. *complexity scheme-rank*) of P.

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We also introduce the following two measures of the complexity of the string of integers $\underline{d}$. Set $\|\underline{d}\|_- := \prod_{i=1}^{s} d_i$ and $\|\underline{d}\|_+ := \prod_{i=1}^{s}(d_i+1)$. The integer $\|\underline{d}\|_-$ is quite natural (when $s = m$ it would give the degree of a zero-dimensional complete intersection $Z \subset \mathbb{P}^m$ cut out by hypersurfaces of degree $d_1, \dots, d_m$). The integer $\|\underline{d}\|_+$ weights more the low degree hypersurfaces. For each zero-dimensional scheme $Z \subset \mathbb{P}^m$ let $\check{c}c(Z)$, resp. $\check{c}\check{c}(Z)$) be the minimal integer $\|\underline{d}\|_-$ (resp. $\|\underline{d}\|_+$) where $d_1 \ge \cdots \ge d_s$ are the degrees of some hypersurfaces $Y_1, \dots, Y_s$ cutting out Z scheme-theoretically. Let $\check{c}z_{m,d}(P)$ (resp. $\check{c}r_{m,d}(P)$) be the minimal integer $\check{c}c(Z)$ for a zero-dimensional scheme (resp. a finite set) $Z \subset \mathbb{P}^m$ such that $P \in \langle \nu_d(Z)\rangle$. Define $\check{c}\check{z}_{m,d}(P)$ and $\check{c}\check{r}_{m,d}(P)$ in the same way using $\| \ \|_+$ instead of $\| \ \|_-$.

We work over an algebraically closed field $\mathbb{K}$ with characteristic zero. For the positive characteristic case, see Remark 2.3.

1. Linear spans of rational normal curves

In this section we first give two preliminary lemmas (Lemma 1.1 and 1.2). Then we shows that schemes evincing either $\widehat{c}z_{m,d}(P)$ or $\check{c}z_{m,d}(P)$ or $\check{c}\check{z}_{m,d}(P)$ for some $P \in \mathbb{P}^r$, $r = \binom{m+d}{m} - 1$, are complete intersection (see Proposition 1.1). Then we consider the case in which, in suitable coordinates, the homogeneous polynomial associated to P is a bivariate polynomial, i.e. the case in which there is a line $L \subset \mathbb{P}^m$ such that $P \in \langle \nu_d(L)\rangle$. The curve $\nu_d(L)$ is a degree d rational normal curve in its linear span $\langle \nu_d(L)\rangle \subset \mathbb{P}^r$ and $\dim(\langle \nu_d(L)\rangle) = d$.

See [12] or [3] for the notion of border rank $b_{m,d}(P)$ of any $P \in \mathbb{P}^r$, $r := \binom{m+d}{m} - 1$, with respect to the Veronese variety $\nu_d(\mathbb{P}^m)$.

Lemma 1.1. *Let X be an integral projective variety. Fix $L \in Pic(X)$ and linear subspaces $V \subset W \subsetneq H^0(X, L)$. Call $B_1, \dots, B_x$ the irreducible components of the set-theoretic base locus of V. Assume that none of them is an irreducible component of the set-theoretic base locus of W. Fix a general $f \in W$. Then the hypersurface $\{f = 0\}$ contains no B_i and hence the base locus of W is either empty or with dimension $\le \max\{dim(B_i)\} - 1$.*

Proof. Use that W is an irreducible variety (and hence that a finite intersection of non-empty open subsets of W is non-empty) and that x is a finite integer. □

Lemma 1.2. *Let $Z \subset \mathbb{P}^m$, $Z \neq \emptyset$, be a zero-dimensional scheme. Let $a_1 \ge \cdots \ge a_s > 0$ be the degrees of a set of polynomials $g_1, \dots, g_s$ defining scheme-theoretically Z. Set $b_i := a_i$ for $1 \le i \le m-1$ and $b_m := a_s$. Fix a general $f_i \in H^0(\mathcal{I}_Z(b_i))$. Then the scheme $\{f_1 = \cdots = f_m = 0\}$ has dimension zero.*

