On some generalized integral transforms and Parseval-Goldstein type relations

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Abstract

In the present article, using the generalized Bessel-Maitland transform, the Laplace transform and the other known transforms, authors obtain new Parseval-Goldstein type relations. Using these relations, some generalized integrals involving Fox-Wright functions are evaluated. Illustrative examples are also given.

1. Introduction, definitions and preliminaries

In 2014, Singh et al [12] defined a new special function which is called generalized Bessel-Maitland function. They considered certain properties of this function and obtained a number of results. They showed Mellin-Barnes integral representation and the relationship with Wright hypergeometric function of generalized Bessel-Maitland function. Then, they calculated images of the function under some integral transforms such as Laplace, Mellin, $\mathcal{K}$- transform.

Albayrak et al [1] introduced a new integral transform whose kernel involves generalized Bessel-Maitland function. The transform is called generalized Bessel-Maitland integral transform. Firstly, new identities for generalized Bessel-Maitland function were obtained. By using these identities, some properties for generalized Bessel-Maitland integral transform were presented. Images of some elementary and special functions under generalized Bessel-Maitland integral transform were calculated and some special cases of them were shown.

In this study, our aim is to investigate Parseval-Goldstein type relations between generalized Bessel-Maitland integral transform and Laplace, Mellin, Hankel transforms which are well known in the literature [5, 6, 9]. By using these relations, we will calculate some generalized integrals involving Fox-Wright functions.

Now, we start with basic definitions and facts for understanding of this study.

Definition 1.1. The generalized hypergeometric series is defined by [7],

$$
{}_rF_s \begin{bmatrix} \alpha_1, \alpha_2, ..., \alpha_r \\ \beta_1, \beta_2, ..., \beta_s \end{bmatrix} z = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n (\alpha_2)_n ... (\alpha_r)_n}{(\beta_1)_n (\beta_2)_n ... (\beta_s)_n} \frac{z^n}{n!} , \tag{1.1}
$$

where $r, s \in \mathbb{Z}^+ \cup \{0\}$ and α_i, $\beta_j \neq 0, -1, -2, ...$ $(1 \leq i \leq r, 1 \leq j \leq s)$ and $(\alpha)_n$ is Pochammer symbol.

Definition 1.2. The Fox-Wright function is a generalization of the hypergeometric function and defined as [16]:

$$
{}_p\Psi_q \begin{bmatrix} (a_1, \alpha_1), (a_2, \alpha_2), ..., (a_p, \alpha_p) \\ (b_1, \beta_1), (b_2, \beta_2), ..., (b_q, \beta_q) \end{bmatrix} z
$$
$$
= \sum_{n=0}^{\infty} \frac{\Gamma(a_1 + \alpha_1 n) \Gamma(a_2 + \alpha_2 n) ... \Gamma(a_p + \alpha_p n)}{\Gamma(b_1 + \beta_1 n) \Gamma(b_2 + \beta_2 n) ... \Gamma(b_q + \beta_q n)} \frac{z^n}{n!} , \tag{1.2}
$$

where $p, q \in \mathbb{Z}^+ \cup \{0\}$, $a_i, b_j \in \mathbb{C}$ and $\alpha_i, \beta_j \in \mathbb{R}$ $(1 \leq i \leq p, 1 \leq j \leq q)$. The generalized hypergeometric series and the Fox-Wright function are related by the following formula [7],

$$
{}_p\Psi_q \begin{bmatrix} (a_1, 1), (a_2, 1), ..., (a_p, 1) \\ (b_1, 1), (b_2, 1), ..., (b_q, 1) \end{bmatrix} z = \frac{\Gamma(a_1) \Gamma(a_2) ... \Gamma(a_p)}{\Gamma(b_1) \Gamma(b_2) ... \Gamma(b_q)} {}_pF_q \begin{bmatrix} a_1, a_2, ..., a_p \\ b_1, b_2, ..., b_q \end{bmatrix} z . \tag{1.3}
$$

Definition 1.3. Singh [12] defined a generalization of the Bessel-Maitland function in 2014 as follows:

$$
\mathcal{J}_{\nu,p}^{\mu,\gamma}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{pn} (-z)^n}{n! \Gamma(\mu n + \nu + 1)} = \frac{1}{\Gamma(\gamma)} {}_1\Psi_1 \begin{bmatrix} (\gamma, p) \\ (\nu + 1, \mu) \end{bmatrix} - z , \tag{1.4}
$$

where $\mu, \nu, \gamma \in \mathbb{C}$; $\mathrm{Re}(\mu) \geq 0$, $\mathrm{Re}(\nu) \geq -1$, $\mathrm{Re}(\gamma) \geq 0$, $p \in (0,1) \cup \mathbb{N}$ and $(\gamma)_0 = 1$, $(\gamma)_{pn} = \dfrac{\Gamma(\gamma + pn)}{\Gamma(\gamma)}$.

Definition 1.4. Riemann-Liouville fractional integral operator of order α is defined as follows [8, 11]:

$$
\mathcal{D}^{-\alpha} \{ f(x); y \} = \frac{1}{\Gamma(\alpha)} \int_0^y (y - x)^{\alpha - 1} f(x) \, dx, \tag{1.5}
$$

where $y \geq 0$, $\alpha \in \mathbb{C}$, $\mathrm{Re}(\alpha) > 0$.

Definition 1.5. Weyl fractional integral operator of order α is defined as follows [8, 11]:

$$
\mathcal{W}^{-\alpha} \{ f(x); y \} = \frac{1}{\Gamma(\alpha)} \int_y^{\infty} (x - y)^{\alpha - 1} f(x) \, dx, \tag{1.6}
$$

where $y \geq 0$, $\alpha \in \mathbb{C}$, $\mathrm{Re}(\alpha) > 0$.

Definition 1.6. Let $f(t)$ be a real- or complex-valued function and y be a real- or complex-valued parameter. Any integral transform of a function $f(t)$ defined in $0 \leq t \leq \infty$ is denoted by

$$
\mathcal{T} \{ f(t); y \} = \int_0^{\infty} K(y, t) f(t) \, dt, \tag{1.7}
$$

where $K(y, t)$ is given function of two variables y and t, is called kernel of transform. The operator $\mathcal{T}$ is called an integral transform operator or simply an integral transformation. The transform of a function is often referred as the image of given object function $f(t)$, and y is called the transform variable. Some integral transforms which have different kernels are given in the Table-1. (see [3, 5, 6])

Table 1. Table of Some Integral Transforms

Transform	Symbol	Kernel	Parameter
Laplace Transform	$\mathcal{L}$	e^{-yt}	$\operatorname{Re} y > 0$
Fourier Sine Transform	$\mathcal{F}_s$	$\sin(yt)$	$\operatorname{Re} y > 0$
Fourier Cosine Transform	$\mathcal{F}_c$	$\cos(yt)$	$\operatorname{Re} y > 0$
Mellin Transform	$\mathcal{M}$	t^{y-1}	$y \in \mathbb{C}$
Widder Potential Transform	$\mathcal{P}$	$\dfrac{t}{t^2 + y^2}$	$\operatorname{Re} y > 0$
Stieltjes Transform	$\mathbb{S}$	$\dfrac{1}{t + y}$	$\lvert \arg y \rvert < \pi$
Generalized Stieltjes Transform	$\mathbb{S}_\alpha$	$\dfrac{1}{(t + y)^\alpha}$	$\lvert \arg y \rvert < \pi$

Definition 1.7. The following generalized Bessel-Maitland integral transform was introduced in [1], as follows:

$$_\alpha \mathcal{H}_{\nu,p}^{\mu,\gamma} \{f(t); s\} = \int_0^\infty (st)^\alpha \mathcal{J}_{\nu,p}^{\mu,\gamma}(st) f(t) \, dt, \tag{1.8}$$

where $\alpha, \mu, \nu, \gamma, s \in \mathbb{C}$; $\operatorname{Re}(\alpha) \geq 0$, $\operatorname{Re}(\mu) \geq 0$, $\operatorname{Re}(\gamma) \geq 0$, $\operatorname{Re}(\nu) \geq -1$, $\operatorname{Re}(s) \geq 0$, and $p \in (0,1) \cup \mathbb{N}$.

