MAPAS

Mathematical Advances in Pure and Applied Sciences, 2019
Received: June 15,2019
Accepted: July 8, 2019
Published online: July 23, 2019

ISSN : 2636-8021
Vol.2, No.1, 25-33
doi: will be added

Weak c-Comparison Function and a Fixed Point Theorem in Cone Metric Spaces over Banach Algebras

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Keywords
Cone Metric Space,
Banach Algebras,
Weak Contraction,
Fixed Point.

Abstract: In this paper, we first give the notion of weak c-comparison function, and prove some associated fixed point theorems containing classes of many well known theorems in the setting of cone metric spaces over Banach algebras with solid cones.

1. Introduction

In the literature, there exist many fixed point theorems for different type mappings such as Banach [1], Kannan [2], Chatterjea [3], Zamfirescu [4]. Furthermore, one can see many significant generalizations and extensions of those fixed point theorems (see [5], [6], [7], [8], [9]). In addition to those studies, many people attempted to extend many well known fixed point theorems by replacing real valued metrics with an ordered topological vector space (or an ordered Banach space) valued metrics. For instance, in 2007, the class of cone metric spaces over Banach spaces was introduced by Huang and Zhang [10] to generalize metric spaces as follows:

Definition 1.1. Let E be an ordered Banach space whose order is obtained by a normal cone P of E and $X \neq \emptyset$. A cone metric space over E is given by a pair (X,d) where d is a mapping $d : X \times X \to E$ satisfying

(cm1) $\theta \preceq d(x,y)$ and $d(x,y) = \theta$ if and only if $x = y$;

(cm2) $d(x,y) = d(y,x)$;

(cm3) $d(x,y) \preceq d(x,z) + d(z,y)$

for all $x,y,z \in E$ and for null vector $\theta \in E$.

One can also see from [10] the fixed point theorems and the corresponding definitions such as convercence, Cauchy sequence and contractive mapping in this setting. Afterwards, many works (see [11], [12], [13]) were devoted to studying the fixed point theory for such (tvs)-cone metric spaces to generalize the fixed point theorems in the usual metric spaces. However, in 2010, Du [12] proved that the Banach's contraction principle and many associated fixed point theorems in the setting of (tvs)-cone metric spaces are equivalent to their counterparts in the usual metric spaces. Later, in 2013, Liu and Xu [14] introduced the notion of cone metric space over a Banach algebra by replacing Banach space E in the underlying cone metric with a Banach algebra $\mathscr{A}$ to obtain proper generalizations of the usual fixed point theorems. They also introduced the notion of generalized contraction in this new setting as follows:

Definition 1.2. For a cone metric space (X,d) over a Banach algebra $\mathscr{A}$, let $T : X \mapsto X$ be a self mapping. T is said to a generalized contraction if there exists a constant vector $k \in \mathscr{A}$ with $\rho(k) < 1$ such that

$$d(Tx,Ty) \preceq kd(x,y) \text{ for all } x,y \in X \tag{1}$$

where $\rho(k)$ stands for the spectral radius of k.

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In addition to the above contraction, Liu and Xu [14] studied fixed point theorems for Kannan [2] and Chatterjea [3] type mappings in this setting by using a solid normal cone in $\mathscr{A}$. They also provided some examples to illustrate that fixed point theorems for such contractive type mappings are proper generalizations of the usual fixed point theorems. Based on the results in this setting, recently, many scholars have paid attention to the fixed point theorems for such contractive type mappings to obtain proper generalizations of the well known results (for example see [15], [16], [17]). In 2014, taking into account some basic results of spectral radius, Xu and Radenovic [17] proved that the main results of [14] can be achieved by omitting the assumption of normality of cone in the underlying Banach algebra. Recently, in [18], the notion of (k,l)-almost contraction has been investigated in the framework of cone metric spaces over Banach algebras by combining the results of [14, 17] with those of [5].
In this study, we first give the definition of weak c-comparison, and introduce a fixed point theorem associated with weak c-comparison function in the setting of cone metric spaces over Banach algebras with solid cones. Finally, we observe that this theorem contains many well known theorems as some special cases .