Proof. Since $Z \neq \emptyset$, we have $s \ge m$ and hence the integers $b_1, \dots, b_m$ are well-defined. For each integer $i \in \{2, \dots, m\}$ set $U_j := \{g_s = \cdots = g_i = 0\}$. Let $A(k)$, $1 \le i \le m$, the statement that $\{f_m = \cdots = f_{m-k+1} = 0\}$ has dimension $m-k$. The lemma is true if $A(m)$ is true. $A(1)$ is true, because $g_s \neq 0$ and hence $f_m \neq 0$. Fix an integer $k \in \{2, \dots, m\}$ and assume $A(k-1)$. Let $B_1, \dots, B_x$ the irreducible components of the scheme $\{f_m = \cdots = f_{m-k+2} = 0\}$. Since $A(k-1)$ is assumed to be true, we have $\dim(B_j) = m-k+1$ for all i. By Lemma 1.1 to prove $A(k)$ it is sufficient to see that the base locus B of the linear subspace of $H^0(\mathcal{I}_Z(b_{m-k+1}))$ spanned by $f_m, \dots, f_{m-k+1}$ has dimension at most $m-k$. Assume that this is not true and take an irreducible component T of B with dimension $> m-k$. We have $T = B_j$ for some j. Since $Z = \{g_1 = \cdots = g_s\}$, we have $\dim(U_k) \le m-k$. Since

U_k is contained in the base locus Δ of $|\mathcal{I}_Z(b_k)|$, Δ has dimension at most k. Hence $f|T \neq 0$ for a general $f \in H^0(\mathcal{I}_Z(b_k)$, a contradiction. □

Proposition 1.1. *Fix positive integers m, d, any $P \in \mathbb{P}^r$, $r := \binom{m+d}{m} - 1$, and any zero-dimensional scheme $Z \subset \mathbb{P}^m$ evincing either $\widehat{cz}_{m,d}(P)$ or $\check{c}z_{m,d}(P)$ or $\check{c}\check{z}_{m,d}(P)$. Then Z is a complete intersection.*

Proof. Let $a_1 \geq \cdots \geq a_s$ be the degrees of a minimal set of generators of the homogeneous ideal of Z. We have $s \geq m$ and $s = m$ if and only if Z is a complete intersection. We have $cc(Z) = d_1 + \cdots + d_s$. By Lemma 1.2 there is a zero-dimensional scheme W containing Z and with $cc(W) = d_s + \sum_{i=1}^{m-1} d_i$. Since $W \supseteq Z$ we have $P \in \langle \nu_d(W) \rangle$. Hence $cc(Z) = \widehat{cz}_{m,d}(P) \leq cc(W)$, i.e. $m = s$.

The same proof works for $\check{c}z_{m,d}(P)$ and $\check{c}\check{z}_{m,d}(P)$. □

Theorem 1.1. *Fix $P \in \mathbb{P}^r$, $r := \binom{m+d}{m} - 1$, for some $m \geq 1$, $d \geq 3$. Assume the existence of a line L such that $P \in \langle \nu_d(L) \rangle$. Then $\widehat{cz}_{m,d}(P) = z_{m,d}(P) + m - 1$ and $\widehat{cr}_{m,d}(P) = r_{m,d}(P) + m - 1$.*

Proof. Set $b := z_{m,d}(P)$. By a theorem of Sylvester the integer b is the border rank of P with respect to the rational normal curve $\nu_d(L)$ ([12], citebgi). Let $Z \subset \mathbb{P}^m$ be any scheme evincing $z_{m,d}(P)$ and $A \subset \mathbb{P}^m$ any scheme evincing $r_{m,d}(P)$. We have $Z \subset L$ and $A \subset L$ ([12], Exercise 3.2.2.2, (for A) and [8], Proposition 2.1 and Corollary 2.2), Z is unique ([11], 1.36 and 1.38, [7], Theorem 1.18, or use [3], Lemma 34) and either $A = Z$ and $r_{m,d}(P) = b$ or $b < (d+2)/2$, $r_{m,d}(P) = d+2-b$ and $A \cap Z = \emptyset$. Since L is the complete intersection of $m-1$ hyperplane and Z (resp. A) A are the complete intersection of L and a hypersurface of degree $\deg(Z)$, resp, $\deg(A)$), we have $cc(Z) \leq b + m - 1$ and $cc(A) \leq r_{m,d}(P) + m - 1$. Hence $\widehat{cz}_{m,d}(P) \leq b + m - 1$ and $\widehat{cr}_{m,d}(P) \leq r_{m,d}(P) + m - 1$. Therefore it is sufficient to prove the inequalities the opposite inequalities.

