International Electronic Journal of Geometry
Volume 6 No. 2 pp. 57-62 (2013) ©IEJG

ALMOST COSYMPLECTIC (κ,μ)-SPACES WITH CYCLIC-PARALLEL RICCI TENSOR

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(Communicated by Josef MIKES)

Abstract. In this study, considering cyclic-parallel Ricci tensor for almost-cosymplectic (κ,μ)-spaces, we show that such type manifolds are locallyRiemannian manifold which is locally the product of a Kähler manifold N and an interval or unit circle S^1.

1. Introduction

The notion of an almost cosymplectic manifold was introduced by Goldberg and Yano in 1969 [14]. The simplest examples of such manifolds are those being the products (possibly local) of almost Kählerian manifolds and the real line $\mathbb{R}$ or the circle S^1. In particular, cosymplectic manifolds in the sense of Blair [5] are of this type. Mikes has some study about this topic ([20],[21]). However the class of almost cosymplectic manifolds is much more wider. There are already many known examples (among others, compact, homogeneous) of such manifolds which are not products (even locally). See Cordero et al. [11] Chinea and Gonzalez [10] and Olszak ([23],[24]).

The topology of cosymplectic manifolds was studied by Blair and Goldberg [6], Chinea et al. ([8], [9]) and others. Most of the results of Libermann [18], Lichnerowicz [19], Fujimoto and Muto [13] also have applications in characterizing of topological and analytical properties of almost cosymplectic manifolds (these authors have used a different terminology).

Curvature properties of almost cosymplectic manifolds were studied mainly by Golberg and Yano [14] ,Olszak ([23], [24]), Kirichenko [16] and Endo [12]. We relate some of them in a historical order.

Blair et al. [2] introduced the notion of (κ,μ)-contact metric manifolds, where κ and μ are real numbers. The full classification of these manifolds was given by Boeckx [7]. Later Koufogiorgos and Tsichlias [17] introduced the generalized (κ,μ)-contact metric manifolds where κ and μ are real functions and they gave several

Date: Received: July 1, 2012 and Accepted: June 11 , 2013.
2010 *Mathematics Subject Classification.* 53D10, 53C15, 53C25, 53C35.
Key words and phrases. Almost Cosmplectic Manifolds, Cyclic-Parallel Ricci Tensor, Almost Cosmplectic (κ,μ)-Spaces .

examples. Finally, the (κ, μ, ν)-contact metric manifolds have been recently introduced by Koufogiorgos, Markellos and Papantoniou where κ, μ, ν are are smooth functions. They proved that these manifolds exist only in the dimension 3, whereas such a manifold in dimension greater than 3 is (κ, μ)-contact metric manifolds. Öztürk et al. studied almost α-cosymplectic (κ, μ, ν)-spaces [26].

Özgür studied almost contact metric manifolds with cyclic-parallel Ricci tensor [25]. Sung-Baik Lee et al. studied Sasakian manifolds with cyclic-parallel Ricci tensor [27].

In this paper we consider almost cosymplectic (κ, μ)-spaces with cyclic-parallel Ricci tensor.

2. Preliminaries

A differentiable $(2n+1)$-dimensional manifold M is said to be a contact manifold if it admits a global differential 1-form η such that $\eta \wedge (d\eta) \neq 0$ everywhere on M.

Given a contact form η, one has a unique vector field ξ, which is called the characteristic vector field, satisfying

$$\eta(\xi) = 1, \quad d\eta(\xi, X) = 0, \tag{2.1}$$

for any vector field X.

It is well-known that, there exists a Riemannian metric g and a $(1,1)$-tensor field φ such that

$$\begin{aligned} &\eta(X) = g(X, \xi), \\ &d\eta(X, Y) = g(X, \varphi Y), \\ &\varphi^2 X = -X + \eta(X)\xi, \end{aligned} \tag{2.2}$$

where X and Y are vector fields on M.