2. Preliminaries

Let us begin by recalling some basic definitions and notions that will be needed to introduce main results in the sequel. Let $\mathscr{A}$ be a real Banach algebra with a multiplicative unit e. It is well known that if the spectral radius of $a \in A$ defined as follows

$$\rho(a) := \lim_{n\to\infty} \|a^n\|^{\frac{1}{n}} \tag{2}$$

is less than 1, then $e-a$ is invertible (see [19]) and

$$(e-a)^{-1} = \sum_{i=0}^{\infty} a^i. \tag{3}$$

It is also known from [17] that if $\rho(a) < 1$, then $\|a^n\| \to 0$ as $n \to \infty$.
Let P be a subset of $\mathscr{A}$ such that $\{\theta, e\} \subset P$. P is called a cone of $\mathscr{A}$ if the following conditions hold:

(c1) P is closed;

(c2) $\lambda P + \mu P \subset P$ for all non-negative real numbers λ and μ;

(c3) $PP \subset P$ and $P \cap (-P) = \theta$.

For a cone P, a partial ordering $\preceq$ w.r.t. P is defined by $x \preceq y$ iff $y - x \in P$. By $x \prec y$ we understand that $x \preceq y$ but $x \neq y$, while $x \ll y$ stands for $y - x \in \text{int}P$, where int$P$ denotes the interior of P. A cone P with int$P \neq \emptyset$ is called solid cone. If there exists a positive real number K such that for all $x, y \in \mathscr{A}$

$$\theta \preceq x \preceq y \text{ implies } \|x\| \leq K\|y\| \tag{4}$$

then a cone P is called normal. The least of K's with the above condition is called the normal constant of P.
Throughtout this work we always suppose that $\mathscr{A}$ is a unital Banach algebra, P is a solid cone of $\mathscr{A}$ and $\preceq$ denotes the partial ordering on $\mathscr{A}$ induced by P.

Definition 2.1. (See [14]) Let (X,d) be a cone metric space over a Banach space E. If E is replaced with a Banach algebra $\mathscr{A}$, then (X,d) is said to be a cone metric space on a Banach algebra $\mathscr{A}$.

Example 2.2. (See [20]) Let $\mathscr{A}$ be the usual algebra of all real valued continious functions on $X = [0,1]$ which also have continious derivations on X. Endowed with the norm $\|f\| = \|f\|_\infty + \|f'\|_\infty$, $\mathscr{A}$ is a Banach algebra with unit $e = 1$. Morever, $P = \{f \in \mathscr{A} | f(t) \geq 0 \text{ for all } t \in X\}$ is a nonnormal cone. Consider a mapping $d : X \times X \to \mathscr{A}$ defined by $d(x,y)(t) = |x-y|e^t$ for all $x, y \in X$. It is obvious that (X,d) is a cone metric space on the Banach algebra $\mathscr{A}$.

Definition 2.3. (See [14]) Let (X,d) be a cone metric space on $\mathscr{A}$ and $\{x_n\}$ be a sequence in X. Then

(i) We say that $\{x_n\}$ converges to $x \in X$ whenever for every $c \gg \theta$ there is a natural number n_0 such that $d(x_n, x) \ll c$ for all $n \geq n_0$. This is denoted by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.

(ii) $\{x_n\}$ is called Cauchy if for all $c \gg \theta$ there is a natural number n_0 such that $d(x_m, x_n) \ll c$ for all $m, n \geq n_0$.

(iii) (X,d) is called complete cone metric if every Cauchy sequence is convergent.

Lemma 2.4. *(See [21]) Let (X,d) be a cone metric space on $\mathscr{A}$. If $\theta \preceq u \ll c$ for each $\theta \ll c$, then $u = \theta$.*

Lemma 2.5. *(See [21]) Let $\{u_n\}$ be a sequence in P. If $\|u_n\| \to 0$ as $n \to \infty$, then for all $c \in intP$, there is one $n_0 \in \mathbb{N}$ such that $u_n \ll c$ whenever $n > n_0$.*

Definition 2.6. (See [17]) Let (X,d) be a cone metric space on $\mathscr{A}$. A sequence $\{u_n\} \subset P$ is said to be a c-sequence if for each $c \gg \theta$ there exists $n_0 \in \mathbb{N}$ such that $\theta \ll u_n$ for $n \geq n_0$.