The Generalized Bessel-Maitland transform and the Laplace transform are related by the following relation [1],

$$_\alpha \mathcal{H}_{0,p}^{p,1} \{f(t); s\} = s^\alpha \mathcal{L} \{t^\alpha f(t); s\}. \tag{1.9}$$

After this, we will use the following notations for the integral transforms:

$$_\alpha \mathcal{H}_{\nu,p}^{\mu,\gamma} = \mathcal{H}, \quad _\alpha \mathcal{H}_{\nu,p}^{\mu,\gamma} \{f(t); x\} = \mathcal{H}\{x\}, \quad \mathcal{T}\{g(y); x\} = \mathcal{T}\{x\}.$$

In the following section, new Parseval- Goldstein type identities are proved. The definitions given in the first part are used to obtain these identities. Triple generalized integrals are evaluated for some elementary and special functions which are not in the integral tables.

2. Main theorems

Lemma 2.1. *The following identity,*

$$\mathcal{H}\{\mathcal{T}\{f(t); x\}; y\} = \int_0^\infty f(t) \ \mathcal{H}\{K(t,x); y\} \, dt, \tag{2.1}$$

holds true, provided that the integrals involved converge absolutely.

Proof. By using the definitions (1.7), (1.8), respectively and changing the order of integration, we obtain (2.1). $\qquad \square$

Lemma 2.2. *The following identity,*

$$\mathcal{T}\{\mathcal{H}\{f(t); x\}; y\} = \int_0^\infty f(t) \mathcal{H}\{K(y,x); t\} \, dt, \tag{2.2}$$

holds true, provided that the integrals involved converge absolutely.

Proof. The proof is similar to the proof of Lemma 2.1. $\qquad \square$

Theorem 2.3. *The following identities,*

$$\int_0^\infty \mathcal{H}\{x\} \mathcal{T}\{x\} \, dx = \int_0^\infty g(y) \left(\int_0^\infty f(t) \mathcal{H}\{K(y,x); t\} \, dt \right) dy \tag{2.3}$$

and

$$\int_0^\infty \mathcal{H}\{x\}\,\mathcal{T}\{x\}\,dx = \int_0^\infty f(t)\left(\int_0^\infty g(y)\,\mathcal{H}\{K(y,x)\,;t\}\,dy\right)dt \tag{2.4}$$

hold true, provided that the integrals involved converge absolutely.

Proof. By using the definition (1.7) and changing the order of integration, then using the definition (1.8), we find

$$\int_0^\infty \mathcal{H}\{x\}\,\mathcal{T}\{x\}\,dx = \int_0^\infty g(y)\,\mathcal{T}\{\mathcal{H}\{f(t)\,;x\}\,;y\}\,dy.$$

The proof of (2.4) follows from (2.2) of Lemma 2.2. Similarly, by using the definition (1.8), then changing the order of integration and using the definition (1.7), we have

$$\int_0^\infty \mathcal{H}\{x\}\,\mathcal{T}\{x\}\,dx = \int_0^\infty f(t)\,\mathcal{H}\{\mathcal{T}\{g(y)\,;x\}\,;t\}\,dt.$$

Now, by using Lemma 2.1, we arrive at (2.3). $\square$

Lemma 2.4. *The following identities,*

$$\mathcal{H}\{\mathcal{M}\{f(t)\,;x\}\,;y\} = y^\alpha \int_0^\infty \frac{f(t)}{t\,(-\ln t)^{1+\alpha}}\,{}_2\Psi_1\left(\frac{y}{\ln t}\right)dt \tag{2.5}$$

and

$$\mathcal{M}\{\mathcal{H}\{f(t)\,;x\}\,;y\} = \frac{\Gamma(\alpha+y)\,\Gamma(\gamma-p(\alpha+y))}{\Gamma(\gamma)\,\Gamma(1+\nu-\mu(\alpha+y))}\mathcal{M}\{f(t)\,;1-y\} \tag{2.6}$$

hold true, provided that the integrals involved converge absolutely, where

$$_2\Psi_1(z) = \frac{1}{\Gamma(\gamma)}\,{}_2\Psi_1\left[\begin{matrix}(\gamma,p),(\alpha+1,1)\\(1+v,\mu)\end{matrix}\bigg|z\right]. \tag{2.7}$$

Proof. By setting $K(x,t) = t^{x-1}$, which is the kernel of the Mellin transform, in Lemma 2.1 and using the following formula for $\mathrm{Re}\,a > 0$, (see [1])

$$\mathcal{H}\{e^{-at}\,;s\} = \frac{s^\alpha}{a^{\alpha+1}\Gamma(\gamma)}\,{}_2\Psi_1\left[\begin{matrix}(\gamma,p),(\alpha+1,1)\\(1+\nu,\mu)\end{matrix}\bigg|\frac{-s}{a}\right], \tag{2.8}$$

we obtain (2.5). Similarly, by setting $K(y,x) = x^{y-1}$, which is the kernel of the Mellin transform, in Lemma 2.2 and using the formula for $\mathrm{Re}\,\beta > -1$, [1]

$$\mathcal{H}\{t^\beta\,;s\} = \frac{\Gamma(\alpha+\beta+1)\,\Gamma(\gamma-p(\alpha+\beta+1))}{\Gamma(\gamma)\,\Gamma(1+\nu-\mu(\alpha+\beta+1))}\frac{1}{s^{\beta+1}}, \tag{2.9}$$

we find (2.6). $\square$

Remark 2.5. If we set $\nu = 0$, $\mu = p$, $\gamma = 1$, $\alpha = 0$ in (2.6), then we obtain the following identity which was obtained earlier [9, p.3, Eq(c')],

$$\mathcal{M}\{\mathcal{L}\{f(t)\,;x\}\,;y\} = \Gamma(y)\,\mathcal{M}\{f(t)\,;1-y\}.$$

Corollary 2.6. *The following identities,*

$$\int_0^\infty \mathcal{H}(x)\,\mathcal{M}(x)\,dx = \int_0^\infty \frac{g(y)}{y\,(-\ln y)^{1+\alpha}}\left[\int_0^\infty t^\alpha f(t)\,{}_2\Psi_1\left(\frac{t}{\ln y}\right)dt\right]dy \tag{2.10}$$

and

$$\int_0^\infty \mathcal{H}\{x\}\,\mathcal{M}\{x\}\,dx = \int_0^\infty t^\alpha f(t)\left(\int_0^\infty \frac{g(y)}{y\,(-\ln y)^{1+\alpha}}\,{}_2\Psi_1\left(\frac{t}{\ln y}\right)dy\right)dt \tag{2.11}$$

hold true, provided that the integrals involved converge absolutely, where ${}_2\Psi_1(z)$ is defined by (2.7).

Proof. By setting $K(x, y) = y^{x-1}$, which is the kernel of the Mellin transform, in (2.3) and (2.4) of Theorem 2.3, and using the formula (2.8), we arrive at (2.10) and (2.11), respectively. $\square$

Lemma 2.7. *The following identities,*

$$\mathcal{H}\left\{\mathcal{L}\left\{f(t);x\right\};y\right\} = y^{\alpha} \int_0^{\infty} \frac{f(t)}{t^{\alpha+1}} \, {}_2\Psi_1\left(-\frac{y}{t}\right) dt \tag{2.12}$$

and

$$\mathcal{L}\left\{\mathcal{H}\left\{f(t);x\right\};y\right\} = \frac{1}{y^{\alpha+1}} \int_0^{\infty} t^{\alpha} f(t) \, {}_2\Psi_1\left(-\frac{t}{y}\right) dt, \tag{2.13}$$

hold true, provided that the integrals involved converge absolutely, where ${}_2\Psi_1(z)$ is defined by (2.7).

