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RIEMANNIAN SUBMERSIONS AND LAGRANGIAN ISOMETRIC IMMERSION 1

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ABSTRACT. In [1], it has shown that if a Riemannian manifold admits a nontrivial Riemannian submersion with totally geodesic fibers, then it cannot be isometrically immersed in any Riemannian manifold of non-positive sectional curvature as a minimal submanifold. In this paper, we consider a nontrivial Riemannian submersion and investigate some properties on Lagrangian isometric immersions using the submersion invariant.

1. INTRODUCTION

Let M and B be Riemannian manifolds of dimension m and b, respectively. A surjective map $\pi : M \to B$ is called a Riemannian submersion if it has maximal rank at any point of M and the differential π_* preserves the length of the horizontal vectors. A vector field on M is called vertical if it is always tangent to fibers and horizontal if it is orthogonal to fibers. A vector field X on M is called basic if X is horizontal and π-related to a vector field X_* on B. i.e. $\pi_* X = X_*$. Let $\mathcal{H}$ and $\mathcal{V}$ be horizontal and vertical distributions. The trivial Riemannian submersion is the projection of a Riemannian product manifold onto one of its factors which has totally geodesic horizontal and vertical distributions. In this paper, a Riemannian manifold M admits a nontrivial Riemannian submersions if there exists a Riemannian submersion $\pi : M \to B$ from M into a Riemannian manifold B such that $\mathcal{H}$ and $\mathcal{V}$ are not both totally geodesic distribution.

Let us assume that M^n admits a Lagrangian isometric immersion $\phi : M \to \tilde{M}^n$ into a Kaehler manifold $\tilde{M}^n$ and we choose a local orthonormal frame $e_1, ..., e_b, e_{b+1},...,$ $e_n, Je_1, ..., Je_n$ such that $e_1, ..., e_b$ are horizontal vector fields, $e_{b+1}, ..., e_n$ are vertical vector fields of M and $Je_1, ..., Je_n$ are normal vector fields of M in $\tilde{M}^n$.

The submersion invariant $\check{A}_\pi$ is defined by

$$\check{A}_\pi = \sum_{i=1}^{b} \sum_{s=b+1}^{n} ||A_{e_i} e_s||^2,$$

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where A is a $(1,2)$ tensor defined as $A_E F = v\nabla_{hE} hF + h\nabla_{hE} vF$. Also, there is an another $(1,2)$ tensor T which is defined as $T_E F = v\nabla_{vE} hF + h\nabla_{vE} vF$. These are called the fundamental tensor fields or the invariants of a Riemannian submersion π. In [1], B. Y. Chen obtained

Theorem 1.1. *If a Riemannian manifold M^n admits a nontrivial Riemannian submersion $\pi : (M^n, g) \to (B^b, g')$ with totally geodesic fibers, then it can not be isometrically immersed in any Riemannian manifold of non-positive sectional curvature as a minimal submanifold.*

In his proof, he found that

$$\check{A}_\pi \leq \frac{n^2}{4} H^2 + b(n-b) max\tilde{K} \tag{1.1}$$

where max$\tilde{K}$ denotes the maximum value of the sectional curvature of the ambient space $\tilde{M}^n$ restricted to plane sections in T_pM for an isometric immersion $\phi : M \to \tilde{M}^n$.

In this paper, we mainly derive two inequalities on Riemannian submersion like (1.1) using the different techniques.

2. Main results

We need the following proposition from the book [4]. Throughout this section, we assume that $\pi : (M, g) \to (B, g')$ is a Riemannian submersion with totally geodesic fibers.

Proposition 2.1. *Let $\pi : (M, g) \to (B, g')$ be a Riemannian submersion with totally geodesic fibers. If M has non-positive sectional curvature, then the horizontal distribution is integrable and B has non-positive sectional curvatures. If M has positive sectional curvatures, then we have the following.*
(a) $\dim M < 2 \dim B$;
(b) B has positive sectional curvature.

In this paper, we define a Riemannian submersion $\pi : (M^n, g) \to (B^b, g')$ is non-trivial if the horizontal and vertical distribution are not both integrable. Moreover, if a Riemannian submersion has totally geodesic fibers, then the vertical distribution is integrable and it is totally geodesic. So, the horizontal distribution of a non-trivial Riemannian submersion with totally geodesic fibers is not integrable.