Lemma 2.7. *(See [22]) Let (X,d) be a cone metric space on $\mathscr{A}$. If $\{x_n\}$ and $\{y_n\}$ are two c-sequences in P, then $\{\alpha x_n + \beta y_n\}$ is a c-sequence for positive real numbers α and β.*

Lemma 2.8. *(See [17]) Suppose that (X,d) is a cone metric space on $\mathscr{A}$ and $k \in P$. If $\{x_n\}$ is a c-sequence in P, then $\{kx_n\}$ is also c-sequence in P.*

Lemma 2.9. *(See [17]) Let (X,d) be a cone metric space on $\mathscr{A}$. If (X,d) is a complete cone metric space over $\mathscr{A}$ and $\{x_n\} \subset X$ is a sequence that converges to $x \in X$, then the following assertions are true:*

(i) $\{d(x_n,x)\}$ is a c-sequence.

(ii) $\{d(x_n,x_{n+m})\}$ is a c-sequence for all $m \in \mathbb{N}$.

Lemma 2.10. *(See [23]) The following conditions hold:*

(i) If $a \preceq b$ and $b \ll c$, then $a \ll c$.

(ii) If $a \preceq b + c$, for each $c \in intP$, then $a \preceq b$.

In [5], Berinde introduced the notion of (k,l)-almost contraction in the setting of metric spaces as follows:

Definition 2.11. Let $T : X \to X$ be a self mapping on a metric space (X,d). If there are $k,l \in \mathbb{R}$ with $0 < k < 1$ and $l \geq 0$ such that for all $x,y \in X$

$$d(Tx,Ty) \preceq kd(x,y) + ld(y,Tx), \tag{5}$$

then T is called a (k,l)-almost contraction.

Afterwards, Berinde extended the class of such mappings to that of (φ,l)- almost mappings using the notion of c-comparison function. Let φ be a self function on $[0,\infty)$. If φ holds the following conditions:

(i) φ is monotone increasing,

(ii) The series $\sum_{i=0}^{\infty} \varphi^i(p)$ is convergent for all $p \in P$ where φ^i stands for i-th iteration of φ,

then it is called a c-comparison function. For more details about c-comparison functions, the reader can refer to [24].

Definition 2.12. (See [25]) Let (X,d) be a metric space. A self mapping $T : X \to X$ is said to be a (φ,l)-almost contraction if there exist a comparison function φ and $l \geq 0$ such that

$$d(Tx,Ty) \preceq \varphi(d(x,y)) + ld(y,Tx) \tag{6}$$

for all $x,y \in X$.

Note that one has the dual of this condition by the symmetry property as follows:

$$d(Tx,Ty) \preceq \varphi(d(x,y)) + ld(x,Ty). \tag{7}$$

Thus one must check both (6) and (7) to check whether a self mapping T is a (φ,l)-almost contraction.
On the other hand, the notion of (k,l)-almost contraction is extended to the setting of cone metric spaces over Banach algebra as follows:

Definition 2.13. (See [18]) Let $T : X \to X$ be a self mapping on a cone metric space (X,d) over a Banach algebra $\mathscr{A}$ with a solid cone P. If there is $k \in P$ with $\rho(k) < 1$ and some $\theta \preceq l$ such that for all $x,y \in X$

$$d(Tx,Ty) \preceq kd(x,y) + ld(y,Tx), \tag{8}$$

then T is called a (k,l)-almost contraction in the setting of cone metric spaces over Banach algebras.

Remark that it is seen from the symmetry property that the condition (8) implicitly contains the following one:

$$d(Tx,Ty) \preceq kd(x,y) + ld(x,Ty) \tag{9}$$

for all $x,y \in X$. Thus, it is necessary to investigate both (8) and (9) for (k,l)-almost contractiveness of a self mapping T. Moreover, the author of [18] shows that any Banach, Kannan and Chatterja mappings are special cases of (k,l)-almost contractions in the setup of cone metric spaces with Banach algebra, and also proves the following theorem:

Theorem 2.14. *Let (X,d) be a complete cone metric space over a Banach algebra $\mathscr{A}$ with a solid cone P. If $T : X \to X$ is a (k,l)-almost contraction, then T has at least one fixed point in X. Additionally, if T satisfies*

$$d(Tx,Ty) \preceq kd(x,y) + ld(x,Tx) \text{ for all } x,y \in X, \tag{10}$$

then it has the unique fixed point, and for any $x \in X$, the iterative sequence $\{T^n x\}$ converges to the fixed point.