Proof. By setting $K(x, t) = e^{-xt}$, which is the kernel of the Laplace transform, in Lemma 2.1 and using the known formula (2.8), we obtain (2.12). Similarly, by setting $K(y, x) = e^{-yx}$, which is the kernel of the Laplace transform, in Lemma 2.2 and using the known formula (2.8), we obtain (2.13). $\square$

Remark 2.8. By setting $\nu = 0, \mu = p, \gamma = 1$ in Lemma 2.7, using the relations (1.9) and the identity,

$$ {}_1\Psi_0\left[\begin{matrix}(\alpha+1,1)\\-\end{matrix}\middle| z\right] = \frac{\Gamma(\alpha+1)}{(1-z)^{\alpha+1}}, \tag{2.14}$$

we get the following formulas which were obtained earlier [14, p.65, Eq(13)],

$$\mathcal{L}\left\{x^{\alpha}\mathcal{L}\left\{f(t);x\right\};y\right\} = \Gamma(\alpha+1)\,\mathcal{S}_{\alpha+1}\left\{f(t);y\right\}, \tag{2.15}$$

$$\mathcal{L}\left\{x^{\alpha}\mathcal{L}\left\{t^{\alpha}f(t);x\right\};y\right\} = \Gamma(\alpha+1)\,\mathcal{S}_{\alpha+1}\left\{t^{\alpha}f(t);y\right\}. \tag{2.16}$$

Remark 2.9. By setting $\alpha = 0$ in (2.15) and (2.16) of Remark 2.8, we obtain the following formula which was obtained earlier [15, p.7],

$$\mathcal{L}\left\{\mathcal{L}\left\{f(t);x\right\};y\right\} = \mathcal{S}\left\{f(t);y\right\}. \tag{2.17}$$

Corollary 2.10. *The following identities,*

$$\int_0^{\infty} \mathcal{H}\left\{x\right\}\mathcal{L}\left\{x\right\} dx = \int_0^{\infty} \frac{g(y)}{y^{\alpha+1}}\left(\int_0^{\infty} t^{\alpha} f(t) \, {}_2\Psi_1\left(-\frac{t}{y}\right) dt\right) dy \tag{2.18}$$

and

$$\int_0^{\infty} \mathcal{H}\left\{x\right\}\mathcal{L}\left\{x\right\} dx = \int_0^{\infty} t^{\alpha} f(t)\left(\int_0^{\infty} \frac{g(y)}{y^{\alpha+1}} \, {}_2\Psi_1\left(-\frac{t}{y}\right) dy\right) dt \tag{2.19}$$

hold true, provided that the integrals involved converge absolutely, where ${}_2\Psi_1(z)$ is defined by (2.7).

Proof. By setting $K(x, y) = e^{-xy}$, which is the kernel of the Laplace transform, in (2.3) and (2.4) of Theorem 2.3, and using the formula (2.8), we arrive at (2.18) and (2.19), respectively. $\square$

Remark 2.11. By setting $\nu = 0, \mu = p, \gamma = 1$ in Corollary 2.10, using the relations (1.9) and (2.14), we have the following Parseval-Goldstein type relations, which were obtained earlier [14, p.65, Eq(15)],

$$\int_0^{\infty} x^{\alpha}\mathcal{L}\left\{t^{\alpha}f(t);x\right\}\mathcal{L}\left\{g(y);x\right\} dx = \Gamma(\alpha+1)\int_0^{\infty} g(y)\,\mathcal{S}_{\alpha+1}\left\{t^{\alpha}f(t);y\right\} dy \tag{2.20}$$

and

$$\int_0^{\infty} x^{\alpha}\mathcal{L}\left\{t^{\alpha}f(t);x\right\}\mathcal{L}\left\{g(y);x\right\} dx = \Gamma(\alpha+1)\int_0^{\infty} t^{\alpha} f(t)\,\mathcal{S}_{\alpha+1}\left\{g(y);t\right\} dt, \tag{2.21}$$

where $\mathrm{Re}(\alpha) > -1$.

Remark 2.12. By setting $\alpha = 0$ in Remark 2.11, we obtain the following formulas which were obtained earlier [13, p.243, Eq(13)],

$$\int_0^\infty \mathcal{L}\left\{f\left(t\right);x\right\}\mathcal{L}\left\{g\left(y\right);x\right\}dx = \int_0^\infty g\left(y\right)\mathcal{S}\left\{f\left(t\right);y\right\}dy \tag{2.22}$$

and

$$\int_0^\infty \mathcal{L}\left\{f\left(t\right);x\right\}\mathcal{L}\left\{g\left(y\right);x\right\}dx = \int_0^\infty f\left(t\right)\mathcal{S}\left\{g\left(y\right);t\right\}dt. \tag{2.23}$$

Corollary 2.13. *The following identities hold true,*

$$\mathcal{H}\left\{\mathcal{F}_s\left\{f\left(t\right);x\right\};y\right\} = y^\alpha \cos\left(\frac{\pi\alpha}{2}\right)\int_0^\infty \frac{f\left(t\right)}{t^{\alpha+1}}\,{}_3\Psi_2\left(-\frac{y^2}{t^2}\right)dt$$

$$+ y^{\alpha+1}\sin\left(\frac{\pi\alpha}{2}\right)\int_0^\infty \frac{f\left(t\right)}{t^{\alpha+2}}\,{}_3\Psi_2^*\left(-\frac{y^2}{t^2}\right)dt \tag{2.24}$$

and

$$\mathcal{F}_s\left\{\mathcal{H}\left\{f\left(t\right);x\right\};y\right\} = \frac{1}{y^{\alpha+1}}\cos\left(\frac{\pi\alpha}{2}\right)\int_0^\infty t^\alpha f\left(t\right)\,{}_3\Psi_2\left(-\frac{t^2}{y^2}\right)dt$$

$$+ \frac{1}{y^{\alpha+2}}\sin\left(\frac{\pi\alpha}{2}\right)\int_0^\infty t^{\alpha+1} f\left(t\right)\,{}_3\Psi_2^*\left(-\frac{t^2}{y^2}\right)dt, \tag{2.25}$$

provided that the integrals involved converge absolutely, where

$$_3\Psi_2\left(z\right) = \frac{1}{\Gamma\left(\gamma\right)}\,{}_3\Psi_2\left[\begin{matrix}\left(\gamma,2p\right),\left(\alpha+1,2\right),\left(1,1\right)\\\left(1+v,2\mu\right),\left(1,2\right)\end{matrix}\middle|z\right], \tag{2.26}$$

$$_3\Psi_2^*\left(z\right) = \frac{1}{\Gamma\left(\gamma\right)}\,{}_3\Psi_2\left[\begin{matrix}\left(\gamma+p,2p\right),\left(\alpha+2,2\right),\left(1,1\right)\\\left(1+v+\mu,2\mu\right),\left(2,2\right)\end{matrix}\middle|z\right]. \tag{2.27}$$

Proof. By setting $K\left(x,t\right) = \sin\left(xt\right)$, which is the kernel of the Fourier Sine transform, in Lemma 2.1 and following the known formula (see [1]),

$$\mathcal{H}\left\{\sin\left(at\right);s\right\} = \frac{s^\alpha}{a^{\alpha+1}}\cos\left(\frac{\pi\alpha}{2}\right)\,{}_3\Psi_2\left(-\frac{s^2}{a^2}\right) + \frac{s^{\alpha+1}}{a^{\alpha+2}}\sin\left(\frac{\pi\alpha}{2}\right)\,{}_3\Psi_2^*\left(-\frac{s^2}{a^2}\right), \quad (2.28)$$

for $\text{Re}\,a > 0$, we arrive at (2.24). By setting $K\left(y,x\right) = \sin\left(yx\right)$, which is the kernel of the Fourier Sine transform, in Lemma 2.2 and using the formula (2.28) for $\text{Re}\,y > 0$, we obtain (2.25). $\qquad\square$

Remark 2.14. If we set $\nu = 0, \mu = p, \gamma = 1, \alpha = 0$ in (2.24) and (2.25), then we obtain the following known formulas,

$$\mathcal{L}\left\{\mathcal{F}_s\left\{f\left(t\right);x\right\};y\right\} = \mathcal{P}\left\{f\left(t\right);y\right\}$$

and

$$\mathcal{F}_s\left\{\mathcal{L}\left\{f\left(t\right);x\right\};y\right\} = y\mathcal{P}\left\{\frac{f\left(t\right)}{t};y\right\}.$$

These relations were obtained earlier in [4, p.4, Eq(23)-Eq(24)] for generalized transforms.