Simply from the above results, we have the following.

Theorem 2.1. *Let $\pi : (M^n, g) \to (B^b, g')$ be a non-trivial Riemannian submersion with totally geodesic fibers. Then*
(a) M has positive sectional curvature, and so has B;
(b) $\dim M < 2 \dim B$.

Proof. The statement (a) and (b) are the immediate result of proposition 2.1 above. □

We need the following for the next result. If $\{U, V\}$ is an orthonormal basis of the vertical 2-plane α, then the sectional curvature of the plane α in T_pM, $p \in M$ is $K(\alpha) = \hat{K}(\alpha) + ||T_U V||^2 - g(T_U U, T_V V)$, where $\hat{K}(\alpha)$ denotes the sectional curvature in the fiber through p. If $\{X,Y\}$ is an orthonormal basis of the horizontal 2-plane α and $K'(\alpha')$ denotes the sectional curvature in (B,g') of the plane α' spanned by π_*X, π_*Y, then $K(\alpha) = K'(\alpha') - 3||A_XY||^2$. Finally, if $X \in \mathcal{H}_p$ and $V \in \mathcal{V}_p$ are unit vectors spanning α, the sectional curvature of the plane α is $K(\alpha) = g((\nabla_X T)(V,V),X) - ||T_VX||^2 + ||A_XV||^2$. Because of our assumption of totally geodesic fibers, tensor T is identically zero so that we have the following.

Theorem 2.2. *Under the same condition in Theorem 2.1, we have*

$$\check{A}_\pi < \tau + 3\sum_{1\le i<j\le b} ||A_{e_i}e_j||^2,$$

where τ is the scalar curvature of M defined by $\tau = \sum_{1\le i<j\le n} K(e_i\wedge e_j)$ for an orthonormal basis $e_1,...,e_n$ at $p\in M$.

Proof. Since $T=0$, the sectional curvature of the plane α spanned by two unit vectors $X\in\mathcal{H}_p$ and $V\in\mathcal{V}_p$ is $K(\alpha) = ||A_XV||^2$ so that its submersion invariant $\check{A}_\pi = \sum_{1\le i\le b, b+1\le\alpha\le n} ||A_{e_i}e_\alpha||^2 = \sum_{i,\alpha} K(e_i\wedge e_\alpha)$. Furthermore,

$$\check{A}_\pi = \tau - \sum_{1\le i<j\le b} K'(\alpha') + 3\sum_{1\le i<j\le b} ||A_{e_i}e_j||^2 - \sum_{b+1\le\alpha<\beta\le n} \hat{K}(e_\alpha\wedge e_\beta).$$

But from Theorem 2.1, $K'(\alpha')$ and $\hat{K}(e_\alpha\wedge e_\beta)$ are all positive so that we have the result. □

Also, we have another inequality for a Riemannian submersion as below. We say a plane α is called the mixed plane if it is spanned by a horizontal vector e_j and a vertical vector e_α for $i=1,...,b$ and $\alpha = b+1,...,n$.

Theorem 2.3. *Again under the same conditions in the previous theorem and if $\phi: M\to\tilde{M}$ is a Lagrangian isometric immersion, then we have another inequality*

$$\check{A}_\pi \ge \tau - \tilde{\tau} + b(n-b)\min\tilde{K} - \frac{1}{2}(b-1)||H||^2_{\mathbb{H}} - \frac{1}{2}(n-b-1)||H||^2_{\mathbb{V}},$$

where $\tilde{\tau}$ is the scalar curvature and $\tilde{K}$ is the sectional curvature of the mixed plane in the ambient space and $||H||^2_{\mathbb{H}}$ is defined as $||H||^2_{\mathbb{H}} = \sum_{r=1}^n\sum_{j=1}^b (h^r_{jj})^2$ and $||H||^2_{\mathbb{V}} = \sum_{r=1}^n\sum_{\alpha=b+1}^n (h^r_{\alpha\alpha})^2$. The equality holds iff the second fundamental form satisfies $h^r_{jj} = \mu$ and $h^r_{\alpha\alpha} = \lambda$ for $j=1,...,b$, $\alpha = b+1,...,n$ and $r=1,...,n$ and $\tilde{K}$ is constant.