Remark 2.15. Since d holds the symmetry rule, we have that the condition (10) holds if and only if

$$d(Tx,Ty) \preceq kd(x,y) + ld(y,Ty) \text{ for all } x,y \in X \tag{11}$$

satisfies. Thus, to show uniqueness of fixed point of a (k,l)-almost contraction, it is necessary to check whether both (10) and (11) hold.

3. Main Results

To extend the notion of usual (k,l)-almost contraction to (φ,l)-almost one in the setting of a Banach algebra $\mathscr{A}$ with solid cone P, we first consider a function $\varphi : P \to P$ and the following items associated with φ:

(iφ) φ is monotone increasing with respect to the ordering induced from P.

(iiφ) $\varphi^n(t)$ is a c-sequence for all $t \in P$.

(iiiφ) The series $\sum_{i=0}^{\infty} \varphi^i(t)$ is convergent for all $t \in P$.

(ivφ) If $\{u_n\}$ is a c-sequence, then $\{\varphi(u_n)\}$ is a c-sequence.

Definition 3.1. (see [26]) A function φ satisfying (iφ), (iiφ) and (ivφ) is said to be a weak comparison function.

Definition 3.2. A function φ satisfying (iφ), (iiiφ) and (ivφ) is said to be a weak c-comparison function.

Proposition 3.3. *Let B be a Banach space with a solid cone P and $\{u_n\}$ be a sequence in P. If $\sum_{k=0}^{\infty} u_k$ is convergent, then $\{u_n\}$ is a c-sequence.*

Proof. It is clear that $\sum_{k=0}^{\infty} u_k$ is convergent $\Leftrightarrow$ for all m,n with $m > n$, $\|\sum_{k=n}^{m} u_k\| \to 0$ as $m,n \to \infty$. Therefore, for the particular case $m = n$, we have that $\|u_n\| \to 0$ as $n \to \infty$. Thus, by Lemma 2.5, it is clear that $\{u_n\}$ is a c-sequence. □

By Proposition 3.3, we can easialy say the following remark:

Remark 3.4. Any weak c-comparison function is a weak comparison function.

Also, it is known from [26] that $\varphi(\theta) = \theta$ where φ is a weak comparison function.
Now we introduce the notion of (φ,l)-almost contraction by using the concept of weak c-comparison function in the setup of cone metric spaces over Banach algebras as follows:

Definition 3.5. Let (X,d) be a cone metric space over a Banach algebra $\mathscr{A}$ with solid cone P. A mapping $T : X \to X$ is said to be a (φ,l)-almost contraction if there exist a weak c-comparison function $\varphi : P \to P$ and $l \in P$ such that

$$d(Tx,Ty) \preceq \varphi(d(x,y)) + ld(y,Tx) \tag{12}$$

for all $x,y \in X$.

Note that one has the dual of the above condition by the symmetry property as follows:

$$d(Tx,Ty) \preceq \varphi(d(x,y)) + ld(x,Ty) \tag{13}$$

Thus we conclude that one must check the both (12) and (13) to show whether a self operator T is a (φ,l)-almost contraction or not.

Proposition 3.6. *Let (X,d) be a complete cone metric space over $\mathscr{A}$ with a solid cone P. If $T : X \to X$ is a (k,l)-almost contraction, then T belongs to the class of (φ,l)-almost contractions in the setting of cone metric spaces over Banach algebras.*

Proof. Let $\varphi : P \to P$ defined by $\varphi(p) = kp$ for all $p \in P$. By the properties of P, it is seen that φ holds the condition (iφ). Since $\rho(k) < 1$, $\sum_{i=0}^{\infty} k^i p$ converges to $(e-k)^{-1}p$ for all $p \in P$, which satisfies the condition (iiiφ). Finally, by Lemma 2.8, we see that φ holds the condition (ivφ). Consequently, a (k,l)-almost contraction T given in Definition 2.13 is also a (φ,l)-almost contraction of the form given in Definition 3.5. □