Let $W \subset \mathbb{P}^m$ be any zero-dimensional scheme evincing $\widehat{cz}_{m,d}(P)$. By Proposition 1.1 W is a complete intersection, say of forms of degree $d_1 \geq \cdots \geq d_m$. To prove that $\widehat{cz}_{m,d}(P) = b + m - 1$ it is sufficient to prove that $d_1 + \cdots + d_m \geq b + m - 1$. Assume $d_1 + \cdots + d_m \leq b + m - 2$.

First assume $W \supseteq Z$. Since W is zero-dimensional, the scheme $W \cap L$ is a zero-dimensional scheme containing Z. Hence $e := \deg(W \cap L) \geq b$. Since at least one of the forms, F_i, does not vanish identically on all L, we have $d_1 \geq b$. Hence $d_1 + \cdots + b_m \geq b + m - 1$.

Now assume $W \not\supseteq Z$. In this case the proof of [2], Lemma 1, gives $h^1(\mathcal{I}_{Z \cup W}(d)) > 0$. Since W is a complete intersection, we have $h^1(\mathcal{I}_W(d_1 + \cdots + d_m - m - 1)) = 1$. Since $\deg(Z) = b$, we have $h^1(\mathcal{I}_{Z \cup W}(d_1 + \cdots + d_m - m - 1)) \leq 1 + b$. For any zero-dimensional scheme $B \subset \mathbb{P}^m$ the map $\mathbb{N} \to \mathbb{N}$ defined by $t \mapsto h^1(\mathcal{I}_B(t))$ is strictly decreasing, until it is zero (e.g., because its different function is the Hilbert function of a graded Artinian ring). Hence $h^1(\mathcal{I}_{Z \cup W}(d_1 + \cdots + d_m - m - 1 + b)) = 0$. Therefore $d \leq -1 + d_1 + \cdots + d_m - m - 1 + b$, i.e. $d_1 + \cdots + d_m \geq d - b + m - 2$. Since $2b \leq d + 2$, we get $cc(W) \geq b + m$ in this case.

Now assume $r_{m,d}(P) \neq b_{r,m}(P)$ and hence $r_{m,d}(P) = d + 2 - b$. Take any $S \subset \mathbb{P}^m$ evincing $\widehat{cr}_{m,d}(P)$ and let $a_1 \geq \cdots \geq a_s$ be the sequence of degrees of forms with S as their scheme-locus and with $a_1 + \cdots + a_s$ minimal. Assume $a_1 + \cdots + a_s = \widehat{cr}_{m,d}(P) \leq d + m - b$. Let $N \supseteq S$ be the zero-locus of general $g_i \in |\mathcal{I}_Z(b_i)|$ with $b_m = d_s$ and $b_i = a_i$ for all $i \in \{1, \ldots, m-1\}$. Lemma 1.2 gives

$\dim(N) = 0$. Since $N \supseteq S$, we have $P \in \langle \nu_d(N) \rangle$. First assume $Z \nsupseteq N$. Since $P \in \langle \nu_d(Z) \rangle \cap \langle \nu_d(N) \rangle$, the proof of [2], Lemma 1, gives $h^1(\mathcal{I}_{N \cup Z}(d)) > 0$. Since N is a complete intersection, we have $h^1(\mathcal{I}_N(b_1 + \cdots + b_m - m - 1)) = 1$. Since $\deg(Z) = b$, we get $h^1(\mathcal{I}_{Z \cup N}(b_1 + \cdots + b_m - m - 1)) \le 1 + b$. As above we get $h^1(\mathcal{I}_{Z \cup N}(b_1 + \cdots + b_m - m - 1 + b + 1)) = 0$. Hence $d \le b_1 + \cdots + b_m - m - 1 + b$. Since $b_1 + \cdots + b_m \le a_1 + \cdots + a_s \le d + m - b$, we get a contradiction.