From (2.2) it follows that

$$\varphi\xi = 0, \quad \eta \circ \varphi = 0, \quad g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y). \tag{2.3}$$

A differentiable manifold M equipped with the structure tensors (φ, ξ, η, g) satisfying (2.2) is said to be a contact metric manifold. The 2-form Φ of M defined by $\Phi(X, Y) = g(\phi X, Y)$, is called the fundamental 2-form. Almost contact metric manifolds such that both η and Φ are closed are called almost cosymplectic manifolds. Finally, a normal almost cosymplectic manifold is called a cosymplectic manifold. On almost cosymplectic manifold M, we can define a $(1,1)$-tensor field h by $h = \frac{1}{2} L_\xi \varphi$, and $lX = R(X, \xi)\xi$ where L and R denote Lie differentiation and curvature tensor respectively. Then we may observe that h is symmetric and satisfies

$$h\xi = 0, \quad l\xi = 0, \quad h\varphi = -\varphi h, \quad \nabla_X \xi = -\varphi h X, \tag{2.4}$$

where ∇ is Levi-Civita connection [2].

3. Almost Cosymplectic Manifold ξ Belonging to the (κ, μ)-Nullity Distribution

Let M be an almost cosymplectic manifold. The (κ, μ)-nullity distribution of M for the pair (κ, μ) is a distribution

$$\begin{array}{l} N(\kappa,\mu): p \to N_p(\kappa,\mu) = \\ \{Z \in T_pM \mid R(X,Y)Z \\ = \kappa[g(Y,Z)X - g(X,Z)Y] \\ +\mu[g(Y,Z)hX - g(X,Z)hY]\}, \end{array} \tag{3.1}$$

where $\kappa,\mu \in \mathbb{R}$ and $\kappa \leq 0$.

If the characteristic vector field ξ belongs to the (κ,μ)-nullity distribution then we have

$$R(X,Y)\xi = \kappa[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY]. \tag{3.2}$$

Then M is called almost cosymplectic (κ,μ)-space. An almost cosymplectic (κ,μ)-space satisfies the following curvature properties: [26]

$$lX = -\kappa\varphi^2 X + \mu hX, \tag{3.3}$$

$$l\varphi X - \varphi lX = 2\mu h\varphi X, \tag{3.4}$$

$$h^2 X = \kappa\varphi^2 X, \text{for } \kappa \leq 0, \tag{3.5}$$

$$(\nabla_\xi h)X = -\mu\varphi hX, \tag{3.6}$$

$$\left(\nabla_\xi h^2\right)X = 0, \tag{3.7}$$

$$\xi(\kappa) = 0, \tag{3.8}$$

$$(\nabla_X\varphi)Y = g(hX,Y)\xi - \eta(Y)hX, \tag{3.9}$$

$$\begin{aligned} (\nabla_X\varphi)Y - (\nabla_Y\varphi)X &= -\kappa(\eta(Y)X - \eta(X)Y) \\ &\quad -\mu(\eta(Y)hX - \eta(X)hY). \end{aligned} \tag{3.10}$$

Now we give some basic properties which we use next.

Lemma 1. *Let M be an almost cosymplectic manifold with ξ belonging to the (κ,μ)-nullity distribution. Then:*

$$\begin{array}{lrl} i) & (\nabla_X h)Y - (\nabla_Y h)X = & \kappa(\eta(Y)\varphi X - \eta(X)\varphi Y + 2g(\varphi X,Y)\xi) \\ & & +\mu(\eta(Y)\varphi hX - \eta(X)\varphi hY) \\ ii) & R(\xi,X)Y = & \kappa[g(X,Y)\xi - \eta(Y)X] + \mu[g(hX,Y)\xi - \eta(Y)hX] \\ iii) & Q\xi = & 2n\kappa\xi \end{array}$$

where X and Y are vector field on M, $\kappa,\mu \in \mathbb{R}$ and Q is the Ricci operator of M [2].

Theorem 1. *On almost cosymplectic (κ,μ)-space of dimension greater than or equal to 5, the functions κ, μ only vary in the direction of ξ, i.e. $X(\kappa) = X(\mu) = 0$ for every vector field X orthogonal to ξ* [26].

Proposition 1. *Let M be an almost cosymplectic manifold. M has Kählerian integral submanifolds if and only if it satisfies the condition*

$$(\nabla_X\varphi)Y = g(hX,Y)\xi - \eta(Y)hX, \tag{3.11}$$

for any vector fields X, Y on M [15].

Remark 1. *From 3.9 it is clear that M has Kählerian integral submanifolds.*

Proposition 2. *Let M be an almost cosymplectic manifold and $\widetilde{M}$ be an integral manifold of $\mathcal{D}$. Then $\widetilde{M}$ is totally geodesic if and only if the operator h vanish* [15].