Now, we are ready to introduce the following theorem, which contains one for (φ,l)-almost contractions in the setting of cone metric spaces over Banach algebras:

Theorem 3.7. *Let (X,d) be a complete cone metric space over a Banach algebra $\mathscr{A}$ with a solid cone P. Suppose that a self-mapping $T : X \to X$ satisfies*

$$d(Tx,Ty) \preceq \varphi(u(x,y)) + ld(y,Tx), \tag{14}$$

where $u(x,y) \in \{d(x,y), d(x,Tx), d(y,Ty)\}$, for all $x,y \in X$. Then T has at least one fixed point in X.

Proof. Let z_0 be an arbitrary element in X. Consider the sequence $\{z_n\}$ with $z_n = Tz_{n-1}$ for all $n \in \mathbb{N}$. By using the facts that T holds (14) and φ is a c-comparison, we obtain

$$d(z_n,z_{n+1}) = d(Tz_{n-1},Tz_n) \preceq \varphi(d(z_{n-1},z_n)) \tag{15}$$

and also

$$u(z_{n-1},z_n) \in \{d(z_{n-1},z_n), d(z_n,z_{n+1})\} \tag{16}$$

for all $n = 1,2,\ldots$. Thus we have the following two cases:

Case1: $d(z_n,z_{n+1}) \preceq \varphi(d(z_n,z_{n+1}))$;

Case2: $d(z_n,z_{n+1}) \preceq \varphi(d(z_{n-1},z_n))$.

If the case 1 satisfies for some $p \in \mathbb{N}$, we get from the monotone property of φ that

$$d(z_n,z_{n+1}) \preceq \varphi^m(d(z_n,z_{n+1})), \tag{17}$$

for all $m \in \mathbb{N}$. Since $\varphi^m(d(z_n,z_{n+1})$ is a c-sequence, for each $\varepsilon \in intP$, there is $N \in \mathbb{N}$ such that

$$d(z_p,z_{p+1}) \preceq \varphi^m(d(z_p,z_{p+1})) \ll \varepsilon \text{ for } m \geq N. \tag{18}$$

Hence, by Lemma 2.4, it follows that $d(z_p,z_{p+1}) = \theta$, implying that z_p is the fixed point of T. So, we can assume that the case 2 holds, that is,

$$d(z_n,z_{n+1}) \preceq \varphi(d(z_{n-1},z_n)) \tag{19}$$

for all $n = 1,2,\ldots$. By the increaseness of φ, the inequality (19) implies the following one:

$$d(z_n,z_{n+1}) \preceq \varphi^n(d(z_0,z_1)). \tag{20}$$

Let $\varepsilon \in intP$. Since $\sum_{n=0}^{\infty} \varphi^n(d(z_0,z_1)$ is convergent, there is $n(\varepsilon) \in \mathbb{N}$ such that

$$\sum_{n \geq n(\varepsilon)} \varphi^n(d(z_0,z_1)) \ll \varepsilon. \tag{21}$$

Let $m > n \geq n(\varepsilon)$. Then, by using the triangle inequality, we obtain

$$\begin{aligned} d(z_n,z_m) &\preceq \sum_{i=n}^{m-1} (z_i,z_{i+1}) \\ &\preceq \sum_{i=n}^{m-1} \varphi^i(d(z_0,z_1)) \\ &\preceq \sum_{n \geq n(\varepsilon)} \varphi^n(d(z_0,z_1)) \ll \varepsilon \end{aligned} \tag{22}$$

implying that $\{z_n\}$ is a Cauchy sequence. Since X is complete, there is $z \in X$ such that $z_n \to z$ as $n \to \infty$. Now we need to show that $Tz = z$. Indeed,

$$\begin{aligned} d(z,Tz) &\preceq d(z,z_{n+1}) + d(z_{n+1},Tz) \\ &= d(z,z_{n+1}) + d(Tz_n,Tz) \\ &\preceq (e+l)d(z,z_{n+1}) + \varphi\left(u(z_n,z)\right), \end{aligned} \tag{23}$$

where $u(z_n,z) \in \{d(z_n,z), d(z_n,z_{n+1}), d(z,Tz)\}$. Hence it is clear that at least one of the following cases satisfies for an infinite subset I of $\mathbb{N}$:

(i) $d(z,Tz) \preceq (e+l)d(z,z_{n+1}) + \varphi\left(d(z_n,z)\right)$;

(ii) $d(z,Tz) \preceq (e+l)d(z,z_{n+1}) + \varphi\left(d(z_n,z_{n+1})\right)$;

(iii) $d(z,Tz) \preceq (e+l)d(z,z_{n+1}) + \varphi\left(d(z,Tz)\right)$,

for all $n \in I$. Assume that (i) holds for all $n \in I$. By Lemma 2.7 and Lemma 2.8, the right side of inequality (i) is a c-sequence. Thus we have that for each $\varepsilon \in intP$, there is $n_0 \in \mathbb{N}$ such that

$$d(z,Tz) \preceq y_n \ll \varepsilon \text{ for } n \geq n_0, \tag{24}$$

where $y_n = (e+l)d(z,z_{n+1}) + \varphi\left(d(z_n,z)\right)$. Hence, by Lemma 2.4, we obtain that $Tz = z$. If (ii) is true for all $n \in I$, we can show that T has a fixed point with similar steps as the case (i). Finally, suppose that (iii) satisfies. By Lemma 2.8, we know that $(e+l)d(z,z_{n+1})$ is a c-sequence, which implies that for each $\varepsilon \in intP$

$$d(z,Tz) \preceq \varepsilon + \varphi\left(d(z,Tz)\right). \tag{25}$$

According to Lemma 2.10, we get that

$$d(z,Tz) \preceq \varphi\left(d(z,Tz)\right). \tag{26}$$

By considering the monotone property of φ, we have

$$d(z,Tz) \preceq \varphi^n\left(d(z,Tz)\right) \tag{27}$$

for all $n \in \mathbb{N}$. Since $\sum_{n=0}^{\infty} \varphi^n(d(z,Tz)$ is convergent, we get that $\varphi^n\left(d(z,Tz)\right)$ is a c-sequence by Proposition 3.3. Thus we have that for each $\varepsilon \in intP$, there is $n_1 \in \mathbb{N}$ such that

$$d(z,Tz) \preceq \varphi^n\left(d(z,Tz)\right) \ll \varepsilon \text{ for } n \geq n_1. \tag{28}$$

Hence, by Lemma 2.4, we obtain that $Tz = z$. □

Taking $u(x,y) = d(x,y)$ in Theorem 3.7, we get the following result:

Corollary 3.8. *Let $T : X \to X$ be a (φ,l)-almost contraction in the setup of a complete cone metric space (X,d) over a Banach algebra $\mathscr{A}$ with a solid cone P. Then T has at least one fixed point in X.*

Remark 3.9. Let $\{u_n\}$ be a sequence in P. It is known from [27] that if $\{u_n\}$ is a θ-sequence, that is, $u_n \to \theta$ as $n \to \infty$, then $\{u_n\}$ is a c-sequence. But its converse is not true unless P is a normal cone. Thus, as a basic result of this fact, one can easily obtain the following lemma, which is necessary to introduce error estimate results for (φ,l)-almost mappings:

Lemma 3.10. *(See [10]) Let (X,d) be a cone metric space over $\mathscr{A}$ with a normal cone P and $\{z_n\} \subset X$ be a sequence converging to $z \in X$. Then, for all $p,n \in \mathbb{N}$, $d(z_n,z_{n+p}) \to d(z_n,z)$ as $p \to \infty$.*

Corollary 3.11. *Suppose that X is a complete cone metric over $\mathscr{A}$, and T is (φ,l)-almost contraction, but P is a normal solid cone. Then,*

(i) Priori error estimate is available as $d(z_n,z) \preceq f(\varphi^n(d(z_0,z_1)))$

(ii) Posteriori error estimate is available as $d(z_n,z) \preceq f(d(z_n,z_{n+1}))$

where $f(t) = \sum_{i=0}^{\infty} \varphi^i(t)$ for $t \in P$ and $Tz = z$ such that $z_{n+1} = Tz_n$, $n = 0,1,2,\ldots$ converges to z.