Now assume $Z \subseteq N$. Since S is reduced, Z is not reduced and S is scheme-theoretically cut-out by forms of degree a_1, there is $F \in |\mathcal{I}_S(a_1)|$ such that $F|Z \ne 0$. Hence for a general $G \in |\mathcal{I}_S(a_1)|$ we have $G|Z \ne 0$. The proof that $A(m-1)$ implies $A(m)$ in the proof of Lemma 1.2 gives $N \nsupseteq Z$, a contradiction. □

Corollary 1.1. *Fix $P \in \mathbb{P}^r$, $r := \binom{m+d}{m} - 1$, with border rank 2. Then $ccr_{m,d}(P) = r_{m,d}(P) + m - 1$ and $ccz_{m,d}(P) = m + 1$.*

Proof. E.g., by the proof of [3], Theorem 32, or by [6], §2.1, $P \in \langle L \rangle$ for some line $L \subset \mathbb{P}^m$. Apply Theorem . □

2. Border rank 3

Lemma 2.1. *Fix a line $L \subset \mathbb{P}^m$, a zero-dimensional scheme $W \subset L$ with $e := \deg(W) \ge 2$ and $O \in \mathbb{P}^m \setminus L$. Then the homogeneous ideal I_A of $A := W \cup \{O\}$ is generated by the equation of a degree e cone F with $F \cap L = W$ and $(m-2)$-dimensional vertex containing O and by two degree reducible quadrics formed by the union of two hyperplanes, one containing L and the other one containing O, but not L and (if $m > 2$) the $m - 2$ linear equations of the plane $\langle L \cup \{O\} \rangle$.*

Proof. It is sufficient to do the case $m = 2$. Let $G \subset \mathbb{P}^2$ be any hypersurface containing A, but not L. Then $G \cap L \supseteq W$ and hence $\deg(G) \ge e$ with equality if and only if $G \cap L = W$. We get that the degree two part of I_A is generated by the equations of two pairs of line through O and (if $e = 2$) the equation of F. We also get that no generators of I_A occurs in degree $< e$. It is easy to check that $h^1(\mathcal{I}_A(e-1)) = 0$ (even if $e = 2$). Hence the Castelnuovo-Mumford lemma gives that I_A is generated in degree $\le e$. Since any two degree e elements of I_A not containing L induce the same degree e divisor on L, we get that I_A is minimally generated by the two reducible conics through O containing L and by the equation of F. □

Theorem 2.1. *Fix $P \in \mathbb{P}^r$, $r := \binom{m+d}{m} - 1$, $m \ge 2$, with border rank 3. Assume $d \ge 7$.*

(a) If there is a line $L \subset \mathbb{P}^m$ such that $P \in \langle \nu_d(L) \rangle$, then $\widehat{cc}r_{m,d}(P) = r_{m,d}(P) + m - 1$ and $\widehat{cc}z_{m,d}(P) = m + 1$.

(b) Assume that there is no line $L \subset \mathbb{P}^m$ such that $P \in \langle \nu_d(L) \rangle$.

(b1) We have $\widehat{cc}z_{m,d}(P) = m + 4$.

(b2) If $r_{m,d}(P) = 3$, then $\widehat{cc}r_{m,d}(P) = m + 4$.

(b3) If $r_{m,d}(P) = d + 1$, then $\widehat{cc}r_{m,d}(P) = m + 4$.

(b4) In all other cases (i.e. if $r_{m,d}(P) = 2d - 1$), then $\widehat{cc}r_{m,d}(P) = 2d + m$.

Proof. By the proof of [3], Theorem 37, we have $r_{m,d}(P) = 2d - 1$ if and only if P is neither as in (a) nor as in (b2) nor as in (b3).

Part (a) is true by Corollary 1.1.

In the set-up of part (b) we fix a scheme Z evincing $z_{m,d}(P)$ (and hence with degree 3 by [6], Proposition 1.2) and a set $S \subset \mathbb{P}^m$ evincing $r_{m,d}(P)$. Z spans a plane, because there is no line $L \subset \mathbb{P}^m$ such that $P \in \langle \nu_d(L) \rangle$. In this case $\langle Z \rangle$ is a plane. We also know that Z is either a union of 3 non-collinear points, or a connected curvilinear scheme or $Z = v \sqcup \{O\}$ with v connected, $\deg(v) = 2$ and $O \notin \langle v \rangle$. In all cases Z is contained in a smooth conic C. It is easy to check first that Z is scheme-theoretically cut out in C by two conics and then that the homogeneous ideal of Z is generated by 3 quadratic equations and (if $m > 2$) the $m-2$ linear equations of $\langle Z \rangle$. Hence $ccz_{m,d}(P) \le 6+m-2 = m+4$. The opposite inequality is obvious, because Z is neither a complete intersection nor contained in a line. We get parts (b1) and (b2). Now assume $r_{m,d}(P) = d+1$. This is the case if and only if $Z = v \sqcup \{O\}$ with v connected, $\deg(v) = 2$ and $O \notin \langle v \rangle$ ([3], proof of Theorem 32). Fix any S evincing $r_{m,d}(P)$. By [1], Theorem 4, we have $S = S' \sqcup \{O\}$ with $O \notin \langle v \rangle$. The case $e = d$ of Lemma 2.1 gives $ccr_{m,d}(P) = m+4$.