Corollary 2.15. *The following identities hold true,*

$$\int_0^\infty \mathcal{H}\left\{x\right\}\mathcal{F}_s\left\{x\right\}dx = \cos\left(\frac{\pi\alpha}{2}\right)\int_0^\infty \frac{g\left(y\right)}{y^{\alpha+1}}\left(\int_0^\infty t^\alpha f\left(t\right)\,{}_3\Psi_2\left(-\frac{t^2}{y^2}\right)dt\right)dy$$

$$+ \sin\left(\frac{\pi\alpha}{2}\right)\int_0^\infty \frac{g\left(y\right)}{y^{\alpha+2}}\left(\int_0^\infty t^{\alpha+1} f\left(t\right)\,{}_3\Psi_2^*\left(-\frac{t^2}{y^2}\right)dt\right)dy \tag{2.29}$$

and

$$\int_0^\infty \mathcal{H}\{x\}\,\mathcal{F}_s\{x\}\,dx = \cos\left(\frac{\pi\alpha}{2}\right)\int_0^\infty t^\alpha f(t)\left(\int_0^\infty \frac{g(y)}{y^{\alpha+1}}\,_3\Psi_2\left(-\frac{t2}{y^2}\right)dy\right)dt$$

$$+ \sin\left(\frac{\pi\alpha}{2}\right)\int_0^\infty t^{\alpha+1}f(t)\left(\int_0^\infty \frac{g(y)}{y^{\alpha+2}}\,_3\Psi_2^*\left(-\frac{t2}{y^2}\right)dy\right)dt, \quad (2.30)$$

provided that the integrals involved converge absolutely, where $_3\Psi_2(z)$ *and* $_3\Psi_2^*(z)$ *are defined in* (2.26) *and* (2.27), *respectively.*

Proof. By setting $K(x,y) = \sin(xy)$, which is the kernel of the Fourier Sine transform, in (2.3) and (2.4) of Theorem 2.3, and using the known formula (2.28), we arrive at (2.29) and (2.30), respectively. $\qquad\square$

Remark 2.16. If we set $\nu = 0, \mu = p, \gamma = 1, \alpha = 0$ in (2.29) and (2.30), then we obtain the following known formulas [4, p.7, Eq(43)-Eq(44)] for generalized transforms,

$$\int_0^\infty \mathcal{L}\{f(t);x\}\,\mathcal{F}_s\{g(y);x\}\,dx = \int_0^\infty yg(y)\,\mathcal{P}\left\{\frac{f(t)}{t};y\right\}dy$$

and

$$\int_0^\infty \mathcal{L}\{f(t);x\}\,\mathcal{F}_s\{g(y);x\}\,dx = \int_0^\infty f(t)\,\mathcal{P}\{g(y);t\}\,dt.$$

Lemma 2.17. *The following identities hold true,*

$$\mathcal{H}\{\mathcal{F}_c\{f(t);x\};y\} = y^{\alpha+1}\cos\left(\frac{\pi\alpha}{2}\right)\int_0^\infty \frac{f(t)}{t^{\alpha+2}}\,_3\Psi_2^*\left(-\frac{y^2}{t^2}\right)dt$$

$$- y^\alpha \sin\left(\frac{\pi\alpha}{2}\right)\int_0^\infty \frac{f(t)}{t^{\alpha+1}}\,_3\Psi_2\left(-\frac{y^2}{t^2}\right)dt \quad (2.31)$$

and

$$\mathcal{F}_c\{\mathcal{H}\{f(t);x\};y\} = \frac{1}{y^{\alpha+2}}\cos\left(\frac{\pi\alpha}{2}\right)\int_0^\infty t^{\alpha+1}f(t)\,_3\Psi_2^*\left(-\frac{t2}{y^2}\right)dt$$

$$- \frac{1}{y^{\alpha+1}}\sin\left(\frac{\pi\alpha}{2}\right)\int_0^\infty t^\alpha f(t)\,_3\Psi_2\left(-\frac{t2}{y^2}\right)dt, \quad (2.32)$$

provided that the integrals involved converge absolutely, where $_3\Psi_2(z)$ *and* $_3\Psi_2^*(z)$ *are defined in* (2.26) *and* (2.27), *respectively.*

Proof. By setting $K(x,t) = \cos(xt)$, which is the kernel of the Fourier Cosine transform, in Lemma 2.1 and using the following known formula (see [1]),

$$\mathcal{H}\{\cos(at);s\} = \frac{s^{\alpha+1}}{a^{\alpha+2}}\cos\left(\frac{\pi\alpha}{2}\right)\,_3\Psi_2^*\left(-\frac{s^2}{a^2}\right) - \frac{s^\alpha}{a^{\alpha+1}}\sin\left(\frac{\pi\alpha}{2}\right)\,_3\Psi_2\left(-\frac{s^2}{a^2}\right), \quad (2.33)$$

for Re$a > 0$, we obtain (2.31). By setting $K(y,x) = \cos(yx)$, which is the kernel of the Fourier Cosine transform, in Lemma 2.2 and using the formula (2.33) for Re$y > 0$, we find (2.25). $\qquad\square$

Remark 2.18. If we set $\nu = 0, \mu = p, \gamma = 1, \alpha = 0$ in (2.24) and (2.25), then we obtain the following known formulas [4, p.4, Eq(25),Eq(26)] for generalized transforms,

$$\mathcal{L}\{\mathcal{F}_c\{f(t);x\};y\} = y\mathcal{P}\left\{\frac{f(t)}{t};y\right\}$$

and

$$\mathcal{F}_c\{\mathcal{L}\{f(t);x\};y\} = \mathcal{P}\{f(t);y\}.$$

Corollary 2.19. *The following identities hold true,*

$$\int_0^\infty \mathcal{H}\{x\}\,\mathcal{F}_c\{x\}\,dx = \cos\left(\frac{\pi\alpha}{2}\right)\int_0^\infty \frac{g(y)}{y^{\alpha+2}}\left(\int_0^\infty t^{\alpha+1}f(t)\ {}_3\Psi_2^*\left(-\frac{t^2}{y^2}\right)dt\right)dy$$

$$-\sin\left(\frac{\pi\alpha}{2}\right)\int_0^\infty \frac{g(y)}{y^{\alpha+1}}\left(\int_0^\infty t^\alpha f(t)\ {}_3\Psi_2\left(-\frac{t^2}{y^2}\right)dt\right)dy \qquad (2.34)$$

and

$$\int_0^\infty \mathcal{H}\{x\}\,\mathcal{F}_c\{x\}\,dx = \cos\left(\frac{\pi\alpha}{2}\right)\int_0^\infty t^{\alpha+1}f(t)\left(\int_0^\infty \frac{g(y)}{y^{\alpha+2}}\ {}_3\Psi_2^*\left(-\frac{t^2}{y^2}\right)dy\right)$$

$$-\sin\left(\frac{\pi\alpha}{2}\right)\int_0^\infty t^\alpha f(t)\left(\int_0^\infty \frac{g(y)}{y^{\alpha+1}}\ {}_3\Psi_2\left(-\frac{t^2}{y^2}\right)dy\right)dt, \qquad (2.35)$$

provided that the integrals involved converge absolutely, where ${}_3\Psi_2(z)$ *and* ${}_3\Psi_2^*(z)$ *are defined in (2.26) and (2.27), respectively.*

Proof. By setting $K(x,y) = \cos(xy)$, which is the kernel of the Fourier Cosine transform, in (2.3) and (2.4) of Theorem 2.3, and using the formula (2.33), we arrive at (2.34) and (2.35), respectively. □

Remark 2.20. If we set $\nu = 0, \mu = p, \gamma = 1, \alpha = 0$ in (2.29) and (2.30), then we obtain the following known formulas [4, p.7, Eq(45),Eq(46)] for generalized transforms,

$$\int_0^\infty \mathcal{L}\{f(t);x\}\,\mathcal{F}_c\{g(y);x\}\,dx = \int_0^\infty g(y)\,\mathcal{P}\{f(t);y\}\,dy$$

and

$$\int_0^\infty \mathcal{L}\{f(t);x\}\,\mathcal{F}_c\{g(y);x\}\,dx = \int_0^\infty tf(t)\,\mathcal{P}\left\{\frac{g(y)}{y};t\right\}dt.$$