Proposition 3. *Under the same situation as in Proposition 2, M is cosymplectic manifold with structure (φ,ξ,η,g) if and only if the integral manifolds of D are tangentially Kähler and the operator h vanish* [15].

Corollary 1. *An almost cosymplectic M^3 such that $\nabla\xi = 0$ is an cosymplectic manifold* [15].

Remark 2. *It is well known that a Cosymplectic manifold M is a locally Riemannian manifold which is locally the product of a Kähler manifold N and an interval or unit circle S^1 (*[3], [22], [4]*).*

Theorem 2. *If M is an almost cosymplectic (κ,μ)-space with $\kappa < 0$, where κ,μ only vary in the direction of ξ, then*

$$\begin{aligned}
R(X_\lambda,Y_\lambda)Z_{-\lambda} &= \kappa\left\{g(\varphi Y_\lambda,Z_{-\lambda})\varphi X_\lambda - g(\varphi X_\lambda,Z_{-\lambda})\varphi Y_\lambda\right\},\\
R(X_{-\lambda},Y-_\lambda)Z_\lambda &= \kappa\left\{g(\varphi Y_{-\lambda},Z_\lambda)\varphi X_{-\lambda} - g(\varphi X_{-\lambda},Z_\lambda)\varphi Y_{-\lambda}\right\},\\
R(X_\lambda,Y_{-\lambda})Z_{-\lambda} &= -\kappa g(X_\lambda,\varphi Z_{-\lambda})\varphi Y_{-\lambda},\\
R(X_\lambda,Y_{-\lambda})Z_\lambda &= -\kappa g(Z_\lambda,\varphi Y_{-\lambda})\varphi X_\lambda,\\
R(X_\lambda,Y_\lambda)Z_\lambda &= 0,\\
R(X_{-\lambda},Y_{-\lambda})Z_{-\lambda} &= 0,
\end{aligned}$$

where $X_{\pm\lambda},Y_{\pm\lambda},Z_{\pm\lambda}$ are eigenvectors of h associated to the eigenvalues $\pm\lambda = \pm\sqrt{-\kappa}$ [1].

Lemma 2. *Let M be an almost cosymplectic manifold with ξ belonging to the (κ,μ)-nullity distribution. For any vector field X, the Ricci operator Q is given by*

$$QX = \mu hX + 2n\kappa\eta(X)\xi. \tag{3.12}$$

Proof. Let $\{e_1,...,e_n,\varphi e_1,...,\varphi e_n,\xi\}$ be a local φ-basis such that $\{e_1,...,e_n\}$ is a basis of D_+, which has positive eigenvalue and let $X = X_+ + X_- \in D_+ \oplus D_-$. From the third and fifth equality in the above theorem and (3.2) we get

$$QX_+ = \mu\lambda X_+ \tag{3.13}$$

where D_- denotes the vectors with negative eigenvalue. On the other hand from forth and last equality in the previous theorem and (3.2) we get

$$QX_- = -\mu\lambda X_-. \tag{3.14}$$

Taking (3.13) , (3.14) and $Q\xi = 2n\kappa\xi$ into account we obtain (3.12). □

Lemma 3. *Let M be an almost cosymplectic manifold with ξ belonging to the (κ,μ)-nullity distribution, then*

$$(\nabla_X S)(Y,Z) = \mu g((\nabla_X h)Y,Z) - 2n\kappa g(Y,\varphi hX)\eta(Z) - 2n\kappa\eta(Y)g(Z,\varphi hX). \tag{3.15}$$

Proof. From (3.12) we get

$$S(Y,Z)=\mu g(hY,Z)+2n\kappa\eta(Y)\eta(Z).$$

Taking covariant derivative by the direction X, we can write

$$(\nabla_X S)(Y,Z)=\nabla_X S(Y,Z)-S(\nabla_X Y,Z)-S(Y,\nabla_X Z).$$

Using (2.1)-(2.4) and (3.15) we get the proof. □

4. Almost Cosymplectic (κ,μ)-Spaces with Cyclic-Parallel Ricci Tensor

The Ricci tensor S of a Riemannian manifold M is said to be cyclic-parallel if

$$(\nabla_Z S)(X,Y)+(\nabla_X S)(Y,Z)+(\nabla_Y S)(Z,X)=0, \tag{4.1}$$

for all vector fields $X,\ Y,\ Z$.