Now assume $r_{m,d}(P) = 2d-1$, i.e. assume that Z is connected and not contained in a line. In this case Z contained in a smooth conic. Fix any $S \subset \mathbb{P}^m$ evincing $r_{m,d}(P)$. By [1], Theorem 4, we have $S \cap Z = \emptyset$ and $S \cup Z$ is contained in a reduced conic T. The homogeneous ideal of S has exactly $m-2$ linearly independent linear forms. First assume that T is smooth. In this case S is the scheme-theoretic intersection of two degree $2d$ divisors of T, because $T \cong \mathbb{P}^1$. Since T is arithmetically Cohen-Macaulay, each of these divisors is the intersection of T with a degree d hypersurface. We get that S is scheme-theoretically cut out by $m-2$ linear forms, a quadratic form and two degree d forms. Hence $\widehat{c}r_{m,d}(P) \le 2d+m$. Bezout theorem gives that every $Y \in |\mathcal{I}_S(d-1)|$ contains T. Hence we get that in any set of forms defining scheme-theoretically S, at least two of these forms have degree at least d. Since S is not a complete intersection, to define scheme-theoretically S we need at least $m+1$ forms. Hence $\widehat{c}cr_{m,d}(P) \ge 2d+m$. Now assume that T is not smooth. Since S is reduced, T is reduced and, calling L, R the components of T with $\sharp(L \cap S) \ge \sharp(R \cap S)$, and $O = L \cap R$ the singular point of T, we have $O \notin S$, $\{O\} = Z_{red}$, $\sharp(L \cap S) = d$ and $\sharp(S \cap L) = d-1$ ([1], part (f) of §4). Since $O \notin S$, there are two plane degree d curves T_1, T_2 in $\langle T \rangle$ such that $T \cap T_1 \cap T_2 = S$ (as schemes); we may take as each T_i a union of d lines each of them spanned by a point of $S \cap L$ and a point of $S \cap R$. □

Remark 2.1. *Take P as in Theorem 2.1. Assume $r_{m,d}(P) > 3$. By the proof of* [3], *Theorem 37, there is a line $L \subset \mathbb{P}^m$ such that $P \in \langle \nu_d(L) \rangle$ if and if $r_{m,d}(P) = d-1$. In all cases the proof of Theorem 2.1 gives that all sets evincing $r_{m,d}(P)$ have the same complexity.*

Remark 2.2. *Assume char$(\mathbb{K}) = 0$. Fix a finite set $S \subset \mathbb{P}^k$, $k \ge 1$, and an integer $s \ge 1$ such that the sheaf $\mathcal{I}_S(x)$ is spanned. Then there are k hypersurfaces $F_i \in |\mathcal{I}_S(x)$ such that the scheme $F_1 = \cdots = F_k = 0$ is a reduced union of x^k points.*

Lemma 2.2. *Fix $S \subset \mathbb{P}^s$ with $\sharp(S) = s+1$ and $\langle S \rangle$. Fix any integer $x \ge 2$. Let $A \subset \mathbb{P}^s$ be the intersection of s general elements of $|\mathcal{I}_S(x)|$. Then A is a reduced zero-dimensional scheme with cardinality x^s.*

Proof. Any two such sets are projectively normal. Hence it is sufficient to note that $h^0(\mathcal{I}_S(x)) = \binom{s+x}{x} - s - 1 \ge s$ and that a general complete intersection of s hypersurfaces of degree x is smooth ([10], Theorem II.8.12). □

Proposition 2.1. *Take a finite set $S \subset \mathbb{P}^m$, $m \geq 1$, which is linearly independent. Set $s := \sharp(S)$. Fix any $P \in \langle \nu_d(S) \rangle$ such that $P \notin \langle \nu_d(S') \rangle$ for any $S' \subsetneq S$. If $s \leq 2$, then $cc(S) = \widehat{c}r_{m,d}(P) = \widehat{c}z_{m,d}(P) = s + m - 1$. If $s \geq 3$, then $cc(S) = m + s + 1$ and $\widehat{c}r_{m,d}(P) = \widehat{c}z_{m,d}(P) = s + m - 1$.*