Lemma 2.21. *The following identities hold true,*

$$\mathcal{H}\{\mathcal{S}\{f(t);x\};y\} = \frac{\pi}{\sin(\pi\alpha)}\int_0^\infty f(t)\ {}_2\Psi_2(ty)\,dt$$

$$-\frac{\pi y^\alpha}{\sin(\pi\alpha)}\int_0^\infty t^\alpha f(t)\ {}_1\Psi_1(ty)\,dt \qquad (2.36)$$

and

$$\mathcal{S}\{\mathcal{H}\{f(t);x\};y\} = \frac{\pi}{\sin(\pi\alpha)}\int_0^\infty f(t)\ {}_2\Psi_2(ty)\,dt$$

$$-\frac{\pi y^\alpha}{\sin(\pi\alpha)}\int_0^\infty t^\alpha f(t)\ {}_1\Psi_1(ty)\,dt, \qquad (2.37)$$

provided that the integrals involved converge absolutely, where

$$_1\Psi_1(z) = \frac{1}{\Gamma(\gamma)}\ {}_1\Psi_1\left[\begin{matrix}(\gamma,p)\\(1+v,\mu)\end{matrix}\middle|z\right], \qquad (2.38)$$

$$_2\Psi_2(z) = \frac{1}{\Gamma(\gamma)}\ {}_2\Psi_2\left[\begin{matrix}(\gamma-p\alpha,p),\ (1,1)\\(1+v-\mu\alpha,\mu),\ (1-\alpha,1)\end{matrix}\middle|z\right]. \qquad (2.39)$$

Proof. By setting $K(x,t) = \dfrac{1}{t+x}$, which is the kernel of the Stieltjes transform, in Lemma 2.1 and using the formula for $\mathrm{Re}\,a > 0$ (see [1]),

$$\mathcal{H}\left\{\frac{1}{a+t};s\right\} = -\frac{\pi(as)^\alpha}{\sin(\pi\alpha)}\ {}_1\Psi_1(as) + \frac{\pi}{\sin(\pi\alpha)}\ {}_2\Psi_2(as), \qquad (2.40)$$

we obtain (2.36). By setting $K(y,x) = \dfrac{1}{x+y}$, which is the kernel of the Stieltjes transform, in Lemma 2.2 and using the formula (2.40) for Re$y > 0$, we find (2.37). □

Remark 2.22. The following relation is a direct result of Lemma 2.21,

$$\mathcal{H}\left\{\mathcal{S}\left\{f\left(t\right);x\right\};y\right\} = \mathcal{S}\left\{\mathcal{H}\left\{f\left(t\right);x\right\};y\right\}. \tag{2.41}$$

Remark 2.23. If we set $\nu = 0$, $\gamma = 1$ and $\mu = p$ in (2.36) and use (1.9), then we obtain

$$\mathcal{L}\left\{x^{\alpha}\mathcal{S}\left\{f\left(t\right);x\right\};y\right\} = \frac{\pi y^{-\alpha}}{\sin\left(\pi\alpha\right)}\int_{0}^{\infty} f\left(t\right) \,_{1}\Psi_{1}\left[\begin{array}{c}(1,1)\\(1-\alpha,1)\end{array}\bigg| ty\right]dt$$
$$- \frac{\pi}{\sin\left(\pi\alpha\right)}\int_{0}^{\infty} t^{\alpha}f\left(t\right)e^{ty}dt.$$

Now, we use the relation (1.3) and the formula [10, p.465, Eq(45:6:2)],

$$_{1}F_{1}\left(1;1-\alpha;ty\right) = -\alpha e^{ty}t^{\alpha}y^{\alpha}\gamma\left(-\alpha,ty\right), \tag{2.42}$$

where $\gamma\left(.\right)$ is incomplete gamma function [10, p.463, Eq(45:3:1)], then we have

$$\mathcal{L}\left\{x^{\alpha}\mathcal{S}\left\{f\left(t\right);x\right\};y\right\} = \frac{\pi}{\sin\left(\pi\alpha\right)\Gamma\left(-\alpha\right)}\int_{0}^{\infty} f\left(t\right)e^{ty}t^{\alpha}\gamma\left(-\alpha,ty\right)dt$$
$$- \frac{\pi}{\sin\left(\pi\alpha\right)}\int_{0}^{\infty} t^{\alpha}f\left(t\right)e^{ty}dt.$$

Now, if we use the relationship between gamma function and incomplete gamma function [10, p.461,Eq(45:0:1)],

$$\Gamma\left(\nu\right) = \Gamma\left(\nu,x\right) + \gamma\left(\nu,x\right), \tag{2.43}$$

where $\Gamma\left(\nu,x\right)$ is complementary gamma function [10, p.463, Eq(45:3:2)], and the known formula [7, p.3,Eq(6)],

$$\frac{\pi}{\sin\left(\pi\alpha\right)} = \Gamma\left(\alpha\right)\Gamma\left(1-\alpha\right), \tag{2.44}$$

we find

$$\mathcal{L}\left\{x^{\alpha}\mathcal{S}\left\{f\left(t\right);x\right\};y\right\} = \Gamma\left(1+\alpha\right)\int_{0}^{\infty} t^{\alpha}e^{ty}\Gamma\left(-\alpha,ty\right)f\left(t\right)dt. \tag{2.45}$$

Remark 2.24. If we set $\nu = 0$, $\gamma = 1$ and $\mu = p$ in (2.37) and use (1.9), (2.42), (2.43) and (2.44), then we obtain

$$\mathcal{S}\left\{x^{\alpha}\mathcal{L}\left\{t^{\alpha}f\left(t\right);x\right\};y\right\} = y^{\alpha}\Gamma\left(1+\alpha\right)\int_{0}^{\infty} t^{\alpha}e^{ty}\Gamma\left(-\alpha,ty\right)f\left(t\right)dt. \tag{2.46}$$

Remark 2.25. If we set $\alpha = 0$ in (2.45) and (2.46), then we get the known relation [2, p.1378,Eq(2.1-2.2)],

$$\mathcal{L}\left\{\mathcal{S}\left\{f\left(t\right);x\right\};y\right\} = \mathcal{S}\left\{\mathcal{L}\left\{f\left(t\right);x\right\};y\right\} = \mathcal{E}_{1}\left\{f\left(t\right);y\right\}, \tag{2.47}$$

where $\mathcal{E}_{1}$ is the exponential integral transform, which has the kernel $K(y,t) = e^{ty}E_{1}\left(ty\right)$ and $E_{1}\left(ty\right) = \Gamma\left(0,ty\right)$ [2, p.1377,Eq(1.1)].

Corollary 2.26. *The following identities hold true,*

$$\int_{0}^{\infty}\mathcal{H}\left\{x\right\}\mathcal{S}\left\{x\right\}dx = \frac{\pi}{\sin\left(\pi\alpha\right)}\int_{0}^{\infty} g\left(y\right)\left(\int_{0}^{\infty} f\left(t\right) \,_{2}\Psi_{2}\left(yt\right)dt\right)dy$$
$$- \frac{\pi}{\sin\left(\pi\alpha\right)}\int_{0}^{\infty} y^{\alpha}g\left(y\right)\left(\int_{0}^{\infty} t^{\alpha}f\left(t\right) \,_{1}\Psi_{1}\left(yt\right)dt\right)dy \tag{2.48}$$

and

$$\int_0^\infty \mathcal{H}\{x\}\,\mathcal{S}\{x\}\,dx = \frac{\pi}{\sin(\pi\alpha)} \int_0^\infty f(t) \left(\int_0^\infty g(y)\ _2\Psi_2(ty)\,dy \right) dt$$

$$- \frac{\pi}{\sin(\pi\alpha)} \int_0^\infty t^\alpha f(t) \left(\int_0^\infty y^\alpha g(y)\ _1\Psi_1(ty)\,dy \right) dt, \qquad (2.49)$$

provided that the integrals involved converge absolutely, where $_1\Psi_1(z)$ *and* $_2\Psi_2(z)$ *are defined in* (2.38) *and* (2.39), *respectively.*

Proof. By setting $K(x,y) = \dfrac{1}{y+x}$, which is the kernel of the Stieltjes transform, in (2.3) and (2.4) of Theorem 2.3, and using the formula (2.40), we arrive at (2.48) and (2.49), respectively. $\qquad\square$