Proof. By using (12) together with the properties of φ, we obtain

$$d(z_{n+i}, z_{n+i+1}) \preceq \varphi^i(d(z_n, z_{n+1})) \text{ for } i, n \in \mathbb{N}. \tag{29}$$

Thus, it follows by the triangle rule that

$$d(z_n, z_{n+p}) \preceq \sum_{i=0}^{n+p-1} \varphi^i(d(z_n, z_{n+1})). \tag{30}$$

Theorefore, by letting $p \to \infty$ in (30), we obtain the posteriori error estimate. Morever, considering (20) with the posteriori error estimate, we get the priori error estimate. □

Theorem 3.12. *Let X and T be as in Theorem 3.7. If there exist a weak c-comparison function $\varphi_1 : P \to P$ and some $l_1 \in P$ such that*

$$d(Tx, Ty) \preceq \varphi_1(d(x, y)) + l_1(d(x, Tx)), \tag{31}$$

then the equation $Tx = x$ holds for only one element x in X.

Proof. Assume that there are two elements x_1 and x_2 such that $Tx_1 = x_1$ and $Tx_2 = x_2$. Then, by (31), we have

$$d(x_1, x_2) = d(Tx_1, Tx_2) \preceq \varphi_1(u(x_1, x_2)) + l_1 d(x_1, Tx_1) \tag{32}$$

which implies that

$$d(x_1, x_2) \preceq \varphi_1(u(x_1, x_2)), \tag{33}$$

where $u(x_1, x_2) \in \{d(x_1, x_2), \theta\}$. If $u(x_1, x_2) = \theta$, we have that

$$d(x_1, x_2) \preceq \varphi_1(\theta) = \theta, \tag{34}$$

which satisfies $x_1 = x_2$. Let suppose that $u(x_1, x_2) = d(x_1, x_2)$. Thus, making use the properties of weak c-comparison function, by induction, we derive the following inequality:

$$d(x_1, x_2) \preceq \varphi_1^n(d(x_1, x_2)). \tag{35}$$

Since $\sum_{n=0}^{\infty} \varphi_1^n(d(x_1, x_2))$ is convergent according to the definition of weak c-comparison, it is clear from Proposition 3.3 that $\{\varphi_1^n(d(x_1, x_2))\}$ is a c-sequence. That is, for each $c \in intP$, there is $n_0 \in \mathbb{N}$ such that

$$d(x_1, x_2) \preceq \varphi_1^n(d(x_1, x_2)) \ll c \text{ for } n \geq n_0. \tag{36}$$

Hence, by Lemma 2.4, it follows that $d(x_1, x_2) = \theta$, implying that $x_1 = x_2$. □

Corollary 3.13. *(see [5]) Suppose that (X, d) is a complete metric space and $T : X \to X$ is a mapping satisfying*

$$d(Tx, Ty) \preceq \delta d(x, y) + Ld(y, Ty) \textit{ for all } x, y \in X \tag{37}$$

for $\delta \in (0, 1)$ and some $L \geq 0$. Then

1) $F(T) = \{x \in X : Tx = x\} \neq \emptyset$;

2) For arbitrary $x_0 \in X$, the Picard iteration $x_{n+1} = Tx_n$ for $n \geq 0$ converges to some $x \in F(T)$;

3) The following estimates

$$\begin{aligned} d(x_n, x) &\leq \frac{\delta^n}{1-\delta} d(x_0, x_1), n = 0, 1, 2, \ldots \\ d(x_n, x) &\leq \frac{\delta}{1-\delta} d(x_{n-1}, x_n), n = 0, 1, 2, \ldots \end{aligned} \tag{38}$$

Proof. By letting $\mathscr{A} = \mathbb{R}$, $k = \delta$ and $l = L$, the proof appears as a special case of that of Corollary 3.11. □

4. Conclusion

We first introduced the notion of weak c-comparison mapping and proved some associated fixed point theorems in cone metric spaces over Banach algebras. We showed that theorems given in this paper contains many well known ones in the frame of cone metric spaces over Banach algebras. Finally we also observed that priori and posteriori error estimates can be constructed when the underlying cone of Banach algebra is normal.

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