Proof. Given an orthonormal basis $e_1,...,e_n$ of the tangent space $T_pM, p\in M$, the scalar curvature τ of M at p is defined to be

$$\tau(p) = \sum_{1\le i<j\le n} K(e_i\wedge e_j).$$

From the definition of a Riemannain submersion, we can get

$$\tilde{A}_\pi = \tau(p) - \tilde{\tau}(p) + \sum_{i,\alpha}\tilde{K}(e_i\wedge e_\alpha) - \sum_{r=1}^n\Big(\sum_{1\le i<j\le b}(h^r_{ii}h^r_{jj} - (h^r_{ij})^2) - \sum_{b+1\le\alpha<\beta\le n}(h^r_{\alpha\alpha}h^r_{\beta\beta} - (h^r_{\alpha\beta})^2)\Big)$$

However, a part of series becomes

$$\sum_{r=1}^n\Big(\sum_{1\le i<j\le b}(h^r_{ii}h^r_{jj} - (h^r_{ij})^2) + \sum_{b+1\le\alpha<\beta\le n}(h^r_{\alpha\alpha}h^r_{\beta\beta} - (h^r_{\alpha\beta})^2)\Big)$$

$$= \sum_{r=1}^{n}\Big(\sum_{2\leq j\leq b}(h^r_{11}h^r_{jj} - (h^r_{1j})^2) + \sum_{2\leq i<j\leq b}(h^r_{ii}h^r_{jj} - (h^r_{ij})^2)\Big)$$

$$+\sum_{r=1}^{n}\Big(\sum_{b+2\leq\beta\leq n}(h^r_{b+1,b+1}h^r_{\beta,\beta} - (h^r_{b+1,\beta})^2) + \sum_{b+2\leq\alpha<\beta\leq n}(h^r_{\alpha\alpha}h^r_{\beta\beta} - (h^r_{\alpha\beta})^2)\Big)$$

The first term in the series becomes inequality

$$\sum_{r=1}^{n}\sum_{2\leq j\leq b}(h^r_{11}h^r_{jj} - (h^r_{1j})^2) \leq \sum_{r=1}^{n}\sum_{j=2}^{b}h^r_{11}h^r_{jj} - \sum_{j=2}^{b}(h^1_{1j})^2 - \sum_{j=2}^{b}(h^j_{1j})^2$$

which means

$$-\sum_{r=1}^{n}\sum_{2\leq j\leq b}(h^r_{11}h^r_{jj} - (h^r_{1j})^2) \geq -\sum_{r=1}^{n}\sum_{j=2}^{b}h^r_{11}h^r_{jj} + \sum_{j=2}^{b}(h^1_{1j})^2 + \sum_{j=2}^{b}(h^j_{1j})^2$$

Using the same type of inequality for every term in the series we get the following.

$$-\sum_{r=1}^{n}\Big(\sum_{2\leq j\leq b}(h^r_{11}h^r_{jj} - (h^r_{1j})^2) - \sum_{2\leq i<j\leq b}(h^r_{ii}h^r_{jj} - (h^r_{ij})^2)\Big)$$

$$-\sum_{r=1}^{n}\Big(\sum_{b+2\leq\beta\leq n}(h^r_{b+1,b+1}h^r_{\beta,\beta} - (h^r_{b+1,\beta})^2) - \sum_{b+2\leq\alpha<\beta\leq n}(h^r_{\alpha\alpha}h^r_{\beta\beta} - (h^r_{\alpha\beta})^2)\Big)$$

$$\geq -\sum_{r=1}^{n}\sum_{j=2}^{b}h^r_{11}h^r_{jj}+\sum_{j=2}^{b}(h^1_{1j})^2+\sum_{j=2}^{b}(h^j_{1j})^2-\sum_{r=1}^{n}\sum_{j=3}^{b}h^r_{22}h^r_{jj}+\sum_{j=3}^{b}(h^2_{2j})^2+\sum_{j=3}^{b}(h^j_{2j})^2+\ldots.$$