Remark 2.27. If we set $\nu = 0$, $\gamma = 1$ and $\mu = p$ in (2.48), (2.49) and use (1.9), (2.42), (2.43) and (2.44), then we obtain the following identities, respectively,

$$\int_0^\infty x^\alpha \mathcal{L}\{t^\alpha f(t);x\}\,\mathcal{S}\{g(y);x\}\,dx$$

$$= \Gamma(1+\alpha) \int_0^\infty y^\alpha g(y) \left(\int_0^\infty t^\alpha e^{ty}\Gamma(-\alpha,ty)f(t)\,dt \right) dy, \qquad (2.50)$$

$$\int_0^\infty x^\alpha \mathcal{L}\{t^\alpha f(t);x\}\,\mathcal{S}\{g(y);x\}\,dx$$

$$= \Gamma(1+\alpha) \int_0^\infty t^\alpha f(t) \left(\int_0^\infty y^\alpha e^{ty}\Gamma(-\alpha,ty)g(y)\,dy \right) dt. \qquad (2.51)$$

Remark 2.28. If we set $\alpha = 0$ in (2.50) and (2.51), we obtain the known relations [2, p.1381,Eq(3.1-3.2)],

$$\int_0^\infty \mathcal{L}\{f(t);x\}\,\mathcal{S}\{g(y);x\}\,dx = \int_0^\infty g(y)\,\mathcal{E}_1\{f(t);y\}\,dy, \qquad (2.52)$$

$$\int_0^\infty \mathcal{L}\{f(t);x\}\,\mathcal{S}\{g(y);x\}\,dx = \int_0^\infty f(t)\,\mathcal{E}_1\{g(y);t\}\,dt, \qquad (2.53)$$

where $\mathcal{E}_1$ is the exponential integral transform, which has the kernel $K(y,t) = e^{ty}E_1(ty)$ and $E_1(ty) = \Gamma(0,ty)$ [2, p.1377,Eq(1.1)].

Example 2.29. We show that

$$\int_0^\infty \frac{y^{1-\mu}}{y+a} \log\left(\frac{y}{a}\right) dy = \frac{\pi^2}{\sin^2(\pi\mu)}a^{1-\mu}, \qquad (2.54)$$

where $1 < \operatorname{Re}\nu < 2$, $\operatorname{Re}x > \operatorname{Re}a$.

If we set $f(t) = e^{-at}, \operatorname{Re}x > \operatorname{Re}a$ and $g(y) = y^{1-\mu}$, $1 < \operatorname{Re}\nu < 2$ in (2.52), then we have

$$\int_0^\infty \mathcal{L}\left\{e^{-at};x\right\}\,\mathcal{S}\left\{y^{1-\mu};x\right\}\,dx = \int_0^\infty y^{1-\mu}\mathcal{E}_1\left\{e^{-at};y\right\}\,dy.$$

By using the formulas [6, p.216 Eq(5)], [5, p.178,Eq(24)] and

$$\mathcal{E}_1\left\{e^{-at};y\right\} = \frac{1}{y+a}\log\left(\frac{y}{a}\right),$$

we arrive at (2.54)

Lemma 2.30. *The following identities hold true,*

$$\mathcal{H}\left\{\mathcal{P}\left\{f\left(t\right);x\right\};y\right\} = \frac{\pi y^{\alpha}}{2\cos\left(\dfrac{\pi\alpha}{2}\right)} \int_{0}^{\infty} t^{\alpha} f\left(t\right) \, _{2}\Psi_{2}\left(-y^{2}t^{2}\right) dt$$

$$+ \frac{\pi y^{\alpha+1}}{2\sin\left(\dfrac{\pi\alpha}{2}\right)} \int_{0}^{\infty} t^{\alpha+1} f\left(t\right) \, _{2}\Psi_{2}^{*}\left(-y^{2}t^{2}\right) dt$$

$$- \frac{\pi y}{\sin\left(\pi\alpha\right)} \int_{0}^{\infty} t f\left(t\right) \, _{2}\Psi_{2}^{**}\left(-y^{2}t^{2}\right) dt \qquad (2.55)$$

and

$$\mathcal{P}\left\{\mathcal{H}\left\{f\left(t\right);x\right\};y\right\} = \frac{\pi y^{\alpha+1}}{2\cos\left(\dfrac{\pi\alpha}{2}\right)} \int_{0}^{\infty} t^{\alpha+1} f\left(t\right) \, _{2}\Psi_{2}^{*}\left(-y^{2}t^{2}\right) dt$$

$$- \frac{\pi y^{\alpha}}{2\sin\left(\dfrac{\pi\alpha}{2}\right)} \int_{0}^{\infty} t^{\alpha} f\left(t\right) \, _{2}\Psi_{2}\left(-y^{2}t^{2}\right) dt$$

$$+ \frac{\pi}{\sin\left(\pi\alpha\right)} \int_{0}^{\infty} f\left(t\right) \, _{2}\Psi_{2}^{***}\left(-y^{2}t^{2}\right) dt, \qquad (2.56)$$

provided that the integrals involved converge absolutely, where

$$_{2}\Psi_{2}\left(z\right) = \frac{1}{\Gamma\left(\gamma\right)} \, _{2}\Psi_{2}\left[\begin{array}{c} \left(\gamma, 2p\right), \left(1, 1\right) \\ \left(1+v, 2\mu\right), \left(1, 2\right) \end{array}\bigg| z\right], \qquad (2.57)$$

$$_{2}\Psi_{2}^{*}\left(z\right) = \frac{1}{\Gamma\left(\gamma\right)} \, _{2}\Psi_{2}\left[\begin{array}{c} \left(\gamma+p, 2p\right), \ \left(1, 1\right) \\ \left(1+v+\mu, 2\mu\right), \ \left(2, 2\right) \end{array}\bigg| z\right], \qquad (2.58)$$

$$_{2}\Psi_{2}^{**}\left(z\right) = \frac{1}{\Gamma\left(\gamma\right)} \, _{2}\Psi_{2}\left[\begin{array}{c} \left(\gamma-p\alpha+p, 2p\right), \ \left(1, 1\right) \\ \left(1+\nu-\mu\alpha+\mu, 2\mu\right), \ \left(2-\alpha, 2\right) \end{array}\bigg| z\right], \qquad (2.59)$$

$$_{2}\Psi_{2}^{***}\left(z\right) = \frac{1}{\Gamma\left(\gamma\right)} \, _{2}\Psi_{2}\left[\begin{array}{c} \left(\gamma-p\alpha, 2p\right), \ \left(1, 1\right) \\ \left(1+\nu-\mu\alpha, 2\mu\right), \ \left(1-\alpha, 2\right) \end{array}\bigg| z\right]. \qquad (2.60)$$

Proof. By setting $K(x,t) = \dfrac{t}{t^{2}+x^{2}}$, which is the kernel of the Widder Potential transform, in Lemma 2.1 and using the following formula for $\text{Re}\,a > 0$ (see [1]),

$$\mathcal{H}\left\{\frac{1}{a^{2}+t^{2}};s\right\} = \frac{\pi a^{\alpha-1}s^{\alpha}}{2\cos\left(\dfrac{\pi\alpha}{2}\right)} \, _{2}\Psi_{2}\left(-a^{2}s^{2}\right)$$

$$+ \frac{\pi a^{\alpha} s^{\alpha+1}}{2\sin\left(\dfrac{\pi\alpha}{2}\right)} \, _{2}\Psi_{2}^{*}\left(-a^{2}s^{2}\right)$$

$$- \frac{\pi s}{\sin\left(\pi\alpha\right)} \, _{2}\Psi_{2}^{**}\left(-a^{2}s^{2}\right), \qquad (2.61)$$

we obtain (2.55). By setting $K(y,x) = \dfrac{x}{x^{2}+y^{2}}$, which is the kernel of the Widder Potential transform, in Lemma 2.2 and using the formula (2.61) and relation

$$_{\alpha}\mathcal{H}_{\nu,p}^{\mu,\gamma}\left\{\frac{t}{a^{2}+t^{2}};s\right\} = s^{-1}\,_{\alpha+1}\mathcal{H}_{\nu,p}^{\mu,\gamma}\left\{\frac{1}{a^{2}+t^{2}};s\right\},$$

we find (2.56). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

Corollary 2.31. *The following identities hold true,*

$$\int_0^\infty \mathcal{H}\{x\}\,\mathcal{P}\{x\}\,dx = \frac{\pi}{2\cos\left(\frac{\pi\alpha}{2}\right)} \int_0^\infty y^\alpha g(y) \left(\int_0^\infty t^\alpha f(t)\,{}_2\Psi_2\left(-y^2 t^2\right)dt\right)dy$$