Let M be an η-Einstein manifold whose Ricci tensor S of the form

$$S(X,Y)=Ag(X,Y)+B\eta(X)\eta(Y), \tag{4.2}$$

where $A,\ B$ are non-zero real numbers and $X,\ Y$ are vector fields on M. So we have;

Theorem 3. *Let M be an almost cosymplectic (κ,μ)-space,η-Einstein manifold of the form (4.2). If the Ricci tensor S of M is cyclic parallel then M is locally the product of a Kähler manifold N and an interval or unit circle S^1.*

Proof. Let us consider M is an almost cosymplectic (κ,μ)-space; η-Einstein manifold of the form (4.2). If the Ricci tensor S of M is cyclic parallel then replacing Z with ξ in 4.1 we can write

$$(\nabla_\xi S)(X,Y)+(\nabla_X S)(Y,\xi)+(\nabla_Y S)(\xi,X)=0.$$

Using (4.2), after some computation we get

$$(\nabla_X S)(Y,Z)=B\left[\eta(Z)g(Y,\nabla_X\xi)+\eta(Y)g(Z,\nabla_X\xi)\right],$$

which implies

$$(\nabla_\xi S)(X,Y)=0, \tag{4.3}$$

$$(\nabla_X S)(Y,\xi)=Bg(Y,\nabla_X\xi), \tag{4.4}$$

$$(\nabla_Y S)(\xi,X)=Bg(X,\nabla_Y\xi). \tag{4.5}$$

So substituting (4.3)-(4.5)

$$B\left[g(Y,\nabla_X\xi)+g(X,\nabla_Y\xi)\right]=0,$$

$$B\left[g(Y,-\varphi hX)+g(X,-\varphi hY)\right]=0,$$

$$g(X,\varphi hY)=0. \tag{4.6}$$

Replacing Y with φY, the equation (4.6) can be written as

$$g\left(X,\varphi^2 hY\right)=g(X,hY)=0,$$

for all vector fields X and Y and hence we have $h=0$ which implies M is cosymplectic. □

Theorem 4. *Let M be an almost cosymplectic (κ,μ)-space if the Ricci tensor S of M is cyclic parallel then M is locally Riemannian manifold which is either locally the product of a Kähler manifold N and an interval or unit circle S^1, or $\kappa=-\dfrac{\mu^2}{4n}$.*

Proof. Let M be an almost cosymplectic (κ,μ)-space. By the use of (3.15) we have

$$(\nabla_\xi S)(X,Y)=\mu g((\nabla_\xi h)X,Y)-2n\kappa g(X,\varphi h\xi)\eta(Y)-2n\kappa g(Y,\varphi h\xi)\eta(Z),$$

$$(\nabla_\xi S)(X,Y)=\mu g((\nabla_\xi h)X,Y).$$

But by making use of $(\nabla_\xi h)X=-\mu\varphi hX$ we get

$$\begin{aligned}(\nabla_\xi S)(X,Y) &= \mu g(-\mu\varphi hX,Y)\\ &= -\mu^2 g(\varphi hX,Y),\end{aligned}\tag{4.7}$$

$$\begin{aligned}(\nabla_X S)(Y,\xi) &= \mu g((\nabla_X h)\xi,Y)-2n\kappa g(Y,\varphi hX)\\ &= \mu g(h\varphi hX,Y)-2n\kappa g(Y,\varphi hX)\\ &= -\mu g\left(-\varphi h^2X,Y\right)-2n\kappa g(\varphi hX,Y),\end{aligned}\tag{4.8}$$

$$(\nabla_Y S)(\xi,X)=-\mu g\left(Y,\varphi h^2X\right)-2n\kappa g(Y,\varphi hX).\tag{4.9}$$

So substituting (4.7)-(4.9)

$$-\left(\mu^2+4n\kappa\right)g(X,\varphi hY)=0.$$

Suppose $g(hX,\varphi Y)=0$. Then replacing Y with φY the last equation becomes $g\left(hX,\varphi^2Y\right)=0$. So we get $g(hX,Y)=0$ for all vector field X and Y, hence we have $h=0$, if $\mu^2+4n\kappa=0$ then we get $\kappa=-\dfrac{\mu^2}{4n}$. □

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