$$-\sum_{r=1}^{n}h^r_{b-1,b-1}h^r_{b,b}+(h^{b-1}_{b-1,b})^2+(h^b_{b-1,b})^2-\sum_{r=1}^{n}\sum_{b+2\leq\beta\leq n}h^r_{b+1,b+1}h^r_{\beta,\beta}+\sum_{\beta=b+2}^{n}(h^{b+1}_{b+1,\beta})^2$$

$$+\sum_{\beta=b+2}^{n}(h^{\beta}_{b+1,\beta})^2) + \ldots. - \sum_{r=1}^{n}h^r_{n-1,n-1}h^r_{nn} + (h^{n-1}_{n-1,n})^2 + (h^n_{n-1,n})^2$$

$$\geq -\frac{1}{2}\sum_{r=1}^{n}[(h^r_{11})^2+(h^r_{22})^2]+\ldots+[(h^r_{11})^2+(h^r_{bb})^2]+[(h^r_{22})^2+(h^r_{33})^2]+\ldots+[(h^r_{22})^2+(h^r_{bb})^2]+\ldots$$

$$+[(h^r_{b-1,b-1})^2+(h^r_{bb})^2]-\frac{1}{2}\sum_{r=1}^{n}[(h^r_{b+1,b+1})^2+(h^r_{b+2,b+2})^2]+\ldots+[(h^r_{b+1,b+1})^2+(h^r_{nn})^2]$$

$$+[(h^r_{b+2,b+2})^2+(h^r_{b+3,b+3})^2]+\ldots+[(h^r_{b+2,b+2})^2+(h^r_{nn})^2]+\ldots+[(h^r_{n-1,n-1})^2+(h^r_{nn})^2]$$

$$+\sum_{j=2}^{b}(h^1_{1j})^2 + \sum_{j=2}^{b}(h^j_{1j})^2 + \sum_{j=3}^{b}(h^2_{2j})^2 + \sum_{j=3}^{b}(h^j_{2j})^2 + \ldots + (h^{b-1}_{b-1,b})^2 + (h^b_{b-1,b})^2$$

$$+\sum_{\beta=b+2}^{n}(h^{b+1}_{b+1,\beta})^2 + \sum_{\beta=b+2}^{n}(h^{\beta}_{b+1,\beta})^2 + \ldots + (h^{n-1}_{n-1,n})^2 + (h^n_{n-1,n})^2$$

by using the simple algebraic inequality for all the mixed terms in the series. Therefore, we now have

$$\tilde{A}_\pi \geq \tau(p) - \tilde{\tau}(p) + b(n-b)\min\tilde{K} - \frac{1}{2}\sum_{r=1}^{n}([(h^r_{11})^2 + (h^r_{22})^2] + \ldots + [(h^r_{11})^2 + (h^r_{bb})^2]$$

$$+[(h^r_{22})^2 + (h^r_{33})^2] + \ldots + [(h^r_{22})^2 + (h^r_{bb})^2] + \ldots + [(h^r_{b-1,b-1})^2 + (h^r_{bb})^2])$$

$$-\frac{1}{2}\sum_{r=1}^{n}([(h^r_{b+1,b+1})^2+(h^r_{b+2,b+2})^2]+...+[(h^r_{b+1,b+1})^2+(h^r_{nn})^2]+[(h^r_{b+2,b+2})^2+(h^r_{b+3,b+3})^2]$$

$$+...+[(h^r_{b+2,b+2})^2+(h^r_{nn})^2]+...+[(h^r_{n-1,n-1})^2+(h^r_{nn})^2])$$

The equality case occurs when $h^k_{ij}=0$ for all $i \neq j$ and $h^r_{11}=...=h^r_{bb}$ and $h^r_{b+1,b+1}=...=h^r_{n,n}$ for $r=1,...,n$ and $\tilde{K}$ becomes a constant. □

Corollary 2.1. *Again under the same conditions in the previous theorem and if* $\phi : M \to \tilde{M}$ *is a Lagrangian isometric immersion, then we have the following inequality*

$$\check{A}_\pi > \tau - \tilde{\tau} + b(n-b)\min\tilde{K} - \frac{1}{2}(b-1)||H||^2$$

where $\tilde{\tau}$ *is the scalar curvature and* $\tilde{K}$ *is the sectional curvature of the mixed plane in the ambient space.*

Proof. By theorem 2.1, we know $n < 2b$ so that $n-b-1 < b-1$ which implies the inequality.

□

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