$$+ \frac{\pi}{2\sin\left(\frac{\pi\alpha}{2}\right)} \int_0^\infty y^{\alpha+1} g(y) \left(\int_0^\infty t^{\alpha+1} f(t)\,{}_2\Psi_2^*\left(-y^2 t^2\right)dt\right)dy$$

$$- \frac{\pi}{\sin(\pi\alpha)} \int_0^\infty y\,g(y) \left(\int_0^\infty t\,f(t)\,{}_2\Psi_2^{**}\left(-y^2 t^2\right)dt\right)dy \qquad (2.62)$$

and

$$\int_0^\infty \mathcal{H}\{x\}\,\mathcal{P}\{x\}\,dx = \frac{\pi}{2\cos\left(\frac{\pi\alpha}{2}\right)} \int_0^\infty t^\alpha f(t) \left(\int_0^\infty y^\alpha g(y)\,{}_2\Psi_2\left(-t^2 y^2\right)dy\right)dt$$

$$+ \frac{\pi}{2\sin\left(\frac{\pi\alpha}{2}\right)} \int_0^\infty t^{\alpha+1} f(t) \left(\int_0^\infty y^{\alpha+1} g(y)\,{}_2\Psi_2^*\left(-t^2 y^2\right)dy\right)dt$$

$$- \frac{\pi}{\sin(\pi\alpha)} \int_0^\infty t\,f(t) \left(\int_0^\infty y\,g(y)\,{}_2\Psi_2^{**}\left(-t^2 y^2\right)dy\right)dt, \qquad (2.63)$$

provided that the integrals involved converge absolutely, where ${}_2\Psi_2(z)$, ${}_2\Psi_2^*(z)$ *and* ${}_2\Psi_2^{**}(z)$ *are defined in* (2.57), (2.58) *and* (2.59), *respectively.*

Proof. By setting $K(x,y) = \dfrac{y}{x^2 + y^2}$, which is the kernel of the Widder Potential transform, in (2.3) and (2.4) of Theorem 2.3, and using the formula (2.61), we arrive at (2.62) and (2.63), respectively. □

Theorem 2.32. *We have the following relations,*

$$\mathcal{H}\left\{\mathcal{D}^{-\beta}\{f(x);t\};s\right\} = \frac{\sin(\pi\alpha)\,s^\alpha}{\sin(\pi(\alpha+\beta))} \int_0^\infty x^{\alpha+\beta}\,{}_2\Psi_2(-sx)\,f(x)\,dx \qquad (2.64)$$

and

$$\mathcal{H}\left\{\mathcal{W}^{-\beta}\{f(x);t\};s\right\} = s^\alpha \int_0^\infty x^{\alpha+\beta}\,{}_2\Psi_2(-sx)\,f(x)\,dx, \qquad (2.65)$$

provided that the integrals involved converge absolutely, where

$${}_2\Psi_2(z) = \frac{1}{\Gamma(\gamma)}\,{}_2\Psi_2\left[\begin{array}{c}(\gamma,p),(\alpha+1,1)\\(1+v,\mu),(\alpha+\beta+1,1)\end{array}\middle| -sx\right]. \qquad (2.66)$$

Proof. By using the definition of generalized Bessel-Maitland integral transform (1.8), the Riemann Fractional integral (1.5) and changing the order of integration, we get

$$\mathcal{H}\left\{\mathcal{D}^{-\beta}\{f(x);t\};s\right\} = \int_0^\infty f(x)\left(\frac{1}{\Gamma(\beta)}\int_x^\infty (st)^\alpha\,\mathcal{J}_{\nu,p}^{\mu,\gamma}(st)(t-x)^{\beta-1}\,dt\right)dx.$$

By using the series representation of generalized Bessel-Maitland function (1.4) and changing the variable of integration from t to u where $t - x = xu$, we obtain,

$$\mathcal{H}\left\{\mathcal{D}^{-\beta}\{f(x);t\};s\right\} = \int_0^\infty f(x)\left(\frac{1}{\Gamma(\beta)}\sum_{n=0}^\infty \frac{(\gamma)_{pn}(-1)^n\,s^{\alpha+n}x^{\alpha+n+\beta}}{\Gamma(1+\nu+\mu n)\,n!}\int_0^\infty \frac{u^{\beta-1}}{(1+u)^{-\alpha-\beta-n}}du\right)dx.$$

By using the definition of Beta function [7, p.9,Eq(1)] and the known formula [7],

$$\frac{\pi}{\sin\pi z} = \Gamma(z)\,\Gamma(1-z), \qquad (2.67)$$

we arrive at (2.64). Now, the assertion (2.65) follows upon inserting the definitions of generalized Bessel-Maitland transform (1.8) and Weyl fractional integral (1.6) and changing the order of integration, we have

$$\mathcal{H}\left\{\mathcal{W}^{-\beta}\left\{f\left(x\right);t\right\};s\right\} = \int_0^\infty f\left(x\right)\left(\frac{1}{\Gamma\left(\beta\right)}\int_0^x \left(st\right)^\alpha \mathcal{J}_{\nu,p}^{\mu,\gamma}\left(st\right)\left(x-t\right)^{\beta-1}dt\right)dx.$$

By using the series representation of generalized Bessel-Maitland function (1.4) and changing the variable of integration from t to u where $t = xu$, we obtain

$$\mathcal{H}\left\{\mathcal{W}^{-\beta}\left\{f\left(x\right);t\right\};s\right\} = \int_0^\infty f\left(x\right)\left(\frac{1}{\Gamma\left(\beta\right)}\sum_{n=0}^\infty \frac{\left(\gamma\right)_{pn}\left(-1\right)^n s^{\alpha+n}x^{\alpha+n+\beta}}{\Gamma\left(1+\nu+\mu n\right)n!}\int_0^1 u^{\alpha+n}\left(1-u\right)^{\beta-1}du\right)dx.$$

By using the definition of Beta function [7, p.9, Eq(2)] and the Fox-Wright function (1.2), we obtain (2.65). □

In the third section of this article, some generalized integrals are calculated by using theorems and corollaries given in the second section. The examples shown in this section are obtained for the first time.

3. Illustrative examples

Example 3.1. We show for $-1 \leq \mathrm{Re}\beta \leq 0$,

$$\int_0^\infty t^{\beta-\alpha-1}\,{}_2\Psi_1\left[\begin{array}{c}\left(\gamma,p\right),\left(\alpha+1,1\right)\\\left(1+v,\mu\right)\end{array}\right| -\frac{y}{t}\right]dt$$
$$= \frac{\Gamma\left(\beta+1\right)\Gamma\left(\alpha-\beta\right)\Gamma\left(\gamma-p\left(\alpha-\beta\right)\right)}{\Gamma\left(1+\nu-\mu\left(\alpha-\beta\right)\right)}\frac{1}{y^{\alpha-\beta}}. \tag{3.1}$$

By setting $f\left(t\right) = t^\beta$ in (2.12) of Corollary 2.10 and using (2.9) and the known formula [5, p.137, Eq(1)] $\mathcal{L}\left\{t^\beta;x\right\} = \dfrac{\Gamma(\beta+1)}{x^{\beta+1}}$, we obtain (3.1).