Proof. We have $s \leq m + 1$. If $s = 1, 2$, then S is a complete intersection. We have $cc(S) = m + s - 1$ (the case $s = 1$ is trivial, the case $s = 2$ by Corollary 1.1). Now assume $s \geq 3$. Since $\mathcal{I}_S(2)|$ is spanned and $\dim(\langle S \rangle) = m + 1 - s$, S is the scheme-theoretic intersection of $m + 1 - s$ linearly independent linear forms and some degree two linear forms. Since S is not a complete intersection, Lemma 2.2 gives $cc(S) = (m + 1 - s) + 2s = m + s + 1$. Lemma 2.2 gives the existence of a reduced set $A \supset S$ which is the complete intersection of $m + 1 - s$ linear forms and $s - 1$ degree two forms. Take $B \subset \mathbb{P}^m$ evincing $\widehat{c}z_{m,d}(P)$. Since P depends exactly on s homogeneous coordinate and $P \in \langle \nu_d(B) \rangle$, we have $\dim(\langle B \rangle) \geq s - 1$. Hence $d_i > 1$ for all $i \leq \dim(\langle B \rangle)$. Hence $cc(B) \geq m + s - 1$. □

In the same way we get the following result.

Proposition 2.2. *Fix $m \geq 2$ and take any linearly independent zero-dimensional scheme $Z \subset \mathbb{P}^m$. Set $s := \deg(Z)$. Fix any $P \in \langle \nu_d(Z) \rangle$ such that $P \notin \langle \nu_d(Z') \rangle$ for any $Z' \subsetneq Z$. If $s \leq 2$, then $cc(Z) = \widehat{c}z_{m,d}(P) = s + m - 1$. If $s \geq 3$, then $cc(Z) = m + s + 1$ and $\widehat{c}z_{m,d}(P) = s + m - 1$.*

Proposition 2.3. *Fix $P \in \mathbb{P}^r$, $r := \binom{m+d}{m} - 1$, for some $m \geq 1$, $d \geq 3$. Assume the existence of a line L such that $P \in \langle \nu_d(L) \rangle$. Then $\check{c}z_{m,d}(P) = z_{m,d}(P)$, $\check{c}r_{m,d}(P) = r_{m,d}(P)$, $\check{c}r_{m,d}(P) = r_{m,d}(P) = 2^{m-1}(z_{m,d}(P) + 1)$ and $\check{c}\check{r}_{m,d}(P) = 2^{m-1}(r_{m,d}(P) + 1)$.*

Proof. Set $b := z_{m,d}(P)$ and take Z evincing $z_{m,d}(P)$. Recall that $Z \subset L$ ([8], Proposition 2.1 and Corollary 2.2). Hence Z is a complete intersection of a degree b hypersurface and $m - 1$ hyperplane. Hence $\|Z\|_- = b$ and $\|Z\|_+ = 2^{m-1}(b + 1)$. Since for any zero-dimensional scheme $W \subset \mathbb{P}^m$ we have $\deg(W) \leq \|W\|_-$ (with equality if and only if W is a complete intersection), then we get $\check{c}z_{m,d}(P) = b$ and that Z is the only scheme evincing $\check{c}z_{m,d}(P)$. If $r_{m,d}(P) = z_{m,d}(P)$, i.e. if Z is reduced, then we also get $\check{c}r_{m,d}(P) = b$. Now assume $r_{m,d}(P) \neq z_{m,d}(P)$. In this case $2b \leq d + 1$, $r_{m,d}(P) = d + 2 - b$ and every set, A, evincing $r_{m,d}(P)$ is contained in A. The set A is the complete intersection of $m - 1$ hyperplanes and a degree $d + 2 - b$ hypersurface and hence $\check{c}c(A) = d + 2 - b$. Since any A evincing $r_{m,d}(P)$ is contained in L ([12], Exercise 3.2.2.2, or [8], Proposition 2.1 and Corollary 2.2), we get $\check{c}r_{m,d}(P) = r_{m,d}(P)$.. □

Remark 2.3. *Assume $p := char(\mathbb{K}) > 0$. It is sufficient to ass assume $p > d$, because Sylvester theorem quoted and proved in characteristic zero in* [9] *and* [3] *is true in positive characteristic if $p > d$ (*[11]*, page 22).*

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