Remark 3.2. If we set $\beta = 0$ in (3.1), then we find

$$\int_0^\infty t^{-\alpha-1}\,{}_2\Psi_1\left[\begin{array}{c}\left(\gamma,p\right),\left(\alpha+1,1\right)\\\left(1+v,\mu\right)\end{array}\right| -\frac{y}{t}\right]dt = \frac{\Gamma\left(\alpha\right)\Gamma\left(\gamma-p\alpha\right)}{\Gamma\left(1+\nu-\mu\alpha\right)}\frac{1}{y^\alpha}. \tag{3.2}$$

Remark 3.3. If we set $g\left(u\right) = 1$ in (2.19) of Corollary 2.10 and use the formula (3.2), then we obtain

$$\int_0^\infty \frac{1}{x}\mathcal{H}\left\{f\left(t\right);x\right\}dx = \frac{\Gamma\left(\alpha\right)\Gamma\left(\gamma-p\alpha\right)}{\Gamma\left(\gamma\right)\Gamma\left(1+\nu-\mu\alpha\right)}\int_0^\infty f\left(t\right)dt. \tag{3.3}$$

Remark 3.4. If we set $f\left(t\right) = e^{-yt}$ in Remark 3.3 and use the formula (2.8), then we obtain

$$\int_0^\infty x^{\alpha-1}\,{}_2\Psi_1\left[\begin{array}{c}\left(\gamma,p\right),\left(\alpha+1,1\right)\\\left(1+v,\mu\right)\end{array}\right| -\frac{x}{y}\right]dx = \frac{\Gamma\left(\alpha\right)\Gamma\left(\gamma-p\alpha\right)}{\Gamma\left(1+\nu-\mu\alpha\right)}y^\alpha. \tag{3.4}$$

Example 3.5. We show for $\left|\arg a\right| < \pi$,

$$\int_0^\infty \frac{e^{-at}}{t^{\alpha+1}}\,{}_2\Psi_1\left[\begin{array}{c}\left(\gamma,p\right),\left(\alpha+1,1\right)\\\left(1+v,\mu\right)\end{array}\right| -\frac{y}{t}\right]dt = \frac{\pi y^{-\alpha}}{\sin\left(\pi\alpha\right)}\,{}_2\Psi_2\left[\begin{array}{c}\left(\gamma-p\alpha,p\right),\ \left(1,1\right)\\\left(1+v-\mu\alpha,\mu\right),\ \left(1-\alpha,1\right)\end{array}\right| ay\right]$$
$$-\frac{\pi a^\alpha}{\sin\left(\pi\left(\alpha\right)\right)}\,{}_1\Psi_1\left[\begin{array}{c}\left(\gamma,p\right)\\\left(1+v,\mu\right)\end{array}\right| ay\right]. \tag{3.5}$$

By setting $f\left(t\right) = e^{-at}$ in (2.12) of Corollary 2.10 and using the formula (2.40), where $\left|\arg a\right| < \pi$ and $\mathcal{L}\left\{e^{-at};x\right\} = \dfrac{1}{x+a}$ for $\mathrm{Re}\left(x-a\right) > 0$, we obtain (3.5).

Remark 3.6. If we set $\gamma = 1, p = \mu, \nu = 0$ in (3.5) and use (2.42), then we obtain

$$\mathcal{S}_{\alpha+1}\left\{e^{-at}; y\right\} = a^{\alpha} e^{ay} \Gamma\left(-\alpha, ay\right).$$

Example 3.7. We show that

$$\cos\left(\frac{\pi\alpha}{2}\right) \int_0^\infty \frac{1}{t^{\alpha+\beta+1}} \; {}_3\Psi_2 \left[\begin{matrix}(\gamma, 2p), (\alpha+1, 2), (1, 1) \\ (1+v, 2\mu), (1, 2)\end{matrix} \; \middle| -\frac{y^2}{t^2}\right] dt$$

$$+ y \sin\left(\frac{\pi\alpha}{2}\right) \int_0^\infty \frac{1}{t^{\alpha+\beta+2}} \; {}_3\Psi_2 \left[\begin{matrix}(\gamma+p, 2p), (\alpha+2, 2), (1, 1) \\ (1+v+\mu, 2\mu), (2, 2)\end{matrix} \; \middle| -\frac{y^2}{t^2}\right] dt$$

$$= \cos\left(\frac{\pi\beta}{2}\right) \frac{\Gamma(1-\beta)\Gamma(\alpha+\beta)\Gamma(\gamma-p(\alpha+\beta))}{\Gamma(1+\nu-\mu(\alpha+\beta))} \frac{1}{y^{\beta+\alpha}}. \tag{3.6}$$

If we set $f(t) = t^{-\beta}$ $(0 < \mathrm{Re}\beta < 1)$ in (2.24) of Corollary 2.13 and use formulas $\mathcal{F}_s\left\{t^{-\beta}; y\right\} = y^{\beta-1}\Gamma(1-\beta)\cos\left(\frac{\pi\beta}{2}\right)$ [5, p.68, Eq(1)] and (2.9), then we obtain (3.6).

Remark 3.8. If we set $\alpha = 0$ in (3.6), then we find for $0 < \mathrm{Re}\beta < 1$,

$$\int_0^\infty \frac{1}{t^{\beta+1}} \; {}_2\Psi_1 \left[\begin{matrix}(\gamma, 2p), (1, 1) \\ (1+v, 2\mu)\end{matrix} \; \middle| -\frac{y^2}{t^2}\right] dt = \frac{\Gamma(\gamma-\beta p)}{\Gamma(1+\nu-\mu\beta)} \frac{\pi}{2y^\beta} \csc\left(\frac{\pi\beta}{2}\right). \tag{3.7}$$

Remark 3.9. If we set $\alpha = 1$ in (3.6), then we obtain for $0 < \mathrm{Re}\beta < 1$,

$$\int_0^\infty \frac{1}{t^{\beta+3}} \; {}_3\Psi_2 \left[\begin{matrix}(\gamma+p, 2p) \;, \; (3, 2) \;, \; (1, 1) \\ (1+v+\mu, 2\mu) \;, \; (2, 2)\end{matrix} \; \middle| -\frac{y^2}{t^2}\right] dt$$

$$= \cos\left(\frac{\pi\beta}{2}\right) \frac{\Gamma(1-\beta)\Gamma(1+\beta)\Gamma(\gamma-p(1+\beta))}{\Gamma(1+\nu-\mu(1+\beta))} \frac{1}{y^{\beta+2}}. \tag{3.8}$$

Example 3.10. We show that

$$y \cos\left(\frac{\pi\alpha}{2}\right) \int_0^\infty t^{\alpha+\beta-1} \; {}_3\Psi_2 \left[\begin{matrix}(\gamma, 2p), (1+\alpha, 2), (1, 1) \\ (1+v, 2\mu), (1, 2)\end{matrix} \; \middle| -\frac{t^2}{y^2}\right] dt$$

$$+ \sin\left(\frac{\pi\alpha}{2}\right) \int_0^\infty t^{\alpha+\beta} \; {}_3\Psi_2 \left[\begin{matrix}(\gamma+p, 2p), (2+\alpha, 2), (1, 1) \\ (1+v+\mu, 2\mu), (2, 2)\end{matrix} \; \middle| -\frac{t^2}{y^2}\right] dt$$

$$= \cos\left(\frac{\pi\beta}{2}\right) \frac{\Gamma(1-\beta)\Gamma(\alpha+\beta)\Gamma(\gamma-p(\alpha+\beta))}{\Gamma(1+\nu-\mu(\alpha+\beta))} y^{\alpha+\beta+1}. \tag{3.9}$$

If we set $f(t) = t^{\beta-1}$, $(0 < \mathrm{Re}\beta < 1)$ in (2.25) of Corollary 2.13 and use formulas (2.9) and [5, p.68, Eq(1)], then we obtain (3.9).

Example 3.11. We show that

$$\int_0^\infty x^{\alpha+\beta+\delta-1} \; {}_2\Psi_2 \left[\begin{matrix}(\gamma, p) \;, \; (\alpha+1, 1) \\ (1+v, \mu) \;, \; (\alpha+\beta+1, 1)\end{matrix} \; \middle| -sx\right] dx$$

$$= \frac{\sin(\pi(\alpha+\beta))}{\sin(\pi\alpha)} \frac{\Gamma(\delta)\Gamma(\alpha+\delta+\beta)\Gamma(\gamma-p(\alpha+\delta+\beta))}{\Gamma(\delta+\beta)\Gamma(1+\nu-\mu(\alpha+\delta+\beta))} \frac{1}{s^{\delta+\beta+\alpha}}. \tag{3.10}$$

If we set $f(x) = x^{\delta-1}$ in (2.64) of Theorem 2.32 and use the formulas,

$$\mathcal{D}^{-\beta}\left\{x^{\delta-1}; y\right\} = \frac{y^{\delta+\beta-1}\Gamma(\delta)}{\Gamma(\delta+\beta)} \tag{3.11}$$

and (2.9), then we find (3.10).

4. Conclusion

We conclude that, many other relations could be found by using different corollaries of this article and many other infinite integrals could be evaluated in this manner by applying the lemmas, theorems and corollaries considered in this